ON THE MULTIPLICATION OF DIRICHLET'S SERIES

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1. In this paper I propose to generalise some results communicated to the Society in 1908.*

As in my former communication, I denote by A and B the series

$$a_1+a_2+a_3+\ldots,\qquad b_1+b_2+b_3+\ldots,$$

and by C the series

$$c_1+c_2+c_3+\ldots,$$

where c_n is a function of the a's and b's, to be defined more precisely in a moment. I shall also use the letters A, B, C to denote the sums of the series, when they are convergent.

I shall denote by

$$A_x$$

the finite sum

$$\sum_{n\leqslant x} a_n;$$

here x is not restricted to be integral. Similarly I define B_x, C_x.

2. When the series C is the product-series of A and B, formed in accordance with Cauchy's rule, we have

$$c_p = a_1b_p+a_2b_{p-1}+\ldots+a_pb_1 = \sum_{(m+n=p+1)} a_m b_n.$$

This was the only case that I considered in my former paper.

Cauchy's rule for multiplication is, however, only one among an infinity. We are led to it by arranging the formal product of the power series

$$\Sigma a_m x^m,\qquad \Sigma b_n x^n$$

in ascending powers of x, and then putting $x=1$. It is the same thing to say that we arrange the formal product of the Dirichlet's series

$$\Sigma a_m e^{-ms},\qquad \Sigma b_n e^{-ns}$$

* "The Multiplication of Conditionally Convergent Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 6, p. 410.

according to the ascending order of the sums

$$m+n,$$

associating together all the terms for which $m+n$ has the same value, and then put $s = 0$. It is clear that we arrive at a generalisation of our conception of multiplication by considering the more general Dirichlet's series

$$\Sigma a_m e^{-\lambda_m s}, \quad \Sigma b_n e^{-\lambda_n s},*$$

and arranging their formal product according to the ascending order of the sums

$$\lambda_m+\lambda_n.†$$

Let

$$\nu_1, \nu_2, \ldots, \nu_p, \ldots$$

be the ascending sequence defined by the possible values of $\lambda_m+\lambda_n$. Then the Dirichlet's product of the series A, B, according to the rule defined by the sequence λ_n, is

$$c_1+c_2+c_3+\ldots,$$

where

$$c_p = \sum_{(\lambda_m+\lambda_n=\nu_p)} a_m b_n.$$

Thus, if $\lambda_m = \log m$, so that the Dirichlet's series are ordinary Dirichlet's series,

$$\nu_p = \log p,$$

and

$$c_p = \sum_{mn=p} a_m b_n = \sum_{(d)} a_d b_{p/d},$$

the summation being extended to all the divisors of p.

3. The three classical theorems relating to ordinary multiplication have their analogues for the general form of Dirichlet's multiplication.

(1) **Analogue of Abel's Theorem.**—*If all three series are convergent, then $C = AB$.*

(2) **Analogue of Cauchy's Theorem.**—*If A and B are absolutely convergent, then C is absolutely convergent.*

(3) **Analogue of Mertens' Theorem.**—*If A is absolutely and B conditionally convergent, then C is convergent.*

* Here, of course, (λ_m) is any increasing sequence whose limit is infinity.

† For a general account of the theory, see Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, Bd. 2, S. 750. Landau remarks that there is no loss of generality in adopting the same sequence (λ_m) in both series: for if we had two sequences (λ_m), (μ_n), we could combine them into one, regarding some of the a's and b's as zero.

Of these results (2) is almost obvious, while (3) was deduced by Landau as a corollary of a more general theorem of Stieltjes.* The analogue of Abel's theorem was also proved by Landau, by considerations drawn from the theory of analytic functions. An elementary proof was afterwards found independently by Phragmen, Bohr, and Riesz,† who deduce it from the following theorem :—

(4) **Analogue of Cesàro's Theorem.**—*If A and B are convergent, then C is summable $(R, 1, \nu)$ to sum AB: that is to say,*

$$\frac{(\nu_2-\nu_1)C_1+(\nu_3-\nu_2)C_2+\ldots+(\nu_p-\nu_{p-1})C_{p-1}}{\nu_p} \to AB,$$

as $p \to \infty$.

4. In my former paper I proved the following theorems (for multiplication by Cauchy's rule).

(5) *If A and B are convergent, and*

$$na_n \to 0, \quad nb_n \to 0,$$

then C is convergent.

