

HARMONIC OSCILLATIONS.*

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PART I. PREAMBLE ON STRUCTURE OF THE EQUATIONS.

INTRODUCTION.

Purpose of Treatment.—In the study of airplane stability one frequently employs the equations of harmonic motion, both linear and angular, and especially of damped harmonic motion. It is therefore useful to have the equations and properties of such motion presented in a form convenient for ready reference and application. Some of the more usual of such equations and properties are given in the present elementary treatment, prepared for the Bureau of Construction and Repair in its Aerodynamical Laboratory at the Washington Navy Yard. The text drawings were made by Mr. G. C. Hill.

Definition.—Harmonic motion in general may be defined as a movement whose acceleration is toward an equilibrium position and is proportional to the displacement of the moving object from that position. Such motion may be either frictionless or retarded by friction, say of the surrounding medium. A simple pendulum bob, for example, when making small excursions about its position of rest, executes harmonic motion, whether performing plane or compound vibrations, whether moving undamped in vacuo or damped in a fluid such as air or water; for, as is well known, its acceleration is always toward the point of rest and proportional to the distance therefrom. Similarly a torsion pendulum gently twisting about its axis performs harmonic motion.

A harmonic oscillation is called “free” when subject only to acceleration within the vibrating system; “forced” when accelerated from without. A free oscillation is called “damped” when impeded by friction or other deadening resistance; “undamped” when void of such resistance.

In this text the damping retardation is assumed proportional to the speed; the external acceleration is taken as either a constant or a sine function of the time.

* Communicated by the Author.

DEVELOPMENT OF THE HARMONIC EQUATIONS.

Equation of Damped Straightaway Motion.—If a moving particle is opposed only by a force proportional and opposite to its velocity, say the resistance of a fluid, it will pursue a straight path till brought to rest. And if ds be the travel in time dt , the damping force and acceleration may be written

$$R = -2ma\dot{s} \dots\dots\dots (1)$$

$$\ddot{s} = -2a\dot{s} \dots\dots\dots (1')$$

in which m is the mass of the particle, and $2a$ is for convenience taken as the “*damping coefficient*.” The quantity $2a$ is positive and may have any fixed value, depending on the opposing agency and the mass of the accelerated particle. It is in fact found from the force equation of the particle by dividing the coefficient of \dot{s} therein by the mass.

Equation of Simple Harmonic Motion.—Similarly if the only opposing force, say an attraction or an elastic resistance, is proportional and opposite to the displacement from rest of the particle, its force and acceleration can be written

$$F = -mb^2s, \dots\dots\dots (2)$$

$$\ddot{s} = -b^2s, \dots\dots\dots (2')$$

in which b^2 is the “*displacement coefficient*” or “*stiffness*.” The quantity b^2 is positive and may have any fixed value, dependent on the restoring force and the mass of the accelerated particle. The movement represented by (2') is known as a “*simple harmonic motion*.” It is, as will be shown by equations (9), (10), the rectilinear oscillation of a point whose acceleration is proportional and opposite to its displacement from rest.

Equation of Damped Harmonic Motion.—If the two foregoing forces act conjointly, the equation of motion becomes

$$\ddot{s} + 2a\dot{s} + b^2s = 0, \dots\dots\dots (3)$$

in which each term is an acceleration, and \ddot{s} , the resultant of the other two, is the actual acceleration of the particle at any instant. The complete treatment of this equation, therefore, comprises both undamped and damped free harmonic oscillations of a point in a straight line.

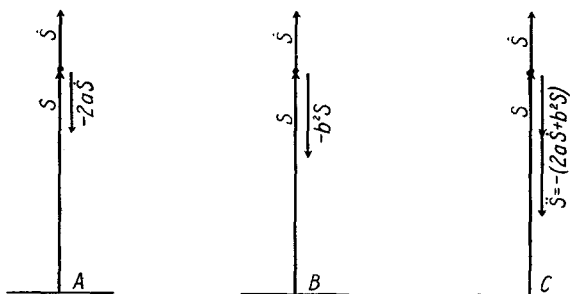
Graphical Composition of s , \dot{s} and \ddot{s} .—Fig. 1 illustrates the relation of the various vector quantities in the foregoing equations. In sketch *A* the particle is seen moving in the positive

direction of s and opposed by a force proportional and opposite to s , which causes the acceleration $-2a\dot{s}$; in sketch *B* the particle moves from its position of rest opposed by a restoring force proportional and opposite to s , which causes the acceleration $-b^2s$; in sketch *C* the particle moves from the point of rest opposed by both said forces, which beget the acceleration $-(2a\dot{s} + b^2s)$. The coefficients are written $2a$ and b^2 , rather than a and b , merely for simplicity in subsequent operations.

METHOD OF SOLVING THE EQUATIONS.

In order to divest of mathematical detail the analysis of the differential equations of motion, a preamble is here given on the usual method of integrating such equations.

FIG. 1.



Component and resultant accelerations of a particle performing rectilinear free harmonic oscillations.

Solution of the General Differential Equation.—The equations of free harmonic motion belong to the class known as linear differential equations with constant coefficients and second member zero, and have the general form

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + P_n y = 0, \dots \dots \dots (4)$$

where P_1, P_2, \dots, P_n , are constants.

To solve this by Euler's method, substitute e^{mx} for y . The first member of the equation then becomes $(m^n + P_1 m^{n-1} + \dots + P_n)e^{mx}$, which equals zero if

$$m^n + P_1 m^{n-1} + \dots + P_n = 0, \dots \dots \dots (5)$$

This latter is called the auxiliary equation to (4). Since each

of its roots $m_1, m_2 \dots m_n$, satisfies (5), $y = e^{m_1 x}$, $y = e^{m_2 x}$, etc., must satisfy (4). Hence the complete solution is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \dots + C_n e^{m_n x} \dots \dots \dots (6)$$

For example, to integrate

$$\frac{d^2 y}{dx^2} - 9y = 0,$$

the roots of its auxiliary equation,

$$m^2 - 9 = 0,$$

are 3 and -3. Hence the complete solution is

$$y = C_1 e^{3x} + C_2 e^{-3x}$$

Case I. Imaginary Roots.—In case the auxiliary equation (5) has imaginary roots, any pair of such roots, say $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, where $i = \sqrt{-1}$, has the corresponding solution

$$\begin{aligned} y &= C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x} = e^{\alpha x} (C_1 e^{i\beta x} + C_2 e^{-i\beta x}) \\ &= e^{\alpha x} [C_1 (\cos \beta x + i \beta \sin x) + C_2 (\cos \beta x - i \beta \sin x)] \\ &= e^{\alpha x} (A \cos \beta x + B \sin \beta x), \end{aligned}$$

in which $A = C_1 + C_2$, and $B = i(C_1 - C_2)$, are arbitrary constants, and are both real if C_1, C_2 , be taken as conjugate imaginaries.

