# DIFFRACTION OF TIDAL WAVES ON FLAT ROTATING SHEETS OF WATER

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1. The methods of the following paper are very similar to those already well known for two-dimensional problems in the diffraction of sound and electric waves. The results, however, are different owing to the different boundary conditions. Also, for the diffraction by objects whose linear dimensions are small compared with a wave length, the interest in the results lies mainly in their form near the objects themselves, which is not usually the case for sound and electric waves.

We shall only consider free tidal motion of sheets of water of uniform depth.

A complete solution is obtained for the case of the diffraction of a plane wave by a circular island, but the remaining solutions are all approximations. They are based on Lord Rayleigh's approximate theory of diffraction,* and the method of conjugate functions is introduced so that Schwarz's method for conformal transformations becomes available. The number of problems which can thus be approximately solved (at least symbolically) is quite large, but, as examples, we shall only consider those of the diffraction of a plane wave by an elliptic island, by a semi-elliptic cape, by a rectangular bay, and by a passage between one sea and another.

## *General Equations.*

2. Suppose the sheet of water to be rotating with constant angular velocity $\omega$ about an axis perpendicular to its plane, and let the depth

* "On the Passage of Waves through Apertures in Plane Screens, and Allied Problems," *Phil. Mag.* (5), Vol. XLIII, p. 259 (1897), [*Sc. Papers*, Vol. IV, p. 283]. Also, "On the Incidence of Aerial and Electric Waves upon Small Obstacles...," *Phil. Mag.* (5), Vol. XLIV, p. 28 (1897) [*Sc. Papers*, Vol. IV, p. 305].

of the water (as rotating in free relative equilibrium) be uniform and equal to $h$. Let $\zeta$ denote the elevation of the free surface at any time.

Then, for a disturbance in which the time $t$ only enters through the factor $e^{i\sigma t}$, we have the equation *

$$(\nabla_1^2+\kappa^2)\,\zeta = 0, \tag{1}$$

there being no disturbing force. Here $\nabla_1^2 = \partial^2/\partial x^2+\partial^2/\partial y^2$ in Cartesian coordinates, and

$$\kappa^2 = \frac{\sigma^2-4\omega^2}{gh}, \tag{2}$$

$g$ being the acceleration due to gravity.

The boundary condition is given by

$$i\sigma\frac{\partial\zeta}{\partial n}+2\omega\frac{\partial\zeta}{\partial s} = 0, \tag{3}$$

where $\partial/\partial n$ denotes differentiation along the outward drawn normal to the boundary, and $\partial/\partial s$ along the positive direction of the arc. We exclude the cases in which $\sigma = 0$ and $\sigma^2 = 4\omega^2$.

When we use polar coordinates $r$, $\theta$, normal solutions of (1) are given by

$$J_m(\kappa r)\,e^{\pm im\theta}, \qquad D_m(\kappa r)\,e^{\pm im\theta},$$

$m$ being any constant. Here $J_m(\kappa r)$, $D_m(\kappa r)$ are Bessel's functions of the first and second kind respectively, the latter being taken to be the form appropriate for the disturbances produced by a finite object in an infinite sea.

For small values of $\kappa r$, taking only principal parts, we have

$$J_0(\kappa r) = 1, \qquad J_m(\kappa r) = \frac{(\kappa r)^m}{2^m m!}, \tag{4}$$

$$D_0(\kappa r) = -\frac{2}{\pi}\left\{\log\tfrac{1}{2}\kappa r+\gamma+\tfrac{1}{2}i\pi\right\}, \qquad D_m(\kappa r) = \frac{2^m(m-1)!}{\pi(\kappa r)^m}, \tag{5}$$

$\gamma$ being Euler's constant. These, with the associated functions of $\theta$, all form two-dimensional harmonic functions.

*Diffraction of a Plane Wave by a Circular Island.*

3. Let the primary wave be given in Cartesians by

$$\zeta = \zeta_0 e^{i\sigma t} \equiv e^{i(\kappa x+\sigma t)}, \tag{6}$$

which obviously satisfies the equation (1).

* See Lamb, *Hydrodynamics*, 3rd ed., p. 303.

Transforming to polar coordinates, we have

$$\zeta_0 = e^{i\kappa r\cos\theta} = J_0(\kappa r) + 2\sum_{n=1}^{\infty} i^n J_n(\kappa r)\cos n\theta. \tag{7}$$

Let us assume that the disturbance produced by the island is given by $\zeta = \zeta_1 e^{i\sigma t}$, where

$$\zeta_1 = A_0 D_0(\kappa r) + 2\sum_{n=1}^{\infty} i^n D_n(\kappa r)\{A_n\cos n\theta + B_n \sin n\theta\}, \tag{8}$$

$A_n$, $B_n$ being constants whose values are to be determined.