(6) *The same result holds under the more general conditions*

$$|na_n| < K, \quad |nb_n| < K.$$

I propose now to establish the analogues of these theorems for the general form of Dirichlet's multiplication. It might be thought that, as (5) is a special case of (6), it would not be worth while to prove it independently, as in my previous paper. This view would, I think, be mistaken. Theorem (5) above, and its generalisation, can be proved by a very much simpler argument than seems to be called for by (6) and its generalisation; and the simpler proof of the less general theorem affords a good deal more information about the behaviour of the product series than can be obtained in the more general case. It therefore seems worth while to keep the two distinct.‡

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* Stieltjes, *Nouvelles Annales*, Sér. 3, t. VI, p. 210; Landau, *Rendiconti di Palermo*, t. XXIV, p. 81; *Handbuch*, S. 752.

† Landau, *Handbuch*, S. 762 and 904; Riesz, *Comptes Rendus*, July 9, 1909; Bohr, *Nachrichten der König. Gesellschaft der Wiss. zu Göttingen*, 1909, S. 247.

‡ These theorems may be compared with Tauber's theorem (the converse of Abel's theorem on the continuity of power series) and its extension given recently in these *Proceedings* by Mr. Littlewood—a similar distinction presents itself in the case of the two latter theorems.

5. THEOREM 1.—*If A and B are convergent, and*

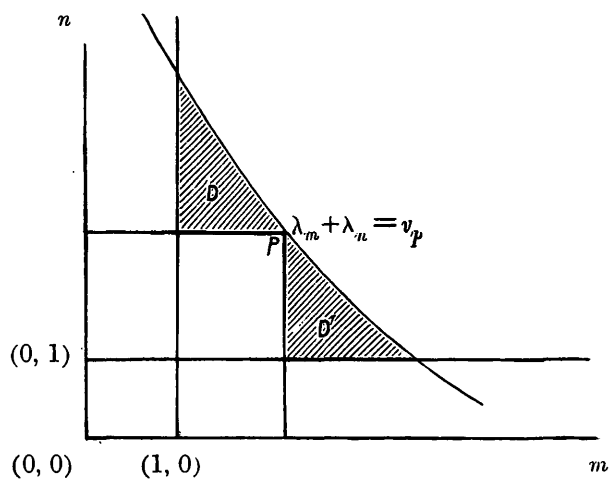
$$\frac{\lambda_n a_n}{\lambda_n - \lambda_{n-1}} \to 0, \qquad \frac{\lambda_n b_n}{\lambda_n - \lambda_{n-1}} \to 0,$$

then the product series C, formed by the rule of Dirichlet's multiplication corresponding to the sequence (λ_n), *is convergent.*

We have
$$c_p = \sum_{\lambda_m + \lambda_n = \nu_p} a_m b_n,$$
and
$$C_p = \Sigma a_m b_n,$$
the summation being bounded by the inequalities

$$m \geqslant 1, \quad n \geqslant 1, \quad \lambda_m + \lambda_n \leqslant \nu_p.$$

I shall suppose that $\lambda_1 = 0$; this hypothesis in no way affects the generality of the result, and simplifies the expression of the proof a little.



Let us draw the curve $\lambda_m + \lambda_n = \nu_p$

in the (m, n) plane (see the figure), and on it take the point P whose coordinates are
$$m_p = n_p = \bar{\lambda}(\tfrac{1}{2}\nu_p),$$

where $\bar{\lambda}$ is the function inverse to λ. Then

$$C_p - A_{m_p} B_{n_p} = \Sigma a_m b_n,$$

where the summation extends to all points (m, n) inside the regions D, D' shaded in the figure, including those on the curved (but not on the straight) boundaries of those regions. Now

$$\sum_{(D)} a_m b_n = \sum_{\bar{\lambda}(\frac{1}{2}\nu_p) < n \leqslant \lambda(\nu_p)} b_n A_{\bar{\lambda}(\nu_p - \lambda_n)}.$$

There is a constant K such that

$$|A_x| < K$$

for all values of x. Moreover we can choose p so that

$$|b_n| < \epsilon(\lambda_n - \lambda_{n-1})/\lambda_n$$

for $n \geqslant \lambda(\frac{1}{2}\nu_p)$. Then

$$\begin{aligned} \left|\sum_{(D)} a_m b_n\right| &< \epsilon K \sum_{\bar{\lambda}(\frac{1}{2}\nu_p) < n \leqslant \bar{\lambda}(\nu_p)} \left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right) \\ &< \epsilon K \left\{ 1 + \sum_{\bar{\lambda}(\frac{1}{2}\nu_p)+1 < n \leqslant \lambda(\nu_p)} \log\left(\frac{\lambda_n}{\lambda_{n-1}}\right) \right\}^* \\ &< \epsilon K \left\{ 1 + \log \frac{\lambda\bar{\lambda}(\nu_p)}{\lambda\bar{\lambda}(\frac{1}{2}\nu_p)} \right\} \\ &= \epsilon K (1 + \log 2). \end{aligned}$$

It follows that
$$\sum_{(D)} a_m b_n \to 0$$
as $p \to \infty$. Similarly we can shew that the sum of the terms inside (D') tends to zero. Hence

$$C_p - A_{m_p} B_{n_p} \to 0, \tag{1}$$

and the theorem follows.

