BIRATIONAL TRANSFORMATIONS OF THE CUBIC VARIETY IN FOUR-DIMENSIONAL SPACE.

By **Virgil Snyder** (Ithaca, N. Y.).

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Many problems in geometry depend for their solution upon the answer to the question whether the general cubic variety V_3 in linear space of four dimensions S_4 can be birationally mapped on ordinary space S_3. While the latter problem has not been solved, various methods have been employed which contribute negative results. I have attempted to determine whether continuous groups of birational transformations exist which leave V_3 invariant. All the transformations of this kind that I have succeeded in finding, while forming an infinite continuous series, do not form a continuous group. If it can be proved that the transformations here considered are the only ones which leave V_3 invariant, then it would follow that V_3 is not invariant under any continuous group of birational transformations, and consequently that it is irrational.

1. The non-singular plane cubic curve cannot be birationally mapped on a straight line, although it is invariant under a continuous group of birational transformations. These transformations are of two kinds, the central and the non-central. The former consist of the projection of the curve upon itself from points upon it, and the latter of the product of two such projections from distinct centers. The former constitute a one-parameter series, one transformation being associated with each point of the curve; the latter constitute a one-parameter group. Given any two central projections a, b and any third a'. A fourth central projection b' can be found such that

$$ab = a' b'.$$

The central is involutorial and has four double elements; the non-central is non-periodic except for particular positions of the centers of the component central projections and has no double elements [1]).

[1]) EM. WEYR, *Ueber eindeutige Beziehungen auf einer allgemeinen ebenen Curve dritter Ordnung* [Sitzungsberichte der mathematisch-naturwissenschaftlichen Klasse der Kaiserlichen Akademie der Wissenschaften (Wien), Abt. II, Vol. LXXXVII (1883), pp. 837-872]. The results of WEYR have been corrected and completed by C. SEGRE, *Le corrispondenze univoche sulle curve ellittiche* [Atti della R. Accademia delle Scienze di Torino, Vol. XXIV (1889), pp. 734-756].

The transformations of both kinds are CREMONIAN, a central projection being expressed as a quadratic inversion of the second kind, that is, a conic C and a point O on it being given, the image P' of P is defined as the harmonic conjugate of P as to O and the residual intersection of OP and C. If the center O is on the cubic curve the invariant conic is the first polar of the point as to the given cubic curve. When the center is a point of inflexion, the central projection is linear.

2. The corresponding problem for a cubic surface of ordinary space has not been considered, since the surface can be mapped birationally on a plane.

The central projection of the cubic surface upon itself from a point on the surface becomes the Geiser involution when the surface is mapped on a plane by the GRASSMANN method. The image of a straight line is a rational curve of order eight, with seven triple points. The entire problem is thus transferred to a plane, by considering such transformations as arise by combinations of Geiser transformations having different fundamental points.

§ 1.

Transformations α.

3. Let A be any point of V_3 in S_4. The operation of projecting V_3 upon itself from A will be denoted by the letter α, with the condition

$$\alpha^2 = 1.$$

Its equations may be written as follows. If $A = (a_1, a_2, a_3, a_4, a_5)$ and

$$H(x) = \sum a_i \frac{\partial V_3}{\partial x_i}, \qquad V_3(a) = 0,$$

and (y) any point in S_4, then (y') has the coordinates

$$y'_i = y_i H(a, y) - a_i H(y),$$

wherein

$$H(a, y) = \frac{1}{2} \sum a_i \frac{\partial H}{\partial y_i}.$$

The transformation α is a quadratic CREMONA transformation, the invariant quadric variety being the first polar of (a) as to V_3.

Every point in which H meets V_3 is invariant under α; it is a surface of order six in S_4. The operation α is linear when and only when $H(x)$ and $H(a, x)$ have a common linear factor; since $H(a, x)$ is itself linear, it follows that $H(a, x)$ must be a factor of $H(x)$. Thus, the discriminant of $H(x)$ is of rank not greater than three. This condition requires more independent relations among the a_i than the number of coordinates, hence is possible only for a particular variety V_3. Such cases will not be included in the following discussion.

The number of parameters in α is three, since any point on V_3 may be chosen as vertex. No two transformations α are equivalent.

4. Since α is a quadratic transformation, the section of V_3 made by an S_3 not passing through A is transformed into the complete intersection of V_3 and the quadratic form V_2 in S_4, image of S_3 in α. The tangent space to V_3 at A meets S_3 in a plane which cuts V_3 in a cubic curve C_3. The image of every point of C_3 is at A, hence we may say: *The image in α of a section of V_3 by a general S_3 is a surface (V_2, V_3) of order six in S_4, having a triple point at A, the tangents all lying on a cubic cone of an S_3* [2]).