For example $\frac{d^2 y}{dx^2} + 8 \frac{dy}{dx} + 25y = 0$

has the auxiliary equation

$$m^2 + 8m + 25 = 0$$

whose roots are $m_1 = -4 + 3i$, $m_2 = -4 - 3i$; hence the solution is

$$y = e^{-4x} (C_1 \cos 3x + C_2 \sin 3x).$$

Case II. Equal Roots.—If some of the roots of (5) are equal the corresponding terms in (6) coalesce. Thus if $m_1 = m_2$, then $C_1 e^{m_1 x} + C_2 e^{m_2 x} = (C_1 + C_2) e^{m_1 x} = C e^{m_1 x}$, and hence (6) lacks one integration constant of the n necessary for a complete solution. It can be shown, however, that if m_1 be a double root of (5), then $x e^{m_1 x}$, as well as $e^{m_1 x}$, is a solution of (4). And in general if m_1 is a root of the order p of (5), the equation (4) has the solutions $e^{m_1 x}$, $x e^{m_1 x}$, $x^2 e^{m_1 x}$, $\dots x^{p-1} e^{m_1 x}$.

For example, the equation

$$\frac{d^3 y}{dx^3} - 3 \frac{d^2 y}{dx^2} + 4y = 0$$

has the auxiliary equation

$$m^3 - 3m^2 + 4 = 0$$

whose roots are $-1, 2, 2$. Hence the complete integral is

$$y = C_1 e^{-x} + C_2 e^{2x} + C_3 x e^{2x}.$$

Similarly if the auxiliary of a 4th degree equation be

$$m^4 - 4m^3 + 10m^2 - 12m + 5 = 0$$

the roots are $1, 1, 1 \pm 2i$, and the complete solution is

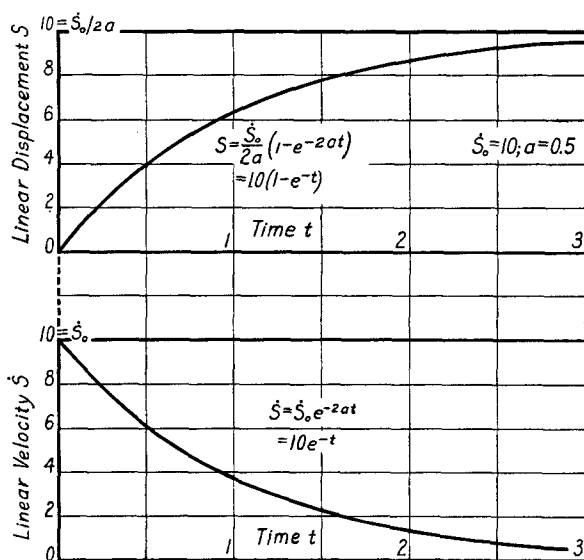
$$y = C_1 e^x + C_2 x e^x + C_3 e^x \cos 2x + C_4 e^x \sin 2x.$$

PART II. FREE OSCILLATIONS.

COMPONENT TERMS OF RECTILINEAR DAMPED HARMONIC MOTION.

Kinematics of Damped Straightaway Motion.—Equation (1')

FIG. 2.



Damped straightaway motion of a particle subject to a resistance proportional to the velocity.

for damped straightaway motion gives, on integration for s and \dot{s} , the general values,

$$s = A e^{-2at} + B$$

$$\dot{s} = -2aA e^{-2at}$$

in which A and B are constants of integration. And if the initial speed is $\dot{s} = \dot{s}_0$, when $s = 0 = t$, the constants are $A = -B = -\dot{s}_0/2a$.

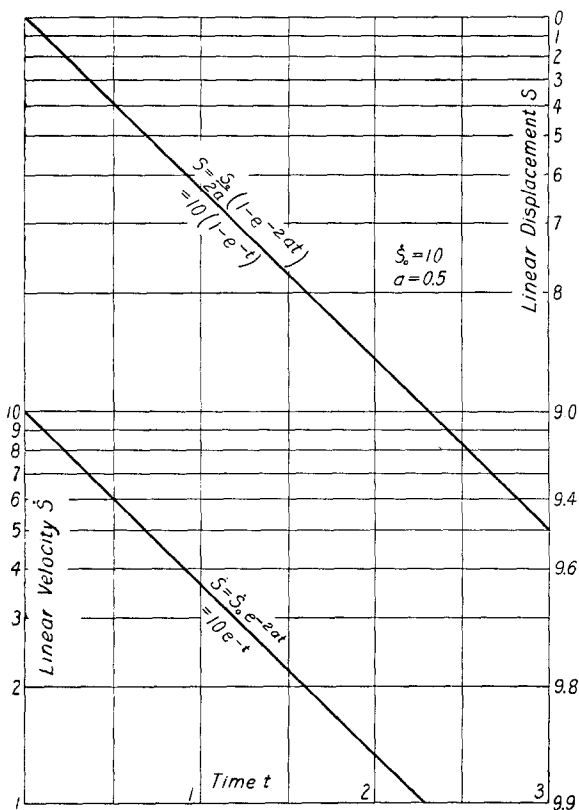
Hence

$$s = \frac{\dot{s}_0}{2a} (1 - e^{-2at}) \dots\dots\dots (7)$$

$$\dot{s} = \dot{s}_0 e^{-2at} = \dot{s}_0 - 2as \dots\dots\dots (8)$$

As t increases indefinitely e^{-2at} vanishes asymptotically; s approaches the limit $\dot{s}_0/2a$; and the speed \dot{s} vanishes asymptotically.

FIG. 2'.



Damped straightaway motion of a particle subject to a resistance proportional to the velocity.

The motion is a species of *subsidence*, or a disturbance which dies out without oscillation.

The graphs of (7) and (8), in Fig. 2, and Fig. 2', illustrate how the travel s , of the particle, asymptotically approaches its limit $\dot{s}_0/2a$, and how similarly the speed dies away in infinite time, as

indicated by the exponential factor e^{-2at} . Such factorial expressions go by the general name of "*damping factor*."

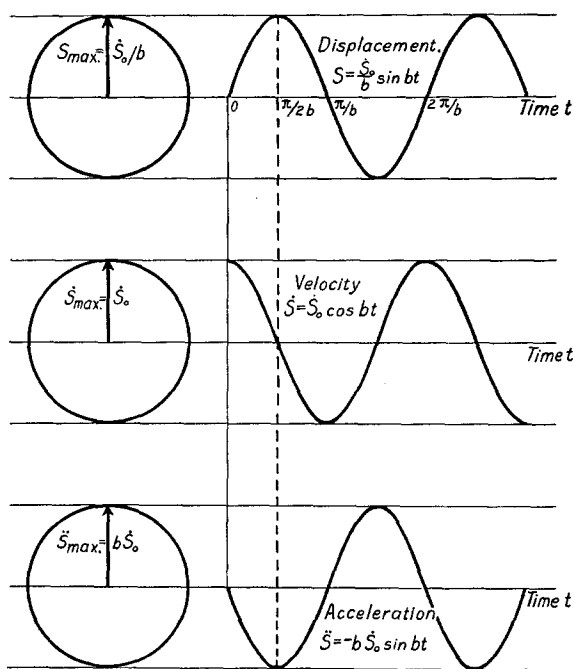
Kinematics of Simple Harmonic Motion.—Equation (2') for simple harmonic motion gives, on integration for s and \dot{s} , the general values

$$s = A \sin bt + B \cos bt$$

$$\dot{s} = b (A \cos bt - B \sin bt)$$

in which A and B are constants of integration. If the initial speed is $\dot{s} = \dot{s}_0$, when $s = 0 = t$, the constants are, $B = 0$, $A = \dot{s}_0/b$.