It should be observed that the same argument proves that

$$C_p - A_{\bar{\lambda}(\alpha\nu_p)} B_{\bar{\lambda}(\beta\nu_p)} \to 0, \tag{2}$$

if α and β are any positive numbers such that $\alpha + \beta = 1$. Moreover the truth of (1) and (2) depends *only* on the existence of an upper limit for $|A_x|$ and $|B_x|$, and not on the actual convergence of A and B. Thus we have in reality proved more than is actually contained in the enunciation of the theorem.

6. I shall now proceed to the proof of the generalised form of Theorem 6 of § 4. Here we find it necessary to pursue an entirely

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* For, if $u = (\lambda_n - \lambda_{n-1})/\lambda_n$, we have $0 < u < 1$, and

$$u < \log\left(\frac{1}{1-u}\right) = \log\left(\frac{\lambda_n}{\lambda_{n-1}}\right).$$

different and much less direct form of argument, and the touch of extra precision, pointed out at the end of the last paragraph, cannot be obtained. We shall also find it necessary to subject the absolute generality of the sequence (λ_n) to a restriction, viz., that

$$\frac{\lambda_n - \lambda_{n-1}}{\lambda_n} \to 0.^*$$

7. THEOREM II.—*If* $\dfrac{\lambda_n - \lambda_{n-1}}{\lambda_n} \to 0,$

then the result of Theorem I is still valid when we assume only that

$$\left|\frac{\lambda_n a_n}{\lambda_n - \lambda_{n-1}}\right| < K, \qquad \left|\frac{\lambda_n b_n}{\lambda_n - \lambda_{n-1}}\right| < K.$$

We know, by Theorem 4 of § 4, that

$$\frac{(\nu_2 - \nu_1) C_1 + (\nu_3 - \nu_2) C_2 + \ldots + (\nu_p - \nu_{p-1}) C_{p-1}}{\nu_p}$$

tends to the limit AB. This expression may also be written in the form

$$C_p - \frac{\nu_1 c_1 + \nu_2 c_2 + \ldots + \nu_p c_p}{\nu_p}.$$

Hence the necessary and sufficient condition that the product series should be convergent is that

$$\frac{\bar{C}_p}{\nu_p} = \frac{\nu_1 c_1 + \nu_2 c_2 + \ldots + \nu_p c_p}{\nu_p} \to 0.$$

Now $$\bar{C}_p = \Sigma\, \nu_p \sum_{\lambda_m + \lambda_n = \nu_p} a_m b_n = \Sigma\, (\lambda_m + \lambda_n)\, a_m b_n,$$

where the summation is bounded by the inequalities

$$m \geqslant 1, \quad n \geqslant 1, \quad \lambda_m + \lambda_n \leqslant \nu_p.$$

That is to say,

$$\bar{C}_p = \sum_{\lambda_n \leqslant \nu_p} \lambda_n b_n \sum_{\lambda_m + \lambda_n \leqslant \nu_p} a_m + \sum_{\lambda_n \leqslant \nu_p} b_n \sum_{\lambda_m + \lambda_n \leqslant \nu_p} \lambda_m a_m$$
$$= {}_1\Gamma_p + {}_2\Gamma_p,$$

* I return to the subject of this restriction later on. It is interesting to observe that Mr. Littlewood has found it necessary to make a similar restriction in one of his theorems concerning the converse of Abel's theorem.