If the shore of the island be given by $r = a$, the boundary condition will be

$$\left(i\sigma\frac{\partial}{\partial r} + 2\omega\frac{\partial}{a\,\partial\theta}\right)(\zeta_0+\zeta_1) = 0, \tag{9}$$

for $r = a$, and all values of $\theta$. Substituting in this from (7) and (8), we obtain

$$i\sigma\kappa J_n'(\kappa a)\cos n\theta - \frac{2\omega n}{a} J_n(\kappa a)\sin n\theta + i\sigma\kappa D_n'(\kappa a)\{A_n\cos n\theta + B_n\sin n\theta\}$$

$$-\frac{2\omega n}{a} D_n(\kappa a)\{A_n \sin n\theta - B_n\cos n\theta\} = 0, \tag{10}$$

which holds for all the values of $n$, including zero, if we take $B_0 = 0$.

On equating to zero the coefficients of $\cos n\theta$ and $\sin n\theta$ in (10) we obtain

$$i\sigma\kappa D_n'(\kappa a) A_n + \frac{2\omega n}{a} D_n(\kappa a) B_n + i\sigma\kappa J_n'(\kappa a) = 0,$$

$$\frac{2\omega n}{a} D_n(\kappa a) A_n - i\sigma\kappa D_n'(\kappa a) B_n + \frac{2\omega n}{a} J_n(\kappa a) = 0.$$

Solving these algebraically, we obtain

$$A_n = -\frac{\sigma^2(\kappa a)^2 J_n'(\kappa a)\, D_n'(\kappa a) - 4\omega^2 n^2 J_n(\kappa a)\, D_n(\kappa a)}{\sigma^2(\kappa a)^2 D_n'^2(\kappa a) - 4\omega^2 n^2 D_n^2(\kappa a)}, \tag{11}$$

$$B_n = -i\frac{2\omega n\sigma\kappa a\,\{J_n(\kappa a)\, D_n'(\kappa a) - J_n'(\kappa a)\, D_n(\kappa a)\}}{\sigma^2(\kappa a)^2 D_n'^2(\kappa a) - 4\omega^2 n^2 D_n^2(\kappa a)}, \tag{12}$$

for all the values of $n$, including zero.

We therefore have

$$-\zeta_1 = \frac{J_0'(\kappa a)}{D_0'(\kappa a)} D_0(\kappa r) + 2\sum_{n=1}^{\infty}\frac{i^n}{\sigma^2(\kappa a)^2 D_n'^2(\kappa a) - 4\omega^2 n^2 D_n^2(\kappa a)}$$

$$\times D_n(\kappa r)\left[\{\sigma^2(\kappa a)^2 J_n'(\kappa a)\, D_n'(\kappa a) - 4\omega^2 n^2 J_n(\kappa a)\, D_n(\kappa a)\}\cos n\theta\right.$$

$$\left. + 2\omega n\sigma\kappa a\,\{J_n(\kappa a)\, D_n'(\kappa a) - J_n'(\kappa a)\, D_n(\kappa a)\}\, i\sin n\theta\right]. \tag{13}$$

Now let us take the principal part of this for points near the island when $\kappa a$ is very small. We have, on substituting from (4) and (5),

$$\zeta_1 = \frac{\kappa a}{\sigma^2 - 4\omega^2} \frac{a}{r} \{(\sigma^2 + 4\omega^2) i \cos\theta - 4\omega\sigma \sin\theta\}. \tag{14}$$

To the same approximation the total elevation is given by

$$\zeta = \left[1 + \left\{\kappa r + \kappa a \frac{\sigma^2 + 4\omega^2}{\sigma^2 - 4\omega^2} \frac{a}{r}\right\} i \cos\theta - \frac{4\omega\sigma\kappa a}{\sigma^2 - 4\omega^2} \frac{a}{r} \sin\theta\right] e^{i\sigma t}, \tag{15}$$

the real part of which may be written

$$\zeta = \left\{1 - \frac{4\omega\sigma\kappa a}{\sigma^2 - 4\omega^2} \frac{a}{r} \sin\theta\right\} \cos(\sigma t - \epsilon), \tag{16}$$

where
$$\epsilon = -\left\{\kappa r + \kappa a \frac{\sigma^2 + 4\omega^2}{\sigma^2 - 4\omega^2} \frac{a}{r}\right\} \cos\theta. \tag{17}$$

Similarly written, and to the same order of approximation, the primary wave is given by

$$\zeta = \cos(\sigma t - \epsilon_0), \quad \epsilon_0 = -\kappa r \cos\theta. \tag{18}$$

The change in the height of high tide can be obtained from (16), and the change in phase from (17).

