

522. If a Triangle Is Inscribed in a Directly Similar Triangle, and If One Triangle Varies while the Other Remains Fixed, the Centre of Similitude Is Either a Fixed Point or on a Fixed Circle

Author(s): T. J. Richards

Source: *The Mathematical Gazette*, Vol. 9, No. 132 (Dec., 1917), pp. 160-161

Published by: The Mathematical Association

Stable URL: <http://www.jstor.org/stable/3604941>

Accessed: 16-06-2016 22:23 UTC

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at

<http://about.jstor.org/terms>

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.



The Mathematical Association is collaborating with JSTOR to digitize, preserve and extend access to *The Mathematical Gazette*

\therefore the triangles ABI , AI_1C are similar ;

$$\therefore AI : AB = AC : AI_1 ;$$

$$\therefore aa_1 = bc.$$

Hence

$$aa_1a_2a_3 = b^2c^2 = 4R^2p^2.$$

$$(2) a^2 + a_1^2 + a_2^2 + a_3^2 = (a^2 + a_2^2) + (a_1^2 + a_3^2)$$

$$= II_2^2 + II_3^2$$

$$= 4OK^2 + 4I_3K^2$$

(O is the circumcentre of $I_1I_2I_3$ and OK perpendicular to I_1I_3)

$$= 4OI_3^2 ; \text{ but } OI_3 = 2R ;$$

$$\therefore a^2 + a_1^2 + a_2^2 + a_3^2 = 16R^2.$$

$$(3) a^{-2} + a_1^{-2} + a_2^{-2} + a_3^{-2} = \frac{a^2 + a_1^2}{a^2a_1^2} + \frac{a_2^2 + a_3^2}{a_2^2a_3^2}$$

$$= \frac{a^2 + a_1^2 + a_2^2 + a_3^2}{4R^2p^2} \text{ (by (1))}$$

$$= \frac{16R^2}{4R^2p^2} = 4p^{-2}.$$

C. H. RICHARD.

522. [K¹. 5. d.] *If a triangle is inscribed in a directly similar triangle, and if one triangle varies while the other remains fixed, the centre of similitude is either a fixed point or on a fixed circle.*

Let ABC , XYZ be two directly similar triangles, X being on BC , Y on CA , Z on AB .

The three circles through AYZ , BZX , CXY have a common point S , at which YZ , ZX , XY subtend angles $\pi - A$, $\pi - B$, $\pi - C$ respectively. Also

$$\angle BSC = \angle BZX + \angle CXY = \angle A + \angle X,$$

and similarly $\angle CSA = \angle B + \angle Y$, $\angle ASB = \angle C + \angle Z$.

Hence, if one of the triangles remains fixed, and the other varies, remaining constant in species, the point S is a fixed point, and bears a fixed relation to each of the triangles.

There are six possible cases according as X , Y and Z are or are not equal respectively to A , B and C .

Case I. Let $X = B$, $Y = C$, $Z = A$.

Then $\angle BSC = A + B$, $\angle CSA = B + C$, $\angle ASB = C + A$.

Also $\angle XSY = \pi - C = \angle BSC$,

and similarly $\angle FSZ = \angle CSA$ and $\angle ZSX = \angle ASB$.

Therefore the fixed point S bears the same relation to the two similar triangles : that is, it is the centre of similitude.

Case II. Let $X = C$, $Y = A$, $Z = B$.

Then, as in Case I., S is the centre of similitude of the two triangles.

It is the point at which BC , CA , AB subtend angles $A + C$, $B + A$, $C + B$ respectively.

For these two cases, see Casey's *Sequel to Euclid*, supplementary chapter, section 2. The two positions of S are the Brocard Points of either triangle.

Case III. Let $X = A$, $Y = B$, $Z = C$.

Then $\angle YSZ = \pi - A = \pi - X$, and $\angle ZSX = \pi - Y$, $\angle XSY = \pi - Z$;

$\therefore S$ is the orthocentre of XYZ .

Also $\angle BSC = A + X = 2A$, and $\angle CSA = 2B$, $\angle ASB = 2C$;

$\therefore S$ is the circumcentre of ABC .

Suppose ABC to remain fixed and XYZ to vary. Then, since XYZ is of constant species and has one point, the orthocentre, fixed, and since X moves along a straight line, the locus of any other point bearing a fixed relation to the triangle is a straight line. In particular, the locus of the centroid H is a straight line. To determine the position of this straight line, let X, Y, Z coincide with D, E, F , the middle points of the sides of the triangle ABC . Then the centroid coincides with G , the centroid of ABC . Therefore S, G, D , three points connected with the triangle DEF , correspond to the points S, H, X in the similar triangle XYZ . $\therefore SDX$ and SGH are similar triangles. $\therefore \angle SGH$ is a right angle; i.e. the locus of H is the straight line through G , perpendicular to SG .

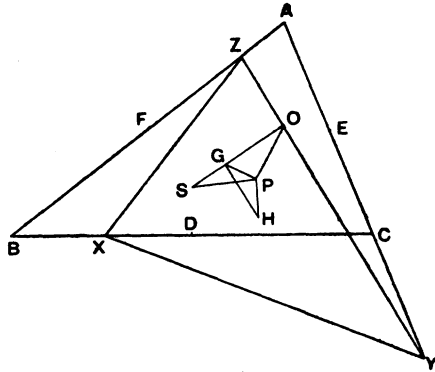


FIG. 1.

Let O be the orthocentre of ABC , and P the double point.

Then since O and S are a pair of corresponding points, and G and H another pair, the triangles SPO and HPG are similar, and $\angle PGH = \angle POS$. $\therefore GH$ touches the circle circumscribing POG : i.e. P lies on the circle on OG as diameter, the orthocentroidal circle of the fixed triangle. From the definition of the double point, it must also be on the orthocentroidal circle of XYZ , which will therefore be the locus if ABC varies and XYZ remains fixed.

Case IV. Let $X=A, Y=C, Z=B$.

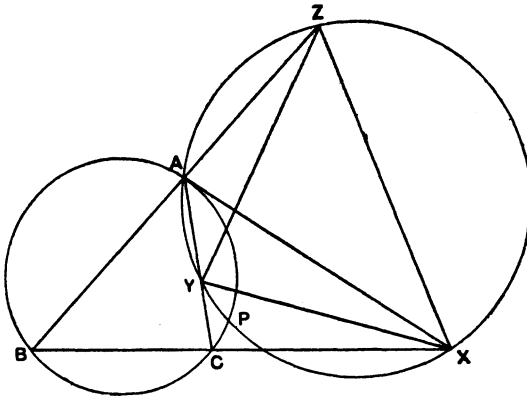


FIG. 2.

In this case X is a fixed point, namely the point where the tangent at A to the circumcircle of ABC meets BC produced. The circumcircles of ABC, XYZ evidently intersect in A : let P be their other point of intersection. Then the sides of the triangle XYZ subtend at P the same angles as the corresponding sides of ABC . P is therefore the double point of the two triangles: and the locus of the double point is the circumcircle of the fixed triangle.

Similarly the locus will evidently be the circumcircle of the fixed triangle in

Case V., where $Y=B, Z=A, X=C$, and in

Case VI., where $Z=C, X=B, Y=A$.

T. J. RICHARDS,