Moreover, the image of any plane section C_3 of V_3 is a sextic curve C_6 lying on a quadric surface F_2 of S_3, as may be seen as follows.

The plane of C_3 and the point A determine an S_3 which cuts V_3 in a cubic surface F_3 passing through C_3 and through A. The cone K_3 having A as vertex and passing through C_3 lies entirely in S_3. Since K_3, S_3, and V_3 are invariant under α, it follows that the image of C_3 in α is the residual intersection of K_3 and F_3. Since part of their intersection is a complete intersection with a plane, the residual C_6 must be a complete intersection with a quadric surface F_2. The tangent plane to F_3 at A meets the plane of C_3 in a straight line; this line meets C_3 in three points, the image of each of which in is at A, hence A is a triple point of C_6, the tangent lines all lying in one plane.

5. Now consider the product $\alpha_i \alpha_l$ of two operations α_i and α_k. Here various cases must be considered. On V_3 lie ∞^2 straight lines. Through every point of V_3 lie six; they are the six liines of the cubic surface passing through the point of contact and cut from V_3 by a tangent space. In an arbitrary space S_3 lie 27 lines of V_3, lying on the cubic surface cut from V by the S_3. Any plane through a line of V_3 is a double tangent plane, the points of contact being the points of intersection of the line and the residual conic.

Let $x_1 = x_2 = x_3 = 0$ be the equations of a general straight line p on V_3. The equation of V may be written in the form

$$V_3 \equiv x_1 \varphi_1 + x_2 \varphi_2 + x_3 \varphi_3 = 0,$$

in which each φ_i is quadratic in $x_1, \ldots, x_5$. The tangent S_3 to V_3 at $(0, 0, 0, y_4, y_5)$ on p has the equation

$$x_1 \varphi_1(y) + x_2 \varphi_2(y) + x_3 \varphi_3(y) = 0,$$

in which $\varphi_i(y) = \varphi_i(0, 0, 0, y_4, y_5)$. The S_3 is tangent to V_3 at one point only.

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[2]) This result could also be obtained from the theorem of G. FANO {*Sulle superficie algebriche contenute in una varietà cubica dello spazio a quattro dimensioni* [Atti della R. Accademia delle Scienze di Torino, Vol. XXXIX (1904), pp. 597-613]} concerning possible surfaces on V_3. The result of FANO has been generalized to apply to V_{r-1} in S_r without multiple points, namely, that the only possible varieties V_{r-2} lying on them are complete intersections. See F. SEVERI, *Una proprietà delle forme algebriche prive di punti multipli* [Rendiconti della R. Accademia dei Lincei (Roma), Vol. XV, 2° semestre 1906, pp. 691-696].

Let

$$\frac{x_1}{A_1} = \frac{x_2}{A_2} = \frac{x_3}{A_3}$$

be the equations of a plane through p. When y_4, y_5 are given, $A_1 : A_2 : A_3$ can be uniquely determined, so that the plane is tangent to V_3 at (y). If $A_1 : A_2 : A_3$ are given. two points of tangency are determined, hence every plane through the line is a bitangent plane. There is a $(1, 1)$ correspondence between the planes through the line and the pairs of points on the line. The tangent spaces to V_3 at the points of p envelope a quadratic variety having p for vertex. In every pencil of spaces through p there are two that are tangent to V_3.

Among the straight lines on V_3 are ∞^1 *special* straight lines l, defined by the property that all the tangent spaces to V_3 at points of l have a plane through the line in common. This basis plane touches V_3 along the entire length of l, and intersects V_3 in a residual line called the *coniugate* of l [3]).

The tangent spaces at points of a special line belong to a pencil; the planes of the pencil and the points of the line l are in $(1, 2)$ correspondence. The points of contact form an involution projective with the pencil of spaces. All the planes through l which belong to the same space cut V_3 in conics which meet l in the same points, hence the conics meet l in points of an involution.

If $x_1 = x_2 = x_3 = 0$ is a special line and $x_2 = 0$, $x_3 = 0$ the basis plane, φ_1 has the form $u_1 x_1 + u_2 x_2 + u_3 x_3$, in which each u_i is linear in $x_1, \ldots, x_5$. The equation of the tangent space at $(0, 0, 0, y_4, y_5)$ has the form

$$x_2 \varphi_2(y) + x_3 \varphi_3(y) = 0.$$

The equations of the planes through the line and lying in the S_3 defined by $x_3 = m x_2$ are of the form $x_3 + k x_1 = 0$, $x_3 = m x_2$.