### *Approximate Theory.*

4. Lord Rayleigh's approximate theory of diffraction depends on the fact that over a region the linear dimensions of which are small compared with a wave-length, a solution of (1) is sensibly a two-dimensional harmonic function. Over such a region, therefore, in the present application, we treat $\zeta$ as such a function, and proceed to satisfy the boundary conditions. This is facilitated by the use of conjugate functions.

Suppose that $\zeta = \zeta_1 + i\zeta_2$, where $\zeta_1$, $\zeta_2$ are real. Take $\zeta' = \zeta_1' + i\zeta_2'$, where $\zeta_1'$, $\zeta_2'$ are real and such that $\zeta_1 + i\zeta_1'$, $\zeta_2 + i\zeta_2'$ are functions of $x + iy$, $x$, $y$ being Cartesian coordinates in the plane of the sheet of water. $\zeta'$ is now determined except to an arbitrary additive constant.

At the boundary we shall have

$$\frac{\partial\zeta_1}{\partial n} = \frac{\partial\zeta_1'}{\partial s}, \quad \frac{\partial\zeta_2}{\partial n} = \frac{\partial\zeta_2'}{\partial s},$$

and consequently $\displaystyle i\sigma \frac{\partial\zeta}{\partial n} + 2\omega \frac{\partial\zeta}{\partial s} = i\sigma \frac{\partial\zeta'}{\partial s} + 2\omega \frac{\partial\zeta}{\partial s} = 0.$

If then we take
$$\psi = i\sigma\zeta' + 2\omega\zeta, \tag{19}$$
$\psi$ will be a two-dimensional harmonic function which is constant along a coast-line. Let us also take
$$\phi = i\sigma\zeta - 2\omega\zeta', \tag{20}$$
so that if $\phi = \phi_1 + i\phi_2$, and $\psi = \psi_1 + i\psi_2$, when $\phi_1$, $\phi_2$ $\psi_1$, $\psi_2$ are real, we shall have
$$\phi_1 = -\sigma\zeta_2 - 2\omega\zeta_1', \qquad \phi_2 = \sigma\zeta_1 - 2\omega\zeta_2',$$
$$\psi_1 = -\sigma\zeta_2' + 2\omega\zeta_1, \qquad \psi_2 = \sigma\zeta_1' + 2\omega\zeta_2.$$

We then see that $\phi_1 + i\psi_1$, $\phi_2 + i\psi_2$ are functions of $x+iy$, and that when either $\phi$ or $\psi$ is known the other is determinate except to an arbitrary additive constant.

From (19) and (20) we have the relation
$$\zeta = -\frac{1}{\sigma^2 - 4\omega^2}(i\sigma\phi + 2\omega\psi). \tag{21}$$

Let us use $\zeta_0$, $\zeta_0'$, $\phi_0$, $\psi_0$ to denote the respective values of $\zeta$, $\zeta'$, $\phi$, $\psi$ for the primary wave only.

We require further conditions for the functions $\phi - \phi_0$, $\psi - \psi_0$, and these are to be determined by the principal parts near the diffracting object of the possible forms for the complete expression of the disturbance.

We shall assume that for the seas in question the possible forms for $\phi - \phi_0$ are such as vanish at infinity or else take the form
$$\log(\tfrac{1}{2}\kappa r) + \gamma + \tfrac{1}{2}i\pi. \tag{22}$$
The form (22) will only be required when the effect at a distance is that due to a source.

Of course, the consideration of forms at infinity is only auxiliary, the results being applicable only over a very small region.

On the above principles the expression (14) can be very easily reproduced.

### *Diffraction by an Elliptic Island.*

5. Let the equation of the shore of the island be given by $\xi = \alpha$, where
$$x = c\cosh\xi\cos\eta, \quad y = c\sinh\xi\sin\eta,$$

and let the primary wave be given by

$$\zeta = e^{i\kappa(lx+my)+i\sigma t}, \tag{23}$$

where $l^2+m^2 = 1$.