FIG. 3.



Simple harmonic motion, graphical elements of.

Hence

$$s = \frac{\dot{s}_0}{b} \sin bt \dots \dots \dots (9)$$

$$\dot{s} = \dot{s}_0 \cos bt \dots \dots \dots (10)$$

$$\ddot{s} = -b \dot{s}_0 \sin bt \dots \dots \dots (11)$$

These general equations for the displacement, velocity and acceleration of the oscillating particle at any instant, are plotted as functions of the time in Fig. 3.

(a) *Maxima*.—It appears both from this figure and from the equations that for $bt = 0, \pi, 2\pi$, etc., the displacement s , and acceleration \ddot{s} , are 0, while the velocity \dot{s} has its greatest absolute value $\dot{s}_0, -\dot{s}_0, \dot{s}_0$, etc. Also that for $bt = \pi/2, 3\pi/2$, etc., the velocity is null, while the displacement and acceleration have their greatest absolute values; viz., $s_1 = \dot{s}_0/b$, $\ddot{s}_1 = -b \dot{s}_0$. The maximum displacement s_1 is called the “*amplitude*.”

(b) *Period and Frequency*.—It is seen also that as bt changes by 2π , the quantities s, \dot{s}, \ddot{s} , run through a complete to-and-fro cycle, so that the *period* T of such a cycle, or oscillation, is

$$T = 2\pi/b \dots\dots\dots (12)$$

That is T is the time of passage of the particle from any point of its path to that point again in the same direction. The *frequency* N of the oscillation is obviously $N = 1/T = b/2\pi$.

Incidentally the force on the particle at any displacement can now be written in terms of the mass and period or frequency. It is, by (2) and (12),

$$F = -\frac{4\pi^2}{T^2}ms = -4\pi^2 N^2 ms \dots\dots\dots (13)$$

(c) *Phase and Epoch*.—If when $s = 0$ in (9) $bt = \epsilon$, the equation becomes $s = \frac{\dot{s}_0}{b} \sin (bt - \epsilon)$. Then $(bt - \epsilon)$ is called the “*phase angle*” or “*phase*,” and ϵ the epoch angle or “*epoch*.” This change of algebraic expression for s would produce in Fig. 3 a shift of the origin along the axis of t .

(d) *Added Motions*.—A number of simple harmonic motions $s_1 = C_1 \sin bt$, $s_2 = C_2 \cos bt$, etc., having the same period $2\pi/b$, can be imposed simultaneously on a particle, giving it a resultant simple harmonic motion of the same period. This statement is proved in textbooks on mechanics, and is merely recalled here as a reminder that such an expression as $s = (C_1 \sin bt + C_2 \cos bt)$ is a simple harmonic motion.

(e) *Projected Circular Motion*.—Simple harmonic motion may be represented as the projection, on the diameter of a circle, of the movement of a point speeding uniformly round the circumference. Thus in Fig. 4, if the point p moves with uniform linear speed \dot{s}_0 , round the circle of radius \dot{s}_0/b , its angular speed is b ; its angular displacement, or *reference angle*, referred to the horizontal radius, is bt in time t ; and its vertical displacement, speed and acceleration are obviously as given in (9), (10), (11).

Also the period of a complete cycle clearly is $2\pi/b$, as given in (12). Hence the kinematic properties, so far found algebraically, for simple harmonic motion, can be read from the *reference circle* whose radius is of length equal to the amplitude of said motion.

(f) *Energy of Simple Harmonic Motion.*—For a particle of mass m , at any displacement s , the force F opposing the motion, and the potential energy W , or work done against F during displacement from rest, are respectively

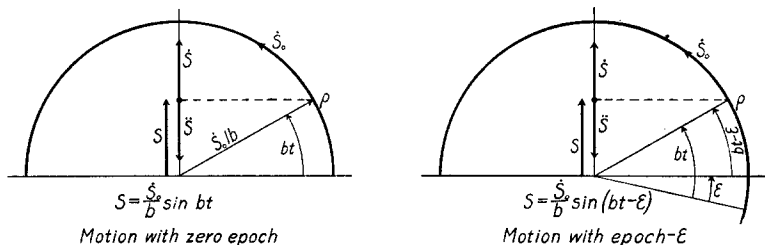
$$F = m\ddot{s} = -m b^2 s,$$

$$W = -\frac{F}{2} s = m b^2 s^2/2 \dots \dots \dots (14)$$

since the mean value of the reversed effective force causing the displacement is $-F/2$. But by (9) and (10), $\dot{s}^2 = \dot{s}_0^2 - b^2 s^2$; hence the kinetic energy, $E = m \dot{s}^2/2$, of the particle is

$$E = \frac{m}{2} (\dot{s}_0^2 - b^2 s^2) \dots \dots \dots (15)$$

FIG. 4.



Simple harmonic motion as the projection of the uniform motion of a point on a reference circle.

Hence the total energy, kinetic and potential, of the particle is

$$E + W = m \dot{s}_0^2/2 = \text{constant}, \dots \dots \dots (16)$$

The mean potential energy throughout the period, $T = 2\pi/b$, is

$$W_m = \frac{1}{T} \int_0^T W \, dt, \text{ which on writing, from (14) and (9),}$$

$$W = b^2 m s^2/2 = \frac{m \dot{s}_0^2}{2} \sin^2 bt, \text{ and integrating, gives}$$

$$W_m = m \dot{s}_0^2/4, \dots \dots \dots (17)$$

or half the total energy. So also the mean kinetic energy is $m \dot{s}_0^2/4$. Both these and the total energy vary as \dot{s}_0^2 ; also as the square of the amplitude or maximum displacement, which was shown to be $s_1 = \dot{s}_0/b$.

Reversed Simple Harmonic Motion.—If in equation (2) the

sign of b^2 be reversed, that is if the force and acceleration be directed away from the point of equilibrium, the equation of motion becomes on integration

$$\begin{aligned}s &= A e^{bt} + B e^{-bt} \\ \dot{s} &= A b e^{bt} - B b e^{-bt}\end{aligned}$$

The first of these equations shows that the space increases indefinitely with the time, and always in one direction, that is without oscillation; the second shows that the speed increases indefinitely.

A motion that increases without oscillation is called a "*divergence*." The displacement may increase indefinitely or it may asymptotically approach a limit. In the latter case it also answers the definition of a subsidence.

KINEMATICS OF RECTILINEAR DAMPED HARMONIC MOTION.

