A PROPERTY OF POLYNOMIALS WHOSE ROOTS ARE REAL

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1. Let $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, be the roots, supposed real, unequal, and in ascending order of magnitude, of a polynomial

$$f(x) \equiv x^n - a_1 x^{n-1} + a_2 x^{n-2} - \ldots,$$

and let $\beta_1, \beta_2, \ldots, \beta_{n-1}$, be the roots, also real and in ascending order of magnitude, of

$$\phi(x) \equiv x^{n-1} - b_1 x^{n-2} + \ldots;$$

further, let the roots of $\phi(x)$ separate those of $f(x)$, so that

$$\alpha_1 < \beta_1 \leqslant \alpha_2 \leqslant \beta_2 \leqslant \ldots \leqslant \beta_{n-1} < \alpha_n.$$

Let $f(x)$ be divided by $\phi(x)$ and let the quotient be $x-\gamma$ and the remainder $-A\psi(x)$, where

$$\psi(x) \equiv x^{n-2} - c_1 x^{n-3} + \ldots.$$

Then denoting the roots of $\psi(x)$ by $\delta_1, \delta_2, \ldots, \delta_{n-2}$, it is immediately seen that the δ's are all real and that they separate the β's, and further that γ and the δ's separate the α's. We can thus take γ and the δ's together as a new set of β's: that is, we take, as a new $\phi(x)$, the polynomial

$$(x-\gamma)(x-\delta_1)(x-\delta_2)\ldots(x-\delta_{n-2})$$

and divide $f(x)$ by it as before, obtaining a new γ and new δ's, which separate the α's as before, and so on, continuing in this way indefinitely. We shall show that $\delta_1, \delta_2, \ldots, \delta_{n-2}$ tend to the limits $\alpha_2, \alpha_3, \ldots, \alpha_{n-1}$.

2. Considering any stage of the process, let the greatest value of

$$|\beta_s - \tfrac{1}{2}(\alpha_1+\alpha_n)| \quad (s = 1, 2, 3, \ldots, n-1)$$

be denoted by g. Then since

$$\beta_1 \leqslant \delta_1 \leqslant \beta_2 \leqslant \delta_2 \leqslant \ldots \leqslant \delta_{n-2} \leqslant \beta_{n-1}, \tag{1}$$

it follows that $$|\delta_s - \tfrac{1}{2}(a_1+a_n)| \leqslant g \quad (s = 1, 2, 3, \ldots, n-2). \tag{2}$$

Also $$\gamma = \sum_{s=1}^{n} a_s - \sum_{s=1}^{n-1} \beta_s,$$

and since $$a_1 \leqslant \beta_1 \leqslant a_2 \leqslant \beta_2 \leqslant \ldots \leqslant \beta_{n-1} \leqslant a_n, \tag{3}$$

we get $$\begin{aligned} g &\geqslant \tfrac{1}{2}(a_1+a_n) - \beta_1 \geqslant \gamma - \tfrac{1}{2}(a_1+a_n) \\ &\geqslant \tfrac{1}{2}(a_1+a_n) - \beta_{n-1} \\ &\geqslant -g. \end{aligned}$$

Hence $$|\gamma - \tfrac{1}{2}(a_1+a_n)| \leqslant g. \tag{4}$$

From (2) and (4) we get that if g' is the greatest value of $|\beta_s - \tfrac{1}{2}(a_1+a_n)|$ for the next stage of the process,

$$g' \leqslant g.$$

Thus at no stage does g increase, and, since it is essentially positive, it tends to a definite limit.