Then, in the neighbourhood of the island, we have

$$\zeta_0 = 1+i\kappa(lx+my), \quad \zeta_0' = i\kappa(ly-mx),$$

on dropping the time factor, so that

$$\phi_0 = i\sigma+(i\sigma l+2\omega m)i\kappa x+(i\sigma m-2\omega l)i\kappa y, \tag{24}$$

$$\psi_0 = 2\omega+(i\sigma l+2\omega m)i\kappa y-(i\sigma m-2\omega l)i\kappa x. \tag{25}$$

In terms of $\xi$, $\eta$, $\psi_0$ becomes

$$\psi_0 = 2\omega+(i\sigma l+2\omega m)i\kappa c \sinh\xi \sin\eta-(i\sigma m-2\omega l)i\kappa c \cosh\xi \cos\eta.$$

For the total disturbance we shall then have

$$\psi = 2\omega+i\kappa c(i\sigma l+2\omega m)(\sinh\xi-e^{a-\xi}\sinh a)\sin\eta$$
$$-i\kappa c(i\sigma m-2\omega l)(\cosh\xi-e^{a-\xi}\cosh a)\cos\eta,$$

since this is constant over $\xi = a$, and differs from $\psi_0$ by a function which tends to vanish as $\xi \to \infty$.

Simplifying, we have

$$\psi = 2\omega+i\kappa c e^a(i\sigma l+2\omega m)\sinh(\xi-a)\sin\eta$$
$$-i\kappa c e^a(i\sigma m-2\omega l)\sinh(\xi-a)\cos\eta, \tag{26}$$

and then $\phi$ must be given by

$$\phi = i\sigma+i\kappa c e^a(i\sigma l+2\omega m)\cosh(\xi-a)\cos\eta$$
$$+i\kappa c e^a(i\sigma m-2\omega l)\cosh(\xi-a)\sin\eta, \tag{27}$$

in accordance with § 4.

Substituting from these into (21), we obtain, after a little reduction,

$$\zeta = 1-\frac{\kappa c e^a}{\sigma^2-4\omega^2}\Big\{2\omega\sigma e^{a-\xi}(l\sin\eta-m\cos\eta)$$
$$-i[\sigma^2\cosh(\xi-a)-4\omega^2\sinh(\xi-a)](l\cos\eta+m\sin\eta)\Big\}. \tag{28}$$

Restoring the time factor, and taking only the real part, we have, to our order of approximation,

$$\zeta = \left\{1-\frac{2\omega\sigma\kappa c e^{2a-\xi}}{\sigma^2-4\omega^2}(l\sin\eta-m\cos\eta)\right\}\cos(\sigma t-\epsilon), \tag{29}$$

where $\epsilon = -\frac{\kappa c e^{a}}{\sigma^2-4\omega^2}\left\{\sigma^2\cosh(\xi-a)-4\omega^2\sinh(\xi-a)\right\}(l\cos\eta-m\sin\eta)$. (30)

Similarly written, the primary wave is

$$\zeta = \cos(\sigma t-\epsilon_0), \qquad \epsilon_0 = -\kappa c(l\cosh\xi\cos\eta+m\sinh\xi\sin\eta). \tag{31}$$

*Diffraction by a Semi-Elliptic Cape.*

6. We consider a cape projecting from a straight coast. Let the equation of the shore of the cape be given by $\xi = a$, where

$$x = c\sinh\xi\cos\eta, \qquad y = c\cosh\xi\sin\eta,$$

for $0 \leqslant \eta \leqslant \pi$, the equation of the coast-line being $y = 0$ or $\eta = 0$ and $\pi$.

Let the primary wave be given by

$$\zeta = e^{-\kappa\sqrt{(\sigma^2-4\omega^2)}.(i\sigma x+2\omega y)+i\sigma t}, \tag{32}^*$$

which satisfies (1) and gives no velocity perpendicular to the straight coast-line.

Near the cape we have, on dropping the time factor,

$$\zeta_0 = 1-\frac{\kappa}{\sqrt{(\sigma^2-4\omega^2)}}(i\sigma x+2\omega y), \qquad \zeta_0' = -\frac{\kappa}{\sqrt{(\sigma^2-4\omega^2)}}(i\sigma y-2\omega x),$$

giving $$\phi_0 = i\sigma+\sqrt{(\sigma^2-4\omega^2)}\,\kappa x, \qquad \psi_0 = 2\omega+\sqrt{(\sigma^2-4\omega^2)}\,\kappa y. \tag{33}$$

As the change in $\phi$ produced by the cape will be real, and the change in $\psi$ will vanish on the straight coast, we see from (21) that on the cape and the neighbouring coast the high tide will have its original coastal height.