say.* I shall first prove that

(1) $$ {}_1\Gamma_p/\nu_p \to 0. $$

Let m_n be the largest integral value of m for which

$$ \lambda_m+\lambda_n \leqslant \nu_p, $$

and q the largest value of n for which $\lambda_n \leqslant \nu_p$, so that $m_1 = q$. Then

$$ {}_1\Gamma_p = \sum_1^q \lambda_n b_n A_{m_n}. $$

Since A is convergent, we may write

$$ A_x = A+\epsilon_x \quad (\epsilon_x \to 0), $$

$$ \frac{{}_1\Gamma_p}{\nu_p} = \Delta_p+\Delta'_p, $$

where $$ \Delta_p = \frac{A}{\nu_p} \sum_1^q \lambda_n b_n \to 0,\dagger $$

and

(2) $$ \Delta'_p < \frac{K}{\nu_p} \sum_1^q (\lambda_n-\lambda_{n-1}) \, | \epsilon_{m_n} | .\ddagger $$

Choose M so that $$ |\epsilon_x| < \epsilon \quad (x \geqslant M). $$

Then we shall have $m_n \geqslant M$, if

$$ \lambda_M+\lambda_n \leqslant \nu_p, $$

or if $$ n \leqslant \bar{\lambda}(\nu_p-\lambda_M). $$

Suppose that this condition is satisfied for $n = 1, 2, \ldots, N$, but that

$$ \lambda_{N+1} > \nu_p-\lambda_M. $$

It is plain that, when M has been fixed, $N \to \infty$ with p.

Then the right-hand side of (2) is less than

$$ \frac{\epsilon K}{\nu_p} \sum_1^N (\lambda_n-\lambda_{n-1}) + K\left(\frac{\lambda_{N+1}-\lambda_N}{\nu_p}\right) + \frac{K'}{\nu_p} \sum_{N+2}^q (\lambda_n-\lambda_{n-1}) $$

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* In the special case when

$$ \lambda_m = m, \quad \lambda_n = n, \quad \nu_p = p+1, $$

${}_1\Gamma_p$ and ${}_2\Gamma_p$ reduce to the expressions X and Y of my former paper (*l.c.*, pp. 415, 416).

† Since B is convergent, so that $(\lambda_1 b_1+\lambda_2 b_2+\ldots+\lambda_q b_q)/\lambda_q \to 0$, and $\lambda_q \leqslant \nu_p$.

‡ It is convenient to agree that $\lambda_0 = \lambda_1 = 0$.

(where K' is any number not less than the product of K by the greatest value of $|\epsilon_x|$)

$$< \epsilon K + K\left(\frac{\lambda_{N+1}-\lambda_N}{\lambda_{N+1}}\right) + \frac{K'}{\nu_p}(\lambda_q - \lambda_{N+1})$$

$$< \epsilon K + K\left(\frac{\lambda_{N+1}-\lambda_N}{\lambda_{N+1}}\right) + \frac{K'\lambda_M}{\nu_p},$$

since $$\lambda_q \leqslant \nu_p, \quad \lambda_{N+1} > \nu_p - \lambda_M.$$

But when M has been fixed, each of the last two terms tends to zero as $p \to \infty$; and so

$$\Delta'_p \to 0,$$

which establishes the truth of the assertion (1).

It remains to prove that

(3) $$ {}_2\Gamma_p/\nu_p \to 0.$$

Now $${}_2\Gamma_p = \sum_{\lambda_n \leqslant \nu_p} b_n \sum_{\lambda_m + \lambda_n \leqslant \nu_p} \lambda_m a_m$$

$$= b_1 \sum_{\lambda_m \leqslant \nu_p} \lambda_m a_m + b_2 \sum_{\lambda_m \leqslant \nu_p - \lambda_2} \lambda_m a_m + \ldots + b_q \sum_{\lambda_m \leqslant \nu_p - \lambda_q} \lambda_m a_m$$

$$= B_1 \sum_{\nu_p - \lambda_2 < \lambda_m \leqslant \nu_p} \lambda_m a_m + B_2 \sum_{\nu_p - \lambda_3 < \lambda_m \leqslant \nu_p - \lambda_2} \lambda_m a_m + \ldots$$

$$+ B_{q-1} \sum_{\nu_p - \lambda_q < \lambda_m \leqslant \nu_p - \lambda_{q-1}} \lambda_m a_m + B_q \sum_{\lambda_m \leqslant \nu_p - \lambda_q} \lambda_m a_m.$$

If in this equation we write

$$B_x = B + \epsilon_x,$$

so that $\epsilon_x \to 0$, we obtain $$\frac{{}_2\Gamma_p}{\nu_p} = \Delta_p + \Delta'_p,$$

where $$\Delta_p = \frac{B}{\nu_p} \sum_{\lambda_m \leqslant \nu_p} \lambda_m a_m \to 0,$$

and $$\nu_p |\Delta'_p| < K|\epsilon_1| \sum_{\nu - \lambda_2 < \lambda_m \leqslant \nu_p} (\lambda_m - \lambda_{m-1}) + \ldots$$

$$+ K|\epsilon_n| \sum_{\nu_p - \lambda_{n+1} < \lambda_m \leqslant \nu_p - \lambda_n} (\lambda_m - \lambda_{m-1}) + \ldots$$

$$+ K|\epsilon_q| \sum_{\lambda_m \leqslant \nu_p - \lambda_q} (\lambda_m - \lambda_{m-1}).$$