3. Again, from the equation

$$f(x) = (x-\gamma)\,\phi(x) - A\psi(x),$$

we get, on equating coefficients of x^{n-2},

$$\begin{aligned} A &= \Sigma\beta_1\beta_2 - \Sigma a_1 a_2 + \gamma\Sigma\beta_1 \\ &= \Sigma\beta_1\beta_2 - \Sigma a_1 a_2 + (\Sigma a_1 - \Sigma\beta_1)\Sigma\beta_1 \\ &= (a_n - \beta_{n-1}) \sum_{s=1}^{n-1} (\beta_s - a_s) + (a_{n-1} - \beta_{n-2}) \sum_{s=1}^{n-2} (\beta_s - a_s) \\ &\quad + \ldots + (a_2 - \beta_1)(\beta_1 - a_1), \end{aligned} \tag{5}$$

so that it follows from (3) that A cannot be negative.

If we denote the value of A at the next stage by A', we have

$$A' = (\Sigma\delta_1\delta_2 + \gamma\Sigma\delta_1) - \Sigma a_1 a_2 + (\gamma + \Sigma\delta_1)(\Sigma a_1 - \gamma - \Sigma\delta_1),$$

since γ and the δ's constitute the new β's.

Thus $$A' = \Sigma\delta_1\delta_2+\gamma\Sigma\delta_1-\Sigma\alpha_1\alpha_2+(\gamma+\Sigma\delta_1)(\Sigma\beta_1-\Sigma\delta_1)$$
$$= \Sigma\delta_1\delta_2-\Sigma\alpha_1\alpha_2+\gamma\Sigma\beta_1+(\Sigma\beta_1-\Sigma\delta_1)\Sigma\delta_1. \tag{6}$$

From (5) and (6), we get

$$A'-A = \Sigma\delta_1\delta_2-\Sigma\beta_1\beta_2+(\Sigma\beta_1-\Sigma\delta_1)\Sigma\delta_1$$
$$= (\beta_{n-1}-\delta_{n-2})\sum_{s=1}^{n-2}(\delta_s-\beta_s)+(\beta_{n-2}-\delta_{n-3})\sum_{s=1}^{n-3}(\delta_s-\beta_s)$$
$$+\ldots+(\beta_2-\delta_1)(\delta_1-\beta_1), \tag{7}$$

so that, from (1), it follows that

$$A' \geqslant A.$$

Since all the β's lie between assignable limits, it is obvious that A cannot exceed an assignable value. Hence A tends to a definite limit.

4. It follows from this result that, ϵ being a given arbitrarily small positive quantity, from and after a certain stage the expression on the right-hand side of (7) will be less than ϵ^2. Since all the products of differences of which this expression is composed are positive, and the expression contains the product $(\beta_{s+1}-\delta_s)(\delta_s-\beta_s)$, we shall then have

$$(\beta_{s+1}-\delta_s)(\delta_s-\beta_s) < \epsilon^2 \quad (s = 1, 2, 3, \ldots, n-2),$$

and hence either $$\beta_{s+1}-\delta_s < \epsilon \quad \text{or} \quad \delta_s-\beta_s < \epsilon. \tag{8}$$

Writing $x = \delta_s$ in the identity

$$f(x) = (x-\gamma)\,\phi(x)-A\psi(x),$$

we get $$f(\delta_s) = (\delta_s-\gamma)\,\phi(\delta_s),$$

and it follows from (8) that

$$|f(\delta_s)| < \epsilon\,(\alpha_n-\alpha_1)^{n-1}$$

Hence from and after a certain stage, δ_s differs from some one of the α's by less than an arbitrarily small ϵ_1. If ϵ_1 is sufficiently small, this one of the α's cannot be either α_1 or α_n. For, if the initial values of $\beta_1-\alpha_1$ and $\alpha_n-\beta_{n-1}$ both exceed h, say, it follows from the result of § 2 that, at every stage of the process, the difference between any δ and either α_1 or α_n also exceeds h.

We thus arrive at the result that, ϵ being given and arbitrarily small, from and after a certain stage in the process, every δ will differ from some one of $a_2, a_3, \ldots, a_{n-1}$, by a quantity less than ϵ.

5. We will next prove that if the process is carried sufficiently far, and if ϵ is sufficiently small, it is impossible that two δ's should differ from the same a by quantities less than ϵ.