In terms of $\xi$, $\eta$, $\psi_0$ becomes

$$\psi_0 = 2\omega+\sqrt{(\sigma^2-4\omega^2)}\,\kappa c\cosh\xi\sin\eta,$$

and then for the total disturbance it is easily seen that we have

$$\psi = 2\omega+\sqrt{(\sigma^2-4\omega^2)}\,\kappa c e^{a}\sinh(\xi-a)\sin\eta, \tag{34}$$

$$\phi = i\sigma+\sqrt{(\sigma^2-4\omega^2)}\,\kappa c e^{a}\cosh(\xi-a)\cos\eta. \tag{35}$$

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* This is the solution first given by Lord Kelvin with reference to a rotating canal. See Lamb, *l.c.*

Substituting from these into (21) we obtain

$$\zeta = 1 - \frac{\kappa c e^{a}}{\sqrt{(\sigma^2-4\omega^2)}} \{ i\sigma \cosh(\xi - a)\cos\eta + 2\omega \sinh(\xi - a)\sin\eta \}. \quad (36)$$

Restoring the time factor and taking only the real part, we have, to our order of approximation,

$$\zeta = \left\{ 1 - \frac{2\omega\kappa c e^{a}}{\sqrt{(\sigma^2-4\omega^2)}} \sinh(\xi - a)\sin\eta \right\} \cos(\sigma t - \epsilon), \quad (37)$$

where
$$\epsilon = \frac{\sigma\kappa c e^{a}}{\sqrt{(\sigma^2-4\omega^2)}} \cosh(\xi - a)\cos\eta. \quad (38)$$

Similarly written, the primary wave is

$$\zeta = \left\{ 1 - \frac{2\omega\kappa c}{\sqrt{(\sigma^2-4\omega^2)}} \cosh\xi \sin\eta \right\} \cos(\sigma t - \epsilon_0), \quad (39)$$

where
$$\epsilon_0 = \frac{\sigma\kappa c}{\sqrt{(\sigma^2-4\omega^2)}} \sinh\xi \cos\eta. \quad (40)$$

*Coast-lines with Projecting Corners.*

7. If we take $a = 0$ in the preceding section, we have a formal solution for the case of a straight narrow *promontory* of length $c$ projecting perpendicularly from a straight coast.

The expressions found, however, lead to an infinite velocity for the water at the projecting end of the promontory, owing to the vanishing there of the Jacobian $\partial(x, y)/\partial(\xi, \eta)$. The question then arises whether the formal solution gives an approximate solution of the physical problem except in the immediate neighbourhood of the end of the promontory, or whether it fails altogether. The question applies to all cases in which an acute-angled portion of land projects into the sea.*

Below are given arguments, tending to show, that with a more stringent condition on the size of the motion in the incident wave, the formal solutions will be approximate representations of the physical facts, except in the immediate neighbourhood of the projecting sharp corners.

* This question was raised by the referee. It would appear to apply also to certain solutions of problems in the diffraction of sound waves given in the papers of Lord Rayleigh already quoted, and to be then answerable in a manner similar to that of the present section.

Except in these neighbourhoods, the formal solutions only undergo small changes in character owing to a small change in the shape of the disturbing object. This remains true even though a round corner changes into a sharp point, as in the case of the promontory.

We assume that the same continuity holds for the actual physical state.

If therefore we can show that it is possible to round off the sharp corners, without materially altering the general configuration of the object, but making the curvatures everywhere small enough for the formal solution in the modified case to be a true representation of the physical facts in this case, we may perhaps assume that the formal solution in the original case (*i.e.*, with the sharp corners), gives an approximate representation of the physical facts in that case, except in the neighbourhood of the corners.

Taking a primary wave of the general type

$$\zeta = Ae^{i(\sigma t+\kappa x)},$$

it is necessary for the validity of the equations of § 2 that the velocity of the water, which is proportional to the gradient of $\zeta$, shall be small compared with $(gh)^{\frac{1}{2}}$. This requires that $A/h$ shall be small.*

Now let $c$ denote the linear order of magnitude of the disturbing object, and $a$ the smallest radius of curvature of the coast-line. We must then have $\kappa c$ small, while at the points of greatest curvature on the coast-line, the order of magnitude of the velocity will bear to that of the primary wave the ratio $c/a$.

We must then have $Ac/ha$ small, while the purpose of the present section requires that $a/c$ shall also be small. Together, these conditions require that $(A/h)^{\frac{1}{2}}$ shall be small.

If then $(A/h)^{\frac{1}{2}}$ and not merely $A/h$ may be considered as small, the modification mentioned above will be a possible one. If, however, $(A/h)^{\frac{1}{2}}$ cannot be considered as small, the partial justification of the present section breaks down.

In the remaining sections we shall assume that the motion in he primary wave is small enough to satisfy the condition just obtained.

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* *Cf.*, Lamb, *l.c.*, p. 243.

*Diffraction by a Rectangular Bay.*

8. Let the three sides of the bay be given in Cartesians by $x = \pm a$, $y = 0$; and the remaining coast-line by the portions of the line $y = b$ for which $|x| > a$ (Fig. 1).