Choose N so that $$|\epsilon_x| < \epsilon \quad (x \geqslant N).$$

Then

$$|\Delta'_p| < \frac{K'}{\nu_p} \sum_{\nu_p-\lambda_N<\lambda_m\leqslant\nu_p} (\lambda_m-\lambda_{m-1}) + \frac{\epsilon K}{\nu_p} \sum_{\lambda_m\leqslant\nu_p-\lambda_N} (\lambda_m-\lambda_{m-1}),$$

where K' is defined as before.

Let M be the largest value of m for which

$$\lambda_m+\lambda_N \leqslant \nu_p,$$

so that

$$\lambda_{M+1} > \nu_p-\lambda_N.$$

Then

$$|\Delta'_p| < \epsilon K + \frac{K'}{\nu_p}(\lambda_{M+1}-\lambda_M) + \frac{K'}{\nu_p}\sum_{M+2}^{q}(\lambda_m-\lambda_{m-1})$$

$$< \epsilon K + K'\left(\frac{\lambda_{M+1}-\lambda_M}{\lambda_{M+1}}\right) + \frac{K'\lambda_N}{\nu_p};$$

and it follows, as in our previous discussion of ${}_1\Gamma_p$, that $\Delta'_p \to 0$ and that the assertion (3) is true. Thus the proof of Theorem II is completed.

8. It will be observed that the condition that $(\lambda_n-\lambda_{n-1})/\lambda_n \to 0$ is only used twice in the above proof, and then in a way that rather suggests the possibility of avoiding it. But I have not been able to free Theorem II of this condition. Nor does the point seem to be of importance.

When the sequence (λ_n) does not satisfy this condition, the conditions

$$\left|\frac{\lambda_n a_n}{\lambda_n-\lambda_{n-1}}\right| < K, \ \ldots$$

tell us nothing (in any interesting case) except that $|a_n|$ and $|b_n|$ have finite upper limits. But this is, of course, already involved in the fact of the convergence of A and B: we know, in fact, that a_n and b_n tend to zero, so that the conditions of Theorem I will be satisfied. Thus there appears to be no particular purpose to be served by attempting a proof of the more general theorem.*

If $(\lambda_n-\lambda_{n-1})/\lambda_n$ tends to a limit other than zero, Theorem I suffices to tell us that *any* two convergent series may be multiplied by the corresponding rule of Dirichlet's multiplication. It is interesting to verify this conclusion in a particular case. Let

$$\lambda_n = 2^n.$$

* In all cases of interest $(\lambda_n-\lambda_{n-1})/\lambda_n$ tends to zero or to some limit (obviously not greater than unity). This limit is zero if

$$\lambda_n \prec e^{\delta n},$$

positive but less than unity if λ_n is (roughly) of increase e^{An}, unity if

$$\lambda_n \succ e^{\Delta n}.$$

Here δ and Δ denote respectively arbitrarily small and arbitrarily large positive numbers.

Then $$\lambda_m + \lambda_n > \lambda_\mu + \lambda_\nu,$$
if $$m \geqslant n, \quad \mu \geqslant \nu, \quad \text{and} \quad m > \mu.$$
The Dirichlet's product of A and B is

$$a_1 b_1 + (a_1 b_2 + a_2 b_1) + a_2 b_2 + (a_1 b_3 + a_3 b_1) + (a_2 b_3 + a_3 b_2) + a_3 b_3 + (a_1 b_4 + a_4 b_1) + \ldots,$$

the principle of the method being that a suffix m does not appear at all until all possible combinations of two lesser suffixes are exhausted.* It will easily be verified that the mere convergence of A and B is enough to ensure the convergence of the product series.†

* The rule is exactly the same for

$$\lambda = 3^n, \; 4^n, \; \ldots, \; 2^{2^n}, \; \ldots \,.$$

† It is perhaps worth pointing out that the reasons which make Theorem II (as an addition to Theorem I) trivial in the case of very "high" indices λ_n, do not apply to the problem of completing Mr. Littlewood's results concerning "Tauber's theorem" in the case of such high indices.