The General Integral.—Equation (3) for damped harmonic motion gives, on integration for s and \dot{s} , the general values

$$\begin{aligned}s &= A e^{\alpha t} + B e^{\beta t} \\ \dot{s} &= A \alpha e^{\alpha t} + B \beta e^{\beta t}\end{aligned}$$

in which A and B are constants of integration, and the roots of the auxiliary equation are

$$\begin{aligned}\alpha &= -a + \sqrt{a^2 - b^2} = -a + r_1 \\ \beta &= -a - \sqrt{a^2 - b^2} = -a - r_1\end{aligned}\quad \dots\dots\dots (18)$$

If $\dot{s} = \dot{s}_0$ when $s = 0 = t$, the constants are $A = \dot{s}_0/2r_1$, $B = -\dot{s}_0/2r_1$. Hence

$$s = \frac{\dot{s}_0}{2r_1} (e^{\alpha t} - e^{\beta t}) \dots\dots\dots (19)$$

$$\dot{s} = \frac{\dot{s}_0}{2r_1} (\alpha e^{\alpha t} - \beta e^{\beta t}) \dots\dots\dots (20)$$

are the general values of the displacement and speed of the oscillating point in terms of the time and given coefficients a , b . The solution takes three forms according as a is less than, equal to, or greater than b ; viz., as $r_1 = \sqrt{a^2 - b^2}$, is imaginary, zero or real.

Case I: $a < b$; Motion Periodic.—When r_1 is imaginary write $r_1 = ir$, where $r = \sqrt{b^2 - a^2}$, and (19) becomes, on using the values of α , β , in (18),

$$s = \frac{\dot{s}_0}{2ir} e^{-at} (e^{irt} - e^{-irt}).$$

Now substituting in this equation Euler's values,

$$e^{irt} = \cos rt + i \sin rt,$$

$$e^{-irt} = \cos rt - i \sin rt,$$

the imaginary i cancels out, and the displacement and speed are found in the form,

$$s = \frac{\dot{s}_0}{r} e^{-at} \sin rt, \dots\dots\dots (21)$$

$$\dot{s} = \dot{s}_0 e^{-at} \left(\cos rt - \frac{a}{r} \sin rt \right). \dots\dots\dots (22)$$

$$= \frac{\dot{s}_0}{\cos \epsilon} e^{-at} \cos (rt + \epsilon), \text{ where } \frac{a}{r} = \tan \epsilon,$$

$$= \frac{\dot{s}_0 b}{r} e^{-at} \cos (rt + \epsilon)$$

(a) *Plot of Factors and Product*.—The right-hand members of (21) and (22) consist each of the product of two factors, an exponential factor e^{-at} , and a harmonic factor. In Fig. 5 these factors and their product are plotted as functions of the time. The sketches show, what is clear from the symbolic expressions, that e^{-at} decreases asymptotically to zero as the time increases indefinitely, while each harmonic factor represents a curve of constant amplitude and pitch. The ordinate of the damped harmonic curve is the product of the coincident ordinates of the component curves, *i.e.*, the exponential and the simple harmonic, and hence dies away in an oscillatory manner to zero as the time increases indefinitely. The amplitude of the simple harmonic motion in (21) is \dot{s}_0/r , which when $a=0$ is \dot{s}_0/b , as given in (9) for undamped motion. Each amplitude is the radius of the corresponding reference circle.

(b) *Period and Reference Angle*.—From equations (21) and (22) can be found the time of describing each stroke of the damped harmonic, and the corresponding value of the reference angle rt . Starting with $s=0=t$, it appears from (21) that $s=0$ again in the successive times given by $rt=\pi, 2\pi, 3\pi \dots n\pi$; that is the time of each out-and-back stroke is

$$T^1 = \pi/r. \dots\dots\dots (23)$$

or the whole period is $T = 2\pi/r$. Also the time T^{11} for any single outstroke to where $s=0$ is given by (22) in the form $\cos(rt^{11} + \epsilon) = 0$, or $rt^{11} + \epsilon = \frac{\pi}{2}$; whence

$$T^{11} = \frac{1}{r} \left(\frac{\pi}{2} - \epsilon \right). \dots\dots\dots (24)$$

Multiplying (23) and (24) by r gives the reference angles. Hence from these equations the chief elements of displacement and time can be summarized as follows:

TABLE I.
Period of Harmonic Movements.

Movement	Duration t .	Reference angle rt
Out stroke	$\frac{1}{r} \left(\frac{\pi}{2} - \epsilon \right)$	$\frac{\pi}{2} - \epsilon$
Back stroke	$\frac{1}{r} \left(\frac{\pi}{2} + \epsilon \right)$	$\frac{\pi}{2} + \epsilon$
Out-and-back stroke.....	$\frac{\pi}{r}$	π
Complete cross stroke.....	$\frac{\pi}{r}$	π
Complete double vibration	$\frac{2\pi}{r} = T = \frac{2\pi}{\sqrt{b^2 - a^2}}$	2π
Simple harmonic vibration.....	$2\pi/b = T_0$	2π

Ratio of last two periods $T/T_0 = b/\sqrt{b^2 - a^2}$

From the foregoing paragraph it appears that damped, like undamped harmonic oscillations, are isochronous, and that their period is to that of the undamped ones in the ratio $b/\sqrt{b^2 - a^2}$. This ratio equals unity when $a = 0$; and is infinity when $a = b$, or when the motion just becomes aperiodic.

(c) *Maxima of Displacement and Speed.*—By (21) for s maximum $\dot{s} = 0$, giving $rt = \frac{\pi}{2} + n\pi - \epsilon$; hence $s = \frac{\dot{s}_0}{r} e^{-at} \cos \epsilon = \frac{\dot{s}_0}{b} e^{-at}$. For \dot{s} maximum the reference angle is $rt = (n+1)\pi - 2\epsilon$; hence $\dot{s} = \dot{s}_0 e^{-at}$. Again since the multipliers of e^{-at} in (21) and (22) remain unchanged as t increases by any number of whole periods, or by nT , the ratio of the n th + 1 maximum to the first is $s_{n+1}/s_1 = e^{-naT} = \dot{s}_n/\dot{s}_0$. Hence the graphs of the equations

$$s = s_1 e^{-at'} \dots \dots \dots (25')$$

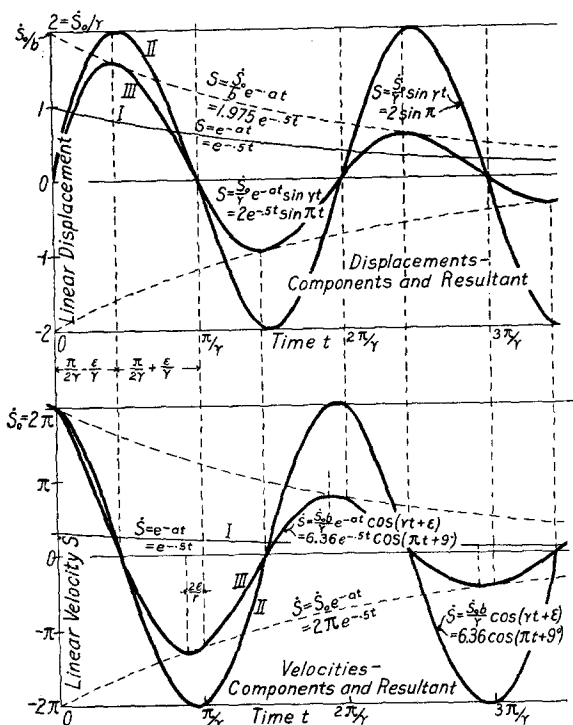
$$= \frac{\dot{s}_0}{b} e^{-at} \dots \dots \dots (25)$$