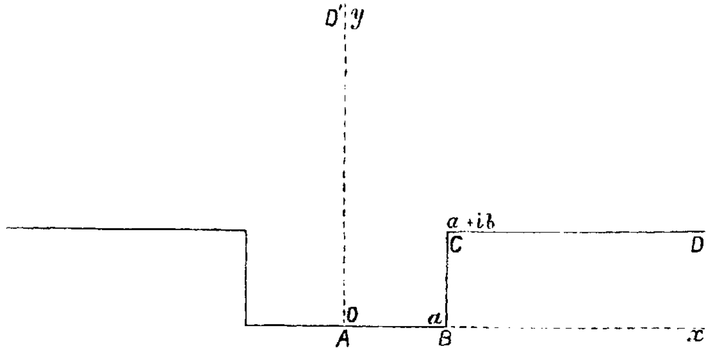


FIG. 1.—$z$ plane.

Let us take the same primary wave as in § 6, so that again

$$\phi_0 = i\sigma + \sqrt{(\sigma^2 - 4\omega^2)}\,\kappa x, \qquad \psi_0 = 2\omega + \sqrt{(\sigma^2 - 4\omega^2)}\,\kappa y. \tag{33}$$

Let us also write $$w = \frac{1}{\sqrt{(\sigma^2 - 4\omega^2)}}\,\{\phi + i\psi - i(\sigma + 2\omega)\}. \tag{41}$$

Then $w$ will be a function of $x+iy$ which is purely imaginary on $x = 0$, while the imaginary part will be equal to $i\kappa b$ on the coast-line, and to $i\kappa y$ at infinity. The correspondence between $w$ and $z$ is shown in Figs. 1 and 2, where corresponding points are similarly lettered.

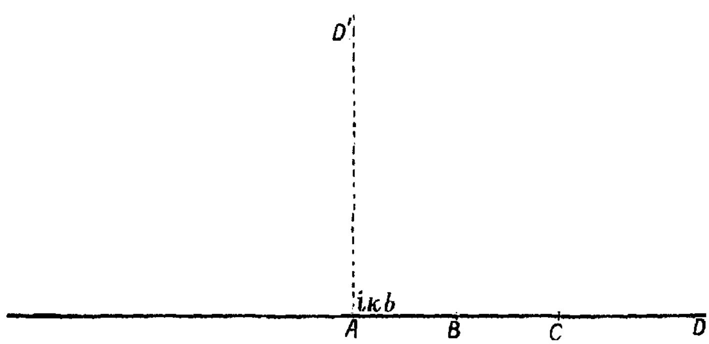


FIG. 2.—$w$ plane.

Taking an auxiliary $t$ plane, to the upper half of which both the regions

$ABCDD'A$ of Figs. 1 and 2 correspond, we may use Schwarz's transformation.

Supposing the points $A$, $B$, $C$, $D$ to correspond respectively to $t = 0,\ 1,\ 1/k^2,\ \infty$, where $k$ has to be found, we have

$$dz = \frac{M(1-k^2t)^{\frac{1}{2}}\,dt}{t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}}, \qquad dw = \frac{N\,dt}{t^{\frac{1}{2}}}, \tag{42}$$

$M$, $N$ being constants whose values are to be determined.

Since $dw/dz = \kappa$ at infinity, we have $N = M\kappa k$.

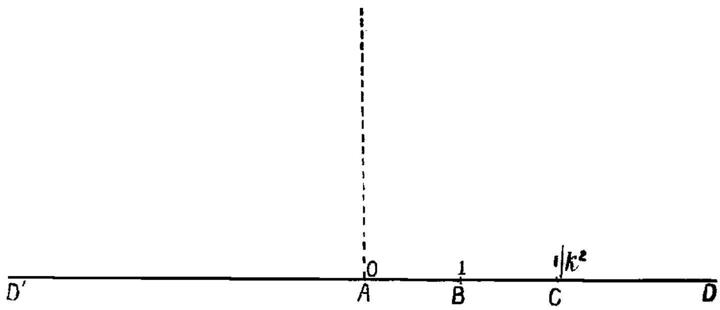


FIG. 3.—$t$ plane.

Let us now write $t^{\frac{1}{2}} = \operatorname{sn} u$, the ordinary Jacobian elliptic function. We shall have

$$w = \iota\kappa b + 2M\kappa k \operatorname{sn} u, \tag{43}$$

and

$$dz = 2M \operatorname{dn}^2 u\, du,$$

giving

$$z = 2ME(u). \tag{44}$$

Here $E(u)$ is the elliptic integral usually so denoted, and the corresponding part of the $u$ plane is shown in Fig. 4.