The residual conic in each of these planes meets l in the points of contact of the bitangent space $x_3 = m x_2$ [4]).

The centers A_1, A_2 may have the following positions:

(1): not on a common line.

(2): on a line p.

(3): on a line l.

In case (1), the ∞^2 plane sections of V_3 through $A_1 A_2$ remain invariant. The curve of order 12, locus of the point of contact of the ∞^1 tangent spaces through $A_1 A_2$ remains point by point invariant. The operation $\alpha_1 \alpha_2$ can not be replaced by another of the form $\alpha'_1 \alpha'_2$.

The image of (V_3, S_3) by α_1 is a sextic surface (V_3, V_2) of S_4, having a triple

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[3]) F. Enriques, *Sugli spazi pluritangenti delle varietà cubiche generali appartenenti allo spazio a 4 dimensioni* [Giornale di Matematiche di Battaglini, Vol. XXXI (1893), pp. 31-35].

[4]) G. Fano, *Ricerche sulla varietà cubica generale dello spazio a quattro dimensioni e sopra i suoi spazi pluritangenti* [Annali di Matematica pura ed applicata, Series III, Vol. X (1904), pp. 251-285].

point at A_1 and no other multiple points. The image of (V_3, V_2) in α_2 is a complete intersection (V_3, V_4) of order 12, having a sixfold point A_2, a triple point at the image of A_1 in α_2, and no other point singularities.

The image of the C_6 on an F_2 having a P_3 at A_1 in α_2 may be found as follows. The lines joining A_2 to points of C_6 from a twodimensional cone K_6 of S_4, of order six. Any S_3 through A_2 cuts six generators from K_6, on each of which lies one point of the image of C_6 in α_2, besides those at A_2. The tangent S_3 to V_3 at A_2 meets C_6 in six points, the image of each of which is at A_2. Hence the image of C_6 is a curve of order twelve, C_{12} having a six fold point at A_2, a triple point at the image of A_1 in α_2, and no other point singularities. Moreover, C_{12} is a complete intersection of V_3, of V_2, the image of S_3 in α_2, and K_2, the three dimensional quadric cone obtained by projecting F_2 from A_2.

Hence the operation $\alpha_1 \alpha_2$ can not be replaced by another of the form $\alpha'_1 \alpha'_2$, unless possibly when A'_1 lies on the line $A_1 A_2$.

Let the line $A_1 A_2$ intersect V_3 in a third point A_3. The only possible combinations are

$$\alpha_1 \alpha_2 = \alpha_3 \alpha_1 \quad \text{and} \quad \alpha_1 \alpha_2 = \alpha_2 \alpha_3 .$$

The operation $\alpha_1 \alpha_2$ transforms the point A_2 into A_1, while the operation $\alpha_3 \alpha_1$ transforms A_2 into the first tangential of A_1. If this point coincides with A_1 for every section through $A_1 A_2$, every tangent to V_3 at A_3 must have three point contact. In this case the polar quadric of A_1 is composite and contains the tangent space to V_3 at A_1 as one component. This was seen to be possible only for particular varieties V_3, hence in the case under consideration no such points exist. Similarly for the hypothesis $\alpha_1 \alpha_2 = \alpha_2 \alpha_3$. This proves the proposition.

(2). Any plane through p cuts from V_3 a conic cutting p in two points P, Q. In this plane $A_1 A_2$ may be replaced by any pair of points $B_1 B_2$ on p which with P, Q have the same cross-ratio. But as the plane turns about p, the points P, Q can come into coincidence with any pair of points whatever of p, it follows that it is impossible that the cross-ratio $(A_1 A_2 P Q)$ shall always be equal to $(B_1 B_2 P Q)$, hence the operation $\alpha_1 \alpha_2$ is not equivalent to $B_1 B_2$ for other planes.

(3). The same argument applies to a special line except for the case in which A_1, A_2 are points of contact of a bitangent plane. In this case $(\alpha_1 \alpha_2)^2 = 1$ and the centers $A_1 A_2$ may be replaced by any other pair of points in the involution formed by the bitangent spaces [5]). Hence.

The only involutions $(\alpha_1 \alpha_2)^2 = 1$ on V_3 are those on the special lines, one on each line l. One center A_1 may be assumed at will, and the other is uniquely defined.