$$\dot{s} = \dot{s}_0 e^{-at} \dots \dots \dots (26)$$

where t' is the value of t for the first maximum s_1 , are curves passing through all the maxima points respectively of s and \dot{s} , as

shown in Fig. 5; for t being continuous assumes in turn all the discreet values of nT , whereupon s and \dot{s} assume all the values of s_n and \dot{s}_n respectively. It may be noted from (25) and (26) that the successive maxima of each curve (21) and (22) form a geometric series whose ratio is e^{-aT} , or that the n th amplitude is $s_n = \frac{\dot{s}_0}{b} e^{-anT}$. It is worth observing also that (25) and (26)

FIG. 5.



Displacement and speed in the periodic damped harmonic oscillation. I and II are components of III. The exponential curves cut the crests of the damped harmonic curves.

represent curves of discreet points, *i.e.*, wave tops, when T is substituted for t . On semilogarithmic paper (25) and (26) plot as straight lines, whose slopes are equal to a , as shown in Fig. 6.

For interpreting stability equations it is useful to note from (25) that $s = s_1/2$ when $a t = \log_e s_1/s = \log_e 2 = 0.693$; that is, the time of damping to one-half amplitude is

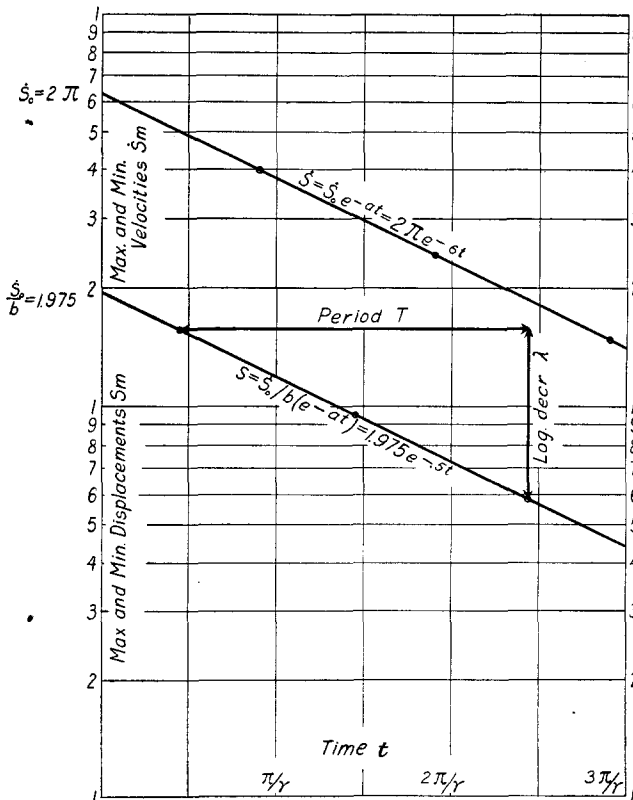
$$t = 0.693/a \dots \dots \dots (27)$$

(d) *Logarithmic Decrement.*—From (25) one derives

$$\log s = \log s_1 - at \dots \dots \dots (28)$$

a linear equation in t , whose straight-line graph, as in Fig. 6, is useful in plotting oscillation data to show the rate of decline of the amplitude. In general the decrement of $\log s$, or $\log s_1 - \log s =$

FIG. 6.



Maxima and minima of displacement and speed in periodic damped harmonic oscillation. Circles mark the maxima and minima.

$a t = \log \frac{s_1}{s}$, is directly proportional to the time, and therefore the same for successive periods T . In particular the *natural* logarithm of the ratio of any maximum to the next succeeding is called the "*logarithmic decrement.*" It is the constant decrease in the logarithm of the amplitude which occurs during each whole

period $2\pi/r$, and may be written in the forms,

$$\lambda = \left| \begin{array}{l} \frac{1}{n} \log \left(\frac{s_1}{s_{n+1}} \right) \\ \log \frac{s_n}{s_{n+1}} = \log e^{aT} = aT = \frac{2a\pi}{r} = \frac{2a\pi}{\sqrt{b^2 - a^2}} \dots \dots \dots (29) \\ \log s_n - \log s_{n+1} = \Delta \log s_n = \frac{\Delta s_n}{s_n} = \left(\frac{s_n - s_{n+1}}{s_n} \right) \end{array} \right.$$

for feeble damping)

The form $\Delta s_n/s_n$ is the fractional, or "percentage," damping in one complete oscillation.

Obviously the logarithmic decrement of the velocity "amplitude," as pictured in Figs. 5 and 6, could be treated in an analogous manner.

(e) *Ratio of Damped and Undamped Periods.*—By Table I the ratio of the periods of the damped and undamped harmonic oscillations is $T/T_0 = b/\sqrt{b^2 - a^2} = 1 + \frac{1}{2} \frac{a^2}{b^2} + \dots$, T_0 being the undamped period. By (29) when a/b is small, $a/b = \lambda/2\pi$. Hence

$$T/T_0 = 1 + \lambda^2/8\pi^2 = 1 + 0.01268\lambda^2 \dots \dots \dots (30)$$

For example, if in one period the relative decrease of amplitude is, $\frac{s_n - s_{n+1}}{s_n} = \lambda = 1/10$, the ratio of the damped and undamped periods is $T/T_0 = 1.0001268$. This shows that when the damping is so great that successive maxima in the harmonic curve differ by $1/10$, the period of the damped vibration is only about $1/8000$ longer than the free period.

(f) *Coefficients in the Equation of Damped Harmonic Motion.*—If the time nT , of n successive oscillations, and the corresponding change of amplitude be observed, the values of a and b can be computed from the formulas already derived. Thus from $\lambda = aT$, and $T = 2\pi/\sqrt{b^2 - a^2}$, there follow

$$a = n\lambda/nT \dots \dots \dots (31)$$

$$b = \sqrt{\lambda^2 + 4\pi^2}/T \dots \dots \dots (32)$$

in which a is the slope and $n\lambda, nT$, the coördinates of the graph of the oscillation data plotted as in Fig. 6.

For a/b small, (32) gives $b \propto 1/T$, on recalling that $a/b = \lambda/2\pi$; and (31) taken with (30) gives $a \propto \lambda$. That is, for a moderately damped oscillation the elastic coefficient is a function chiefly of the observed period; the damping coefficient chiefly of the rate of decay. Both are still acceleration coefficients, as illustrated in Fig. 1. The corresponding frictional and elastic forces, as given in (1) and (2), are respectively $-2m a \dot{s}$, $-m b^2 s$, m being the mass of the oscillating particle.