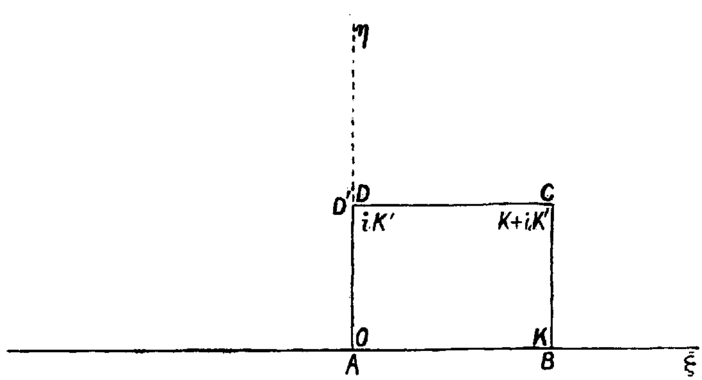


FIG. 4.—$u$ plane.

Now from the correspondences for $B$ and $C$, we have

$$a = 2ME(K) = 2ME,$$

$$a+ib = 2ME(K+iK') = 2M\{E+i(K'-E')\},$$

$K$, $K'$, $E$, $E'$ being the constants usually so denoted in the theory of elliptic functions.

Thus $$M = a/2E,$$

and $$\frac{K'-E'}{E} = \frac{b}{a}, \tag{45}$$

the latter determining the modulus $k$.

If we take $u = \xi+i\eta$, where $\xi$, $\eta$ are real, we shall have

$$w = i\kappa b+\kappa a\,\frac{k}{E}\,\frac{\operatorname{sn}\xi\,\operatorname{dn}\eta}{1-\operatorname{dn}^2\xi\,\operatorname{sn}^2\eta}+i\kappa a\,\frac{k}{E}\,\frac{\operatorname{cn}\xi\,\operatorname{dn}\xi\,\operatorname{sn}\eta\,\operatorname{cn}\eta}{1-\operatorname{dn}^2\xi\,\operatorname{sn}^2\eta}, \tag{46}$$

from the ordinary addition formula, and the imaginary transformation, $k$ being the modulus for the functions of $\xi$, and $k'$ that for the functions of $\eta$.

For the tidal problem we have therefore

$$\phi = i\sigma+\sqrt{(\sigma^2-4\omega^2)}\,\kappa a\,\frac{k}{E}\,\frac{\operatorname{sn}\xi\,\operatorname{dn}\eta}{1-\operatorname{dn}^2\xi\,\operatorname{sn}^2\eta}, \tag{47}$$

$$\psi = 2\omega+\sqrt{(\sigma^2-4\omega^2)}\left\{\kappa b+\kappa a\,\frac{k}{E}\,\frac{\operatorname{cn}\xi\,\operatorname{dn}\xi\,\operatorname{sn}\eta\,\operatorname{cn}\eta}{1-\operatorname{dn}^2\xi\,\operatorname{sn}^2\eta}\right\}, \tag{48}$$

and consequently

$$\zeta = 1-\frac{1}{\sqrt{(\sigma^2-4\omega^2)}}\left\{i\sigma\kappa a\,\frac{k}{E}\,\frac{\operatorname{sn}\xi\,\operatorname{dn}\eta}{1-\operatorname{dn}^2\xi\,\operatorname{sn}^2\eta}+2\omega\kappa b\right.$$
$$\left.+2\omega\kappa a\,\frac{k}{E}\,\frac{\operatorname{cn}\xi\,\operatorname{dn}\xi\,\operatorname{sn}\eta\,\operatorname{cn}\eta}{1-\operatorname{dn}^2\xi\,\operatorname{sn}^2\eta}\right\}, \tag{49}$$

$\xi$, $\eta$ being given by $$x+iy = \frac{a}{E}\,E(\xi+i\eta),$$

and the modulus $k$ by (45).

Restoring the time factor, and taking only the real part, we have, to our order of approximation,

$$\zeta = \left\{1-\frac{2\omega}{\sqrt{(\sigma^2-4\omega^2)}}\left(\kappa b+\kappa a\,\frac{k}{E}\,\frac{\operatorname{cn}\xi\,\operatorname{dn}\xi\,\operatorname{sn}\eta\,\operatorname{cn}\eta}{1-\operatorname{dn}^2\xi\,\operatorname{sn}^2\eta}\right)\right\}\cos(\sigma t-\epsilon), \tag{50}$$

where $$\epsilon = \frac{\sigma\kappa a}{\sqrt{(\sigma^2-4\omega^2)}}\,\frac{k}{E}\,\frac{\operatorname{sn}\xi\,\operatorname{dn}\eta}{1-\operatorname{dn}^2\xi\,\operatorname{sn}^2\eta}. \tag{51}$$

It is interesting to notice such particular cases as are easily evaluated. The changes in phase at $A$ and $B$ are instances.