We have established the two results.

The transformation $\alpha_1 \alpha_2$ contains six parameters. The transformations $\alpha_1 \alpha_2$ cannot form a continuous group, nor can they include any subset which forms a continuous group.

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5) C. STÉPHANOS, *Mémoire sur la représentation des homographies binaires par des points de l'espace avec application à l'étude des rotations sphériques* [Mathematische Annalen, Vol. XXII (1883), pp. 299-367].

6. Now consider the product of three projections α_1, α_2, α_3. The plane determined by the three vertices remains invariant, and in this plane the product can be replaced by one central projection, having four invariant points. It has no invariant points except those in the plane $A_1 A_2 A_3$. If another operation $\alpha'_1 \alpha'_2 \alpha'_3$ can be found equivalent to $\alpha_1 \alpha_2 \alpha_3$, the vertices $A'_1 A'_2 A'_3$ must lie in the plane $A_1 A_2 A_3$.

Consider the image of the section (V_3, S_3) made on V_3 by any S_3, when transformed by $\alpha_1 \alpha_2 \alpha_3$. It has been seen that the image of (V_3, S_3) in $\alpha_1 \alpha_2$ is a surface of S_4 of order 12, the complete intersection of V_3 with V_4. This surface is transformed by α_3 into (V_3, V_8) of order 24; it has a conical point of order 12 at A_3, a point of order 6 at the image of A_2 in α_3, a triple point at the image of A_1 in $\alpha_2 \alpha_3$, and no other point singularities. If the same section were transformed by $\alpha'_1 \alpha'_2 \alpha'_3$, the resulting surface would have similar singularities at A'_3, etc., hence when $A_1 A_2 A_3$ do not lie by twos on lines of V_3, no operation $\alpha'_1 \alpha'_2 \alpha'_3$ can be found which is equivalent to the given one $\alpha_1 \alpha_2 \alpha_3$.

It has been seen that the image of a general plane section in $\alpha_1 \alpha_2$ is a curve C_{12} of S_4 which is also a complete intersection. The image of C_{12} in α_3 is not a complete intersection, nor can any general conclusions be made concerning images of curves in S_4.

Among the lines which lie on V_3 and meet a given special line of V_3 are 28 special lines. These are arranged in 14 pairs, such that the lines of a pair intersect each other. The triangle formed by three special lines has points of contact of a tritangent space for vertices [6]).

Let A_1, A_2, A_3 be the vertices of a triangle of special lines. The operation $\alpha_1 \alpha_2 \alpha_3$ is equivalent to that expressed by any permutation of these same factors, or to such combinations as $\alpha'_1 \alpha'_2 \alpha'_3$, in which A'_2, A'_3 are the points of contact of any bitangent space through $A_2 A_3$. It is also involutional.

When A_1, A_2, A_3 are vertices of a triangle the only involutions are those defined by the points of contact of tritangent spaces.

Finally, if A_1, A_2, A_3 all lie on a line and $\alpha'_1 \alpha'_2 \alpha'_3$ is equivalent to $\alpha_1 \alpha_2 \alpha_3$, evidently A'_1, A'_2, A'_3 must lie on the same line. Three cases arise:

The line $A_1 A_2 A_3$ is (*a*) not on V_3; (*b*) a non-special line p; (*c*) a special line l. In case (*a*) the only possibilities are

$$\alpha'_1 \alpha'_2 \alpha'_3 = \alpha_2 \alpha_3 \alpha_1 \quad \text{or} \quad \alpha'_1 \alpha'_2 \alpha'_3 = \alpha_3 \alpha_1 \alpha_2 \quad \text{or} \quad \alpha'_1 \alpha'_2 \alpha'_3 = \alpha_3 \alpha_2 \alpha_1 .$$

In any case this is possible only when the tangents at two of the vertices meet on the curve of section for every plane through the line, but this is not possible unless the tangent spaces intersect on V_3, and the general V_3 does not contain a plane. In cases (*b*), (*c*) no equivalence is possible from Nos. 4 and 5. Hence we may say.

The operation $\alpha_1 \alpha_2 \alpha_3$ defines a nine-parameter transformation. The transformations do not form a continuous group nor include any continuous group with fewer parameters.

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[6]) G. Fano, loc. cit. [4]).