Case II: $a=b$; Critical Periodicity.—For the reason given in the mathematical preamble, the solution (19) and (20) fail when the roots of (3) are equal; but this on direct integration gives, if $\dot{s} = \dot{s}_0$, when $s = 0 = t$,

$$s = \dot{s}_0 t e^{-at} \dots\dots\dots (33)$$

$$\dot{s} = \dot{s}_0 e^{-at} (1 - at) \dots\dots\dots (34)$$

These values of the displacement and speed are plotted as functions of the time in Fig. 7. Starting from zero, s increases till $\dot{s} = 0$, $t = 1/a$, when it attains its maximum value $s_m = \dot{s}_0/ae = 0.368 \dot{s}_0/a$; then diminishes asymptotically to zero in infinite time. Hence the period is infinite, and the motion is called "*aperiodic.*" The speed starting with the value \dot{s}_0 , becomes zero when $t = 1/a$; becomes a minimum $\dot{s}_m = -\dot{s}_0/e^2 = -0.135 \dot{s}_0$, in time $t = 2/a$; then dies out asymptotically to zero in infinite time. After its first reversal each of the graphs represents a subsidence, the one of displacement, the other of velocity. The sinusoids of Fig. 6 become straight lines in Fig. 7.

Case III: $a > b$; Aperiodic Motion.—The general values (19), (20), of the displacement and speed, when $a > b$, can, by use of (18), be reduced directly to

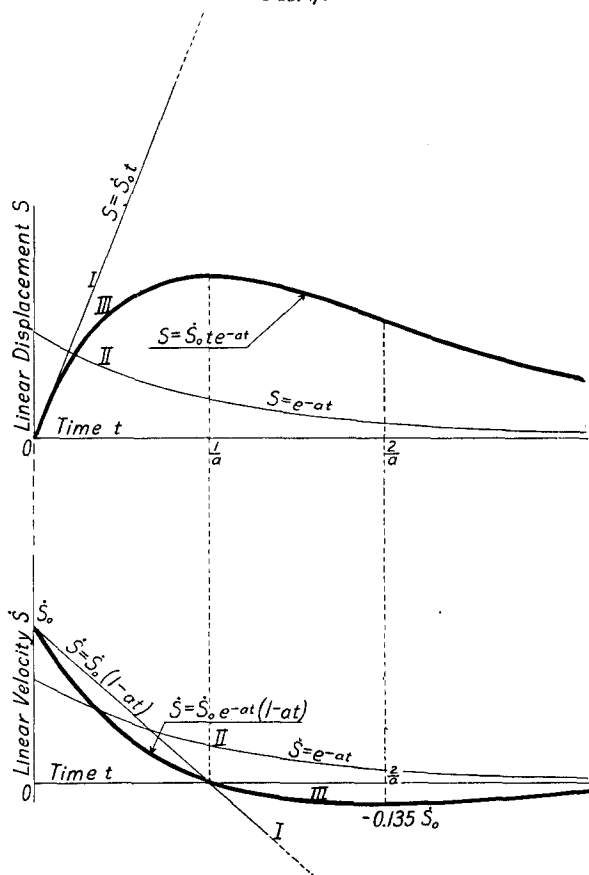
$$s = \frac{\dot{s}_0}{2 r_1} e^{-at} (e^{r_1 t} - e^{-r_1 t}) \dots\dots\dots (35)$$

$$\dot{s} = \frac{\dot{s}_0}{2 r_1} e^{-at} \left[(-a + r_1) e^{r_1 t} + (a + r_1) e^{-r_1 t} \right] \dots\dots\dots (36)$$

The graphs of these equations simulate those of Fig. 7. Starting from zero, s increases till $\dot{s} = 0$, $t = \frac{1}{2 r_1} \log \frac{a + r_1}{a - r_1}$, when it attains its maximum value s_m , found by putting this value of t in (35); then diminishes asymptotically to zero in infinite time. The motion is therefore aperiodic. The speed, starting with the value

\dot{s}_0 , falls to zero at the time t just found; changes sign and becomes a minimum when t equals twice this value; then dies out asymptotically in infinite time. After its first reversal each of the graphs for Case III, as for Case II, represents a subsidence. For $b = 0$,

FIG. 7.



Displacement and speed in a periodic damped harmonic oscillation. I and II are components of III.

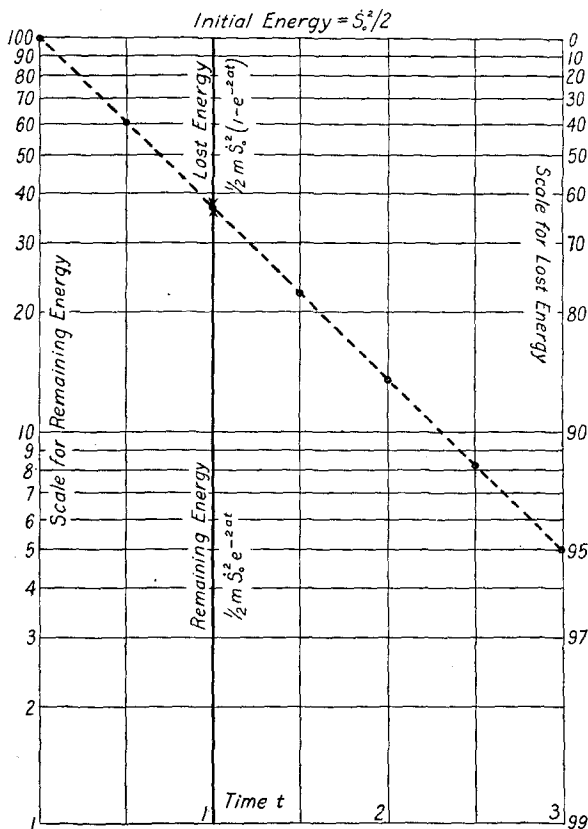
(35) and (36) are easily seen to become (7) and (8) respectively, since then $r_1 = \sqrt{a^2 - b^2} = a$.

Influence of a on Type of Motion.—The foregoing text shows that for $a = 0$, the motion is sinusoidal; for $a < b$, it is oscillatory with decaying amplitude; for $a \geq b$ it is aperiodic, *i.e.*, dies away without oscillation.

ENERGY OF DAMPED HARMONIC MOTION.

General Equation of Energy.—Except for a friction term, the energy expression for damped harmonic motion is like that for simple harmonic motion. If the particle m starts at $s = 0 = t$, with speed \dot{s}_0 , its initial energy is $m \dot{s}_0^2/2$; at any later time its

FIG. 8.