At $A$, $\xi = \eta = 0$, and therefore $\epsilon = 0$, so that there is no change in phase.

At $B$ we have, for $\epsilon$,

$$\frac{\sigma\kappa a}{\sqrt{(\sigma^2-4\omega^2)}}\frac{k}{E},$$

as against

$$\frac{\sigma\kappa a}{\sqrt{(\sigma^2-4\omega^2)}},$$

for the primary wave.

*Passage between Two Seas.*

9. Let the two seas be given by the upper and lower portions of the $z$ plane, the passage lying between the two points $z = \pm c$, $c$ being real.

Let us take

$$x = c\cosh\xi\cos\eta, \qquad y = c\sinh\xi\sin\eta,$$

and suppose that $\xi$ ranges from $-\infty$ to $\infty$, while $\eta$ is restricted to lie between 0 and $\pi$. Then the coast-lines are given by each side of $\eta = 0$ and of $\eta = \pi$, and the passage by $\xi = 0$.

Let us take the primary wave to be given by (32) for the upper sea only. Then we have

$$\phi_0 = i\sigma+\sqrt{(\sigma^2-4\omega^2)}\,\kappa c\cosh\xi\cos\eta, \quad \psi_0 = 2\omega+\sqrt{(\sigma^2-4\omega^2)}\,\kappa c\sinh\xi\sin\eta, \tag{52}$$

for the upper sea, while over the lower sea we may conveniently take $\zeta_0' = 2\omega/i\sigma$, so that

$$\phi_0 = -\frac{4\omega^2}{i\sigma}, \quad \psi_0 = 2\omega. \tag{53}$$

When $|\xi|$ is large, we have

$$\log\tfrac{1}{2}\kappa r = |\xi| + \log(\tfrac{1}{4}\kappa c),$$

so that at infinity the only effect of the passage on the value of $\phi$ must be a constant multiple of

$$|\xi| + \log(\tfrac{1}{4}\kappa c)+\gamma+\tfrac{1}{2}i\pi.$$

It is then easily seen that the solution is given by

$$\phi = -\frac{1}{2i\sigma}(\sigma^2+4\omega^2)+\frac{1}{2i\sigma}(\sigma^2-4\omega^2)\frac{\xi}{\log(\frac{1}{4}\kappa c)+\gamma+\frac{1}{2}i\pi} + \tfrac{1}{2}\sqrt{(\sigma^2-4\omega^2)}\,\kappa c e^{\xi}\cos\eta, \tag{54}$$

$$\psi = 2\omega+\frac{1}{2i\sigma}(\sigma^2-4\omega^2)\frac{\eta}{\log(\frac{1}{4}\kappa c)+\gamma+\frac{1}{2}i\pi}+\tfrac{1}{2}\sqrt{(\sigma^2-4\omega^2)}\,\kappa c e^{\xi}\sin\eta. \tag{55}$$

These give

$$\zeta = \tfrac{1}{2}\left[1 - \frac{\xi + \dfrac{2\omega}{i\sigma}\eta}{\log(\frac{1}{4}\kappa c) + \gamma + \frac{1}{2}i\pi} - \frac{\kappa c e^{\xi}}{\sqrt{(\sigma^2 - 4\omega^2)}}(i\sigma\cos\eta + 2\omega\sin\eta)\right]e^{i\sigma t}. \quad (56)$$

In the middle of the passage we have $\xi = 0$, $\eta = \frac{1}{2}\pi$, and consequently

$$\zeta = \tfrac{1}{2}\left[1 - \frac{\omega}{i\sigma}\,\frac{\pi}{\log(\frac{1}{4}\kappa c) + \gamma + \frac{1}{2}i\pi} - \frac{2\omega\kappa c}{\sqrt{(\sigma^2 - 4\omega^2)}}\right]e^{i\sigma t},$$

the height of high tide being changed in the ratio

$$1 : \tfrac{1}{2}\left[1 + \frac{\omega}{2\sigma}\,\frac{\pi^2}{\{\log(\frac{1}{4}\kappa c) + \gamma\}^2 + \frac{1}{4}\pi^2} - \frac{2\omega\kappa c}{\sqrt{(\sigma^2 - 4\omega^2)}}\right],$$

and the phase being changed by

$$\frac{\pi\omega}{\sigma}\,\frac{\log(\frac{1}{4}\kappa c) + \gamma}{\{\log(\frac{1}{4}\kappa c) + \gamma\}^2 + \frac{1}{4}\pi^2},$$

providing this be small.