7. Now consider the operation $\alpha_1\alpha_2\alpha_3\alpha_4$. Various cases are to be considered, according as the space of fewest dimensions which contains $A_1 - A_4$ is an S_3, and S_2 or an S_1. If $A_1 - A_4$ do not lie in a plane, and four points $A'_1 A'_2 A'_3 A'_4$ can be found such that $\alpha'_1\alpha'_2\alpha'_3\alpha'_4 = \alpha_1\alpha_2\alpha_3\alpha_4$, then the smallest space through $A_1 - A_4$ is also an S'_3. If $S'_3 \neq S_3$, the F_3 cut from V_3 by S_3 is invariant and the F'_3 cut from S'_3 is invariant. But F'_3 is transformed into an F'_6 by α_1, etc., which cannot return to F'_3 unless $\alpha_1\alpha_2\alpha_3\alpha_4$ is the identity, by No. 5. But in case $A_1 - A_4$ are not coplanar $\alpha_1 \ldots \alpha_4$ cannot be the identity, by No. 4. Hence this case is impossible. If $S'_3 = S_3$ we can proceed exactly as in No. 7 to obtain the following results.

The operation $\alpha_1\alpha_2\alpha_3\alpha_4$ is a 12 parameter transformation. These transformations do not form a continuous group nor include a continuous group of fewer parameters.

The only involution of the form $\alpha_1\alpha_2\alpha_3\alpha_4$ is defined by the points of contact of a quadritangent space. V_3 has 495 such quadritangent spaces.

8. It is now easy to generalize.

That the operation $\prod_{j=1}^{k}\alpha_j$ is a transformation involving $3k$ parameters, that the transformations do not form a continuous group nor include a continuous group with fewer parameters. The only transformations that are of finite period are the involutions α_1, $\alpha_1\alpha_2$, $\alpha_1\alpha_2\alpha_3$ and $\alpha_1\alpha_2\alpha_3\alpha_4$ of multitangent spaces.

§ 2.

The Transformations π.

9. A second birational transformation leaving V_3 invariant may be defined as follows. Pass a plane S_2 through a line p of V_3. The residual intersection of V_3 by S_2 is a conic. Let P be the pole of p as to the conic. The transformation is the harmonic homology in each plane S_2 determined by p and P.

This transformation π is also CREMONIAN. When p is an ordinary line of V_3, the ∞^2 planes through it determine ∞^2 pairs of points upon it and also ∞^2 positions of P. Hence the locus of P is a rational surface in S_4, since any plane through p meets it in just one point not on p.

To determine the equations of the transformation, let $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ be the equations of p.

$$V_3 = x_1\varphi_1 + x_2\varphi_2 + x_3\varphi_3.$$

The equations of a plane through p are

$$\frac{x_1}{\lambda} = \frac{x_2}{\mu} = \frac{x_3}{\nu}$$

and the residual conic is defined by

$$\lambda\varphi_1\left(x_1, \frac{\mu x_1}{\lambda}, \frac{\nu x_1}{\lambda}, x_4, x_5\right) + \mu\varphi_2 + \nu\varphi_3 = 0.$$

By putting $x_1 = 0$ we have a quadratic equation $x_4 : x_5$ whose roots define the points of intersection of p and the conic. Let these points be

$$K' \equiv (0, 0, 0, k'_4, k'_5), \quad K'' \equiv (0, 0, 0, k''_4, k''_5).$$

The equations of the tangents to the conic at K', K'' are of the form

$$(A k'_4 + B k'_5) x_1 + (B k'_4 + C k'_5) x_4 + (D k'_4 + E k'_5) x_5 = 0,$$
$$(A k''_4 + B k''_5) x_1 + (B k''_4 + C k''_5) x_4 + (D k''_4 + E k''_5) x_5 = 0.$$

The coordinates of the point of intersection are all multiples of $(k'_4 k''_5 - k'_5 k''_4)$. Discarding this factor we have

$$x_1 = (BE - CD), \quad x_4 = (DB - AE), \quad x_5 = (AC - B^2),$$

in which A, B, C are distinct linear functions of λ, μ, ν and D, E are independent quadratic functions of λ, μ, ν. The coordinates of P are thus

$$P \equiv (\lambda g_2, \mu g_2, \nu g_2, \psi_3, f_3),$$

wherein ψ_i, f_i, g_i are polynomials of degree i in λ, μ, ν. The equations of the locus of P are therefore

$$\psi_3(x_1, x_2, x_3) = x_4 g_2(x_1, x_2, x_3),$$
$$f_3(x_1, x_2, x_3) = x_5 g_2(x_1, x_2, x_3).$$

The section of this surface by an arbitrary S_3 is a space curve of order nine. An S_3 through p cuts it in p counted six times and in a space cubic curve cutting p twice.