Energy in periodic damped harmonic oscillation.

kinetic energy is $E = m\dot{s}^2/2$; its potential energy is $W = m b^2 s^2/2$, by (14); and their sum equals the initial energy less that consumed by the friction $f = -2m a \dot{s}$. Since the rate of frictional loss is $f \dot{s} dt = -2m a \dot{s}^2 dt$, the total energy at any time t is

$$E + W = m \dot{s}_0^2/2 - 2m a \int_0^t \dot{s}^2 dt \dots \dots \dots (37)$$

which has the form of (16) except for the last term.

Integral of Lost Energy.—The friction term can be integrated for any portion of the movement, since \dot{s} has been found in terms of t , in earlier parts of the text. For any single stroke the lost energy can be found without integration, since at mid-stroke the total energy is $m\dot{s}^2/2$; and at the end of the stroke it is $m b \dot{s}^2/2$; while s and \dot{s} are given by (25) and (26).

For example, if the motion is periodic, the energy at mid-stroke and full stroke, respectively, is

$$E = \frac{1}{2} m \dot{s}^2 = \frac{1}{2} m \dot{s}_0^2 e^{-2at}, \quad W = \frac{1}{2} m b^2 \dot{s}^2 = \frac{1}{2} m \dot{s}_0^2 e^{-2at}$$

and these values plotted against time, on semilogarithmic paper, lie on a straight line, as in Fig. 8. For the particular values of a/b that make $\cos \epsilon = 1$ in (25), the graph in Fig. 8 delineates at every point the value of the integral in (37); but otherwise only at the marked points, for which the energy is all kinetic or all potential.

INFLUENCE OF MASS IN DAMPED HARMONIC MOTION.

Kinematic Equation Containing Mass Explicitly.—The general equation (3) contains implicitly the mass m , of the particle, and may be rewritten in the form

$$\ddot{s} + \frac{2a_1}{m} \dot{s} + \frac{b_1^2}{m} s = 0 \dots\dots\dots (38)$$

in which all the terms are still accelerations. The following definitions may be useful:

$$\begin{aligned} \frac{2a_1}{m} &\equiv \text{coefficient of damping acceleration,} \\ \frac{2a_1 \dot{s}}{m} &\equiv \text{damping acceleration,} \\ 2a_1 &\equiv \text{coefficient of damping force,} \\ 2a_1 \dot{s} &\equiv \text{damping force.} \end{aligned}$$

Kinematic Effect of Mass.—The effect of m in modulating the motion can now be seen by placing these two coefficients, respectively, in place of $2a$ and b^2 in the previous equations of the text. Thus, if they be substituted in Table I, they show that increase of m prolongs the periods $2\pi/b$, $2\pi b/\sqrt{b^2 - a^2}$, respectively, of undamped and damped motions. Similarly m is seen to diminish the rate of damping $e^{-\frac{a_1 t}{m}}$, and the rate of dissipation of energy $2 m \frac{a_1}{m} \dot{s}^2$ where $\dot{s} \propto e^{-\frac{a_1}{m} t}$, as seen by (22).

APPLICATION TO ANGULAR MOTION.

Equation of Angular Damped Harmonic Motion.—All the foregoing equations can be directly applied to the vibrations of a body about a fixed axis, usually taken as the principal centroidal axis, if m be replaced by I , the moment of inertia of the body for that axis, and if s, \dot{s}, \ddot{s} be replaced by the corresponding angular values. Thus the angular analogue of (38) may be written

$$\ddot{\theta} + \frac{2\mu}{I} \dot{\theta} + \frac{K_1}{I} \theta = 0 \dots \dots \dots (39)$$

and the coefficients of $\ddot{\theta}$ and $\dot{\theta}$ may be respectively substituted for $2a$ and b^2 in the equations (1) to (37), to derive the corresponding expressions for angular vibration. For example, the period $T = 2\pi/b$, for undamped motion, becomes

$$T = 2\pi\sqrt{I/K_1} \dots \dots \dots (40)$$

while a similar expression is found for damped motion.

Formula for Moment of Inertia in Torsional Vibration.—Equation (40) is sometimes used to determine I for a model. If T is observed in a vibration test, and the elastic coefficient K_1 is known from a torsion test, I can be computed directly from (40). It may be recalled that in such torsion test the torsional moment M , and the ensuing displacement θ , in radians, are measured to determine $K_1 = M/\theta$.

If K_1 is not supplied, T_1 may be observed in a second vibration test with the same model having an added moment of inertia I_1 , and will evidently satisfy the formula $T_1^2 = 2\pi\sqrt{I + I_1/K_1}$. From this and (40) one derives

$$I = I_1 \frac{T^2}{T_1^2 - T^2} \dots \dots \dots (41)$$

which can be used to find I from two vibration tests, when the elasticity of the spring is unknown.

A similar operation may be used to determine I when the motion is damped.

PART III. FORCED OSCILLATIONS.

KINEMATICS OF RECTILINEAR FORCED OSCILLATIONS.

The foregoing text treats briefly the theory of the free rectilinear oscillation of a particle both undamped and damped, and indicates a like treatment for the theory of oscillation of a rigid

body about a fixed axis. In closing, attention may be given also to a simple case of a forced oscillation superposed upon the free.

Equation of Forced Oscillation and Resultant Oscillation.—When a particle is performing free vibrations as represented by (3), let it be subjected to a single additional acceleration from without, parallel to the given motion, and a continuous function of the time. This latter may have any one of numerous forms. For the present let the applied acceleration be taken as a simple harmonic function of the time; so that (3) becomes

$$\ddot{s} + 2a\dot{s} + b^2s = h \sin \omega t, \dots\dots\dots (42)$$

The complete solution of this equation can be written as the sum of its complementary function (19), and its particular integral, which is easily shown to have the form

$$s = \frac{h}{\sqrt{(b^2 - \omega^2)^2 + 4a^2\omega^2}} \sin (\omega t - \epsilon) \dots\dots\dots (43)$$

$$= \frac{h \sin \epsilon}{2a\omega} \sin (\omega t - \epsilon) \dots\dots\dots (43')$$

where

$$\tan \epsilon = 2a\omega / (b^2 - \omega^2) \dots\dots\dots (44)$$

And if the complementary function has the particular form (21), for a periodic oscillation, the complete integral is

$$s = \frac{\dot{s}_0}{r} e^{-at} \sin rt + \frac{h}{\sqrt{(b^2 - \omega^2)^2 + 4a^2\omega^2}} \sin (\omega t - \epsilon) \dots\dots\dots (45)$$

where the last term represents the forced oscillation, which is superposed upon the free one given by the other term; and ϵ is the phase difference between the applied acceleration and the consequent oscillation.

Magnitude of the Resultant Oscillation.—Since the right member of (45) has only t variable, its first term, if $a \neq 0$, is graphically represented by a damped, its second term by an undamped, sine curve; and hence their resultant gradually becomes an undamped sine curve represented by the second term. That is, the free vibration gradually dies out, and the oscillation of the particle has for steady motion that due to the applied acceleration only. But in the ideal case when $a = 0$, the motion is the resultant of a permanent free simple harmonic oscillation added to a forced oscillation, also assumed permanent, whose peculiarities are still

to be examined. In general the periods $2\pi/r$ and $2\pi/\omega$, of the two motions, and their frequencies $r/2\pi$, $\omega/2\pi$, are different.