For suppose that 2ϵ is less than the difference between any two consecutive a's, and that δ and δ' both differ from a_r by less than ϵ. Since the interval between two consecutive a's cannot contain more than one δ, it follows that δ and δ' are consecutive δ's. Let β, β', β'', be the consecutive β's which they separate. Then we shall have either

$$\text{(i)} \quad \beta < a_{r-1} < \delta < \beta' < a_r < \delta' < \beta'',$$

or

$$\text{(ii)} \quad \beta < \delta < a_r < \beta' < \delta' < a_{r+1} < \beta''.$$

We can suppose the process carried so far that the expression on the right-hand side of (7) is less than either $\frac{1}{2}\epsilon(a_r - a_{r-1})$ or $\frac{1}{2}\epsilon(a_{r+1} - a_r)$. Hence $(\beta'' - \delta')(\delta - \beta)$ is less than either of these quantities.

Now, in case (i),

$$\delta - \beta > \tfrac{1}{2}(a_r - a_{r-1}), \quad \beta'' - \delta' < \epsilon, \quad \beta'' - a_r < 2\epsilon,$$

and in case (ii),

$$\beta'' - \delta' > \tfrac{1}{2}(a_{r+1} - a_r), \quad \delta - \beta < \epsilon, \quad a_r - \beta < 2\epsilon.$$

Thus in either case we shall have two of the δ's and two of the β's all differing from a_r by less than 2ϵ.

Hence from the equation

$$f(x) = (x - \gamma)\,\phi(x) - A\psi(x),$$

we get, observing that the right-hand side must be divisible by $x - a_r$,

$$(x - a_1)(x - a_2)\ldots(x - a_n) = (x - a_r)^2 F(x) + \epsilon(x - a_r)\,G(x),$$

where $|G(a_r)|$ cannot exceed an assignable fixed value, independent of ϵ.

Dividing by $x - a_r$ and putting $x = a_r$, we get

$$f'(a_r) = \epsilon G(a_r),$$

which is impossible if ϵ is sufficiently small. This proves the result required.

6. If, then, ϵ_1 is sufficiently small, and we choose any ϵ less than ϵ_1, we shall have, from and after a certain stage, every δ differing from one of $a_2, a_3, \ldots, a_{n-1}$ by less than ϵ, and no two δ's differing from the same a by less than ϵ. Thus we shall have

$$|\delta_1 - a_2| < \epsilon, \quad |\delta_2 - a_3| < \epsilon, \quad \ldots, \quad |\delta_{n-2} - a_{n-1}| < \epsilon.$$

Thus $\delta_1, \delta_2, \ldots, \delta_{n-2}$ tend to the limits $a_2, a_3, \ldots, a_{n-1}$.

If γ and γ' are the values of γ obtained in two consecutive stages of the operation, it immediately follows from this result, combined with the equation

$$\gamma = \sum_{s=1}^{n} a_s - \sum_{s=1}^{n-1} \beta_s,$$

that $\gamma + \gamma'$ tends to the limit $a_1 + a_n$. Hence the alternate values of γ tend to limits l, l', such that $l + l' = a_1 + a_n$, and we evidently have that if L is the limiting value of A,

$$(x - a_1)(x - a_n) = (x - l)(x - l') - L,$$

so that
$$L = ll' - a_1 a_n.$$

The values of l, l', L, depend on the initial choice of $\beta_1, \beta_2, \ldots, \beta_n$.

7. We have

$$L = (l - a_1)(a_n - l) = (l - a_1)(l' - a_1) = (a_n - l)(a_n - l').$$

It may be observed that

$$|(a_r - l)(a_r - l')| \leqslant (l - a_1)(l' - a_1) \quad (r = 2, 3, \ldots, n-1).$$

For suppose, if possible, that

$$\left| \frac{(a_r - l)(a_r - l')}{(a_1 - l)(a_1 - l')} \right| = 1 + \lambda,$$

where λ is positive.