The equations of the transformation are found by determining (x'), the harmonic conjugate of a given point (x) as to P and the intersection of $(x)P$ with p. They are

$$\sigma x'_1 = -g_2 x_1, \quad \sigma x'_2 = -g_2 x_2, \quad \sigma x'_3 = -g_2 x_3,$$
$$\sigma x'_4 = 2\psi_3(x_1, x_2, x_3) - g_2 x_4, \quad \sigma x'_5 = 2 f_3(x_1, x_2, x_3) - g_2 x_5.$$

The line p is fundamental. An arbitrary S_3 is transformed into a cubic cone of the first kind having the point in which S_3 meets p as vertex. The line p is a double line on the cone. An S_3 through p goes into itself. If the line p is a special line, the basis plane is fundamental. We may now put

$$V_3 = x_1 \varphi_a + x_2 \varphi_b + x_3 \varphi_c = 0,$$
$$\varphi_a = \sum a_{ik} x_i x_k, \quad \varphi_b = \sum b_{ik} x_i x_k, \quad \varphi_c = \sum c_{ik} x_i x_k.$$

If the basis plane is $x_2 = 0$, $x_3 = 0$, then $a_{44} = a_{45} = a_{55} = 0$. In this case we have

$$A = \mu b_{44} + \nu c_{44}, \quad B = \mu b_{45} + \nu c_{45}, \quad C = \mu b_{55} + \nu c_{55}.$$
$$D = \lambda(\lambda a_{41} + \mu b_{41} + \nu c_{41}) + \mu(\mu b_{42} + \nu c_{42}) + \nu^2 c_{43},$$
$$E = \lambda(\lambda a_{51} + \mu b_{51} + \nu c_{51}) + \mu(\mu b_{52} + \nu c_{52}) + \nu^2 c_{53}.$$

In case of a non special line π cannot be expressed in terms of the transformations α.

Consider the image of an arbitrary S_3 by $\pi_1 \pi_2$. By π_1, S_3 is transformed into a three dimensional cubic cone, having a point on p_1 as vertex and the line p_1 as a double line. By π_2 this cone is transformed into a variety of S_4, containing a double curve and containing p_2 as a sixfold line. If p_1, p_2 intersect, the double curve is replaced by a simple plane curve lying in a tangent S_2 to V_3, but p_2 is still a six fold line. Hence no pair of operators $\pi_1 \pi_2$ can be replaced by another pair. In the same way it is seen that the product of three operations $\pi_1 \pi_2 \pi_3$ can not be replaced by any other product of three, etc., hence the operations π_i can not form a continuous group nor any combination of them generate a continuous group.

Moreover, by combining the preceding results, we see that no combination of operations α, π can form a continuous group.

In case of a special line l

$$\pi = \alpha_1 \alpha_2 = \alpha_2 \alpha_1,$$

A_1, A_2 being the points of contact of any bitangent space on l.

§ 3.

The variety $\sum x_i^3 = 0$.

10. It would not be appropriate to this discussion to consider all the special or particular forms of the cubic variety, but one form of considerable interest was mentioned by FANO [7]).

The variety $V \equiv \sum x_i^3 = 0$ contains no planes and no double points, but has the maximum number (thirty) of points at which the tangent space intersects the variety in a cubic cone. If i, k, l, m, n represent any permutation of the intergers 1, 2, 3, 4, 5, and ω is any number which satisfies the equation $\omega^3 = 1$, the equations of these particular spaces are of the form $x_i + \omega x_k = 0$ and section cut from the variety is $x_l^3 + x_m^3 + x_n^3 = 0$. It was shown by FANO that the variety is invariant under a collineation group of order $3^4\, 5!$ made up of replacing each x_i by ωx_i, each coefficient being independent, and of the permutation of the coordinates.

These transformations can all be expressed in terms of projections from the vertices of the cones. Thus, from the point $(0, 0, 0, \omega, 1)$ the operation α has the equations

$$x_1' = x_1, \quad x_2' = x_2, \quad x_3' = x_3, \quad x_4' = \omega x_5, \quad x_5' = \omega^2 x_4.$$

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Cornell University, September 1913.

VIRGIL SNYDER.

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7) G. FANO, *Sopra una varietà cubica particolare dello spazio a quattro dimensioni* [Rendiconti del R. Istituto Lombardo di Scienze e Lettere (Milano), Series II, Vol. XXXVII (1904), pp. 554-566].