Period and Magnitude of the Forced Oscillation.—The forced oscillation (43) has the same period $2\pi/\omega$ as the applied acceleration, and contains four arbitrary parameters, h , a , b , ω , in the expression for its amplitude. This latter varies directly as h , the coefficient of intensity of the applied acceleration, and inversely as $(b^2 - \omega^2)^2 + 4a^2\omega^2$, which has three arbitrary parameters. For a given value of h , therefore, the amplitude will be small when this binomial is large, and *vice versa*.

Conditions Making Forced Oscillation Small.—The condition that, for a given intensity h , the forced vibration shall be feeble is that $(b^2 - \omega^2)^2 + 4a^2\omega^2$ be large, where a , b and ω are positive. This can occur when $b - \omega$ is large; that is, when there is a large difference between the frequencies of the undamped free oscillation and the applied acceleration; also when $a\omega$ is large, which may be due to large damping or to high frequency.

Conditions Making Forced Oscillation Large.—The general condition that the forced oscillation shall be large is that $(b^2 - \omega^2)^2 + 4a^2\omega^2$ be small, which means that both $b - \omega$ and $a\omega$ are small. As $b - \omega$ tends toward zero $\tan \epsilon = 2a\omega/(b^2 - \omega^2)$, tends toward $\pm \infty$, that is ϵ approaches $\pi/2$ from either side, according as b is greater or less than ω . The condition $a\omega$ small indicates that, even with considerable damping, the oscillation can be large if the frequency, $\omega/2\pi$, of the applied acceleration is sufficiently small, while nearly equalling the frequency, b/π , of the undamped free oscillation.

For $a = 0$, if $b - \omega \neq 0$, $\tan \epsilon = 0$, and the forced oscillation (43) becomes $s = \frac{h}{b^2 - \omega^2} \sin \omega t$, showing that the phases of the motion and of the applied acceleration agree in sign if $b > \omega$, and *vice versa*. But if $b - \omega = 0$, $\tan \epsilon = 0/0$, and the integral of (42) for this special case can better be written

$$s = \frac{-h}{2\omega} \cos \omega t \dots \dots \dots (46)$$

which indicates a divergent harmonic motion whose period is that of the undamped free oscillation, and whose amplitude $h t/2\omega$ increases directly as the time, having infinity as its "steady" value. For large values of t the speed of the particle, $\dot{s} = \frac{h}{2} \sin \omega t$,

has the same phase as the applied acceleration $h \sin \omega t$. This case is treated somewhat differently by Lord Rayleigh, in his "Theory of Sound," § 41, which is recommended for present reference, and from which is taken the expression (51) below.

Special Forms of the General Equation.—The general equation (42) may be given various interesting special forms by assigning certain particular values to the constant coefficients, a, b, h, ω . Thus if a alone is zero the equation is that of a forced undamped harmonic oscillation $\ddot{s} - b^2 s = h \sin \omega t$, an interesting case of which is given in (46). Again, if in (42) $h \sin \omega t$ is constant, the equation is that of a damped harmonic oscillation with a constant disturbing force. In this case, which is of some interest in aeronautic experiments, (42) becomes

$$\ddot{s} + 2a\dot{s} + b^2s = c \dots\dots\dots(47)$$

whose solution is

$$s = s_1 + c/b^2 \dots\dots\dots(48)$$

where s_1 is the solution of (3).

It may be recalled that, for a free damped periodic motion, the plot of s_{max} on semilogarithmic paper is a dotted straight line, as in Fig. 6. The form of (48) shows that the graph ceases to be straight when a constant disturbing acceleration occurs.

More General Forced Oscillation.—By Fourier's theorem any single-valued periodic function $f(t)$, of period $T = 2\pi/\omega$, can be expanded, between definite limits of the variable, in a series of the form

$$f(t) = h_0 + h_1 \sin(\omega t + \epsilon_1) + h_2 \sin(\omega \cdot 2t + \epsilon_2) \\ + h_3 \sin(\omega \cdot 3t + \epsilon_3) + \dots\dots\dots(49)$$

Hence if an applied acceleration of this form be substituted for the second member of (42), the equation of the forced oscillation of the particle becomes

$$\ddot{s} + 2a\dot{s} + b^2s = \sum_n h_n \sin(\omega \cdot nt + \epsilon_n) \dots\dots\dots(50)$$

The integral of this takes the form of (45), but with its last term summational. Thus the rectilinear motion of the particle consists of a damped free oscillation, of period $2\pi/r$, upon which is superposed a complex harmonic motion composed of n forced oscillations of periods $T, T/2, T/3, \dots T/n$, and phases $\epsilon_1, \epsilon_2, \dots \epsilon_n$.

ENERGY OF FORCED OSCILLATIONS.

The kinetic energy of the forced oscillation, whether damped or undamped, is proportional to \dot{s}^2 . For the damped one (43') gives, if the speed is \dot{s}_0 , when $s = 0$,

$$\text{Energy} \propto \dot{s}_0^2 \propto \sin^2 \epsilon \dots \dots \dots (51)$$

This is a maximum when $\epsilon = \pi/2$, or by (44), when $\omega = b$. That is, the forced oscillation attains its greatest kinetic energy when the period of the applied acceleration equals the natural period of the undamped free oscillation. For the undamped forced oscillation, when $\omega = b$, the kinetic energy, as shown by (46), increases indefinitely with the time.

APPLICATION TO ANGULAR MOTION.

The foregoing equations for the rectilinear forced oscillations of a particle can be applied to the motion of a rigid body oscillating about a fixed axis, in the manner already explained for free oscillations.

The Anomaly of the Nickel-Steels. C. E. GUILLAUME, Director of the International Bureau of Weights and Measures, Sevres. (*Proc. Phys. Soc. London*, August 15, 1920.)—In this lecture the distinguished author, to whom the Nobel Prize in Physics for 1920 was awarded, traces the course of the investigations which led to the discovery of the alloy invar. This name is applied to alloys of steel and nickel in which the latter metal enters to the extent of about 35.6 per cent. The characteristic of invar is its low coefficient of linear expansion, .00000119.

From a scientific point of view the new alloy has especially proved its worth in the measurement of geodetic bases. The employment of invar wires scarcely changing their length with temperature has wonderfully facilitated the accurate determination of lengths on the earth's surface. The number of requisite observers is reduced to one-fifth of the previous quota, the time expended is only one-tenth as much, while the expense falls to 2 per cent. of the former amount. In addition it is possible to work on terrain where the older type of apparatus could not have been set up.

Additional applications of nickel-steel alloys are: (a) To compensate the pendulums of clocks for temperature changes; (b) to render constant the rate of watches. When the balance-wheel is made of only one metal a nickel-steel spring keeps the daily gain or loss very small. (c) For the mountings of glass and for leading-in wires to incandescent lamps. An alloy for these purposes is selected with the same expansion coefficient as the glass in question.

G. F. S.