From the equation

$$(x - a_1)(x - a_2) \ldots (x - a_n)$$
$$= (x - \gamma')(x - \gamma)(x - \delta_1) \ldots (x - \delta_{n-2}) - A(x - \delta_1')(x - \delta_2') \ldots (x - \delta_{n-2}'),$$

we get, putting $x = a_r$,

$$\frac{a_r - \delta_{r-1}'}{a_r - \delta_{r-1}} = \frac{(a_r - \gamma')(a_r - \gamma)}{A} \prod_s \left(\frac{a_r - \delta_s}{a_r - \delta_s'} \right),$$

where the product is taken for every value of s from 1 to $n-2$ inclusive, except $r-1$. Since $\delta_1, \delta_2, \ldots$ tend to limits $a_2, a_3, \ldots$, it follows that this product tends to the limit unity. Also

$$\text{Lim}\left|\frac{(a_r-\gamma')(a_r-\gamma)}{A}\right| = 1+\lambda.$$

Hence from and after a certain stage, we shall have

$$\left|\frac{(a_r-\gamma')(a_r-\gamma)}{A}\right| > 1+\tfrac{1}{2}\lambda, \quad \text{and} \quad \left|\Pi_s\left(\frac{a_r-\delta_s}{a_r-\delta'_s}\right)\right| > \frac{1}{1+\frac{1}{2}\lambda},$$

so that from and after this stage

$$\left|\frac{a_r-\delta'_{r-1}}{a_r-\delta_{r-1}}\right| > 1,$$

which is impossible, since $\text{Lim}\,\delta_{r-1} = a_r$.

8. It has been supposed that the roots of $f(x)$ are unequal, but we may evidently dispense with this condition, for if $f(x)$ contains a factor $(x-a)^r$, we must suppose the original $\phi(x)$ to contain a factor $(x-a)^{r-1}$, so that $\psi(x)$ also contains a factor $(x-a)^{r-1}$, and so on. In fact, the polynomial $\psi(x)$ obtained at any stage is $(x-a)^{r-1}\psi_1(x)$, where $\psi_1(x)$ is the polynomial obtained at the corresponding stage of the process of repeated division of $f(x)/(x-a)^{r-1}$. The theorem thus holds in all cases when the roots of $f(x)$ are real.

If we divide $f(x)$ by the successive polynomials $\psi(x)$, we get a succession of quadratics, whose roots tend to the limits a_1 and a_n. Taking a polynomial $\psi(x)$, whose roots are sufficiently close to $a_2, a_3, \ldots, a_{n-1}$, we can apply the same process to it, and in this way we can obtain a quadratic whose roots are as close as we please to a_2 and a_{n-1}, and so on. We can thus obtain a set of equations, which are all quadratic if n be even, or one linear and the rest quadratic if n be odd, whose roots approximate as closely as we please to a_1 and a_n, a_2 and a_{n-1}, a_3 and a_{n-2}, &c.

In particular, we can take, for the initial $\phi(x)$, the first derived $f'(x)$ of $f(x)$. In this case, the coefficients in the quadratic equations are rational functions of the coefficients of $f(x)$. The process can thus be applied to an algebraic function y defined by an equation $f(y, x) = 0$, and we get the following result:—If for all real values of x in the range $a \leqslant x \leqslant b$, the values of y given by the equation $f(y, x) = 0$ are real, these values may be represented, to an arbitrarily close approximation, as the values of rational functions of x, or the roots of quadratic equations whose coefficients are rational functions of x; these equations may be obtained by the process described above, the initial divisor being $f'_y(y, x)$. As a trivial example, if $|x| \leqslant 1$, successive approximations to one value of y given by the equation

$$y^3 - 3y^2 + 4x^2 = 0$$

are

$$y = 2x^2, \quad y = (2x^4 - 4x^2)/(2x^4 - 2x^2 - 1),$$

and so on.