Philosophical Magazine Series 6

Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tphm17

XXXVII. Reversion of power series

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Published online: 21 Apr 2009.

XXXVII. *Reversion of Power Series.*

By C. E. VAN ORSTRAND *.

A LARGE number of the series employed in pure and applied mathematics are special cases of the integral power series,

$$y = a_0 x + a_1 x^2 + a_2 x^3 + \dots \quad . \quad . \quad . \quad . \quad (1)$$

In the numerous applications of this series, it is oftentimes necessary to express x as a function of y by means of the integral power series

$$x = A_0 y + A_1 y^2 + A_2 y^3 + \dots \quad . \quad . \quad . \quad . \quad (2)$$

The usual method of procedure is to substitute the value of y in the second equation, equate coefficients, and then solve for A_0, A_1, A_2, ... in terms of a_0, a_1, a_2, ... The first three or four coefficients may be determined in this way without much difficulty, but the coefficients of the higher powers of y are so complicated that this method is almost useless for the determination of their values.

To obviate this difficulty, Professor McMahon † bases the development of the second equation on Lagrange's series. He puts

$$z = \frac{y}{a_0}, \quad b_1 = -\frac{a_1}{a_0}, \quad b_2 = -\frac{a_2}{a_0} \dots,$$

$$x = z + b_1 x^2 + b_2 x^3 + \dots = z + \phi(x),$$

$$n^{r/1} = n(n+1)(n+2)\dots(n+r-1),$$

and obtains

$$b_{n-2} + \Sigma \frac{n^{r/1}}{p!\, q! \dots} b_i^p b_j^q \dots, \quad . \quad . \quad . \quad (a)$$

as the coefficient of z^{n-1} in the reverted equation. The exponents and subscripts of the b's in this expression satisfy the conditions

$$p + q + \dots = r + 1,$$

$$pi + qj + \dots = n - 2.$$

Another method of obtaining the general term of a reverse series has been suggested by Professors Harkness and Morley. They differentiate (2) with respect to x and divide by y^n.

* Read before the Philosophical Society of Washington, D.C., Oct. 9, 1909. Communicated by the Author.

† "On the General Term in the Reversion of Series," Bull. Am. Math. Soc. p. 170, April 1894.

These operations give

$$\frac{1}{y^n}=y'\left[\frac{A_0}{y^n}+\frac{2A_1}{y^{n-1}}+\ldots\frac{nA_{n-1}}{y}\right.$$
$$\left.+(n+1)A_n+(n+2)A_{n+1}y+\ldots\right]. \quad . \quad . \quad (3)$$

It may be shown by substitution of y from (1) that $nA_{n-1}y^{-1}y'$ is the only term in the right-hand member which contains x^{-1}. That this is true is shown also by means of the equation

$$\frac{y'}{y^n}=\frac{1}{-(n-1)}\frac{d}{dx}\left(\frac{1}{y^{n-1}}\right)$$
$$=\frac{1}{-(n-1)}\frac{d}{dx}\left[(a_0x)^{-(n-1)}\left(1+\frac{a_1}{a_0}x+\frac{a_2}{a_0}x^2+\ldots\right)^{-(n-1)}\right]$$
$$=\frac{d}{dx}\left[\frac{\alpha_1}{x^{n-1}}+\frac{\alpha_2}{x^{n-2}}+\ldots\frac{\alpha_{n-1}}{x}+\alpha_n+\alpha_{n+1}x+\ldots\right],$$

a series which after differentiation contains no terms in x^{-1} for integral values of n other than unity. Hence, by equating the coefficients of x^{-1} in (3), we obtain *

$$A_{n-1}=\frac{1}{n}\left[\frac{1}{y^n}\right]_{\frac{1}{x}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where the expression in the brackets means the coefficient of x^{-1} in the development of y^{-n} as a function of x. This coefficient may be found without much difficulty. Performing the indicated expansion

$$y^{-n}=(a_0x+a_1x^2+a_2x^3+\ldots)^{-n}$$
$$=(a_0x)^{-n}\left(1+\frac{a_1}{a_0}x+\frac{a_2}{a_0}x^2+\ldots\right)^{-n}$$
$$=(a_0x)^{-n}(1-b_1x-b_2x^2+\ldots)^{-n}$$
$$=(a_0x)^{-n}\left[1+n(b_1x+b_2x^2+\ldots)+\ldots\right.$$
$$\left.+\frac{n(n+1)\ldots(n+r-1)}{r!}(b_1x+b_2x^2+\ldots)^r+\ldots\right].(5)$$

* Harkness and Morley, 'Introduction to Analytic Functions, p. 144.

The polynomial theorem gives

$$(b_1x+b_2x^2+\ldots)^r=\Sigma\frac{r!}{p!\,q!}b_1^pb_2^q\ldots x^{p+2q+\ldots}, \quad . \quad (6)$$

the exponents being subject to the conditions

$$\left.\begin{array}{l} p+q+\ldots=r, \\ p+2q+\ldots=n-1. \end{array}\right\} \quad . \quad . \quad . \quad . \quad (7)$$

The first condition arises from the homogeneity of the b terms in the expanded equation, and the second is imposed by the condition that the terms in x must be of degree $(n-1)$ in order that the complete expansion of y^{-n} contain a term in x^{-1}. These conditions therefore require that the b terms are of order r and weight $(n-1)$ instead of order $(r+1)$ and weight $(n-2)$ as in expression (a). This difference in the order and weight is due to the fact that (a) is the coefficient of z^{n-1} instead of z^n. Finally, by substituting (6) in (5) and (5) in (4), we obtain

$$A_{n-1}=\frac{\Sigma(n+1)(n+2)\ldots(n+r-1)}{p!\,q!}b_1^pb_2^q\ldots, \quad . \quad (8)$$

as the coefficient of $y^na_0^{-n}=z^n$.

Formulas (7) and (8) hold for all positive integral values of r and n. The seemingly exceptional case, $r=1$, is readily seen from (4) and (5) to give a numerical coefficient unity for all values of n. Since the b terms are of a given order and weight, they may be taken in part from tables, such as Bruno's table, "Symmetrische Funktionen der Wurzeln einer Gleichung," contained in his treatise, 'Binäre Formen,' which contains all terms of successive orders and weights from 1 to 11 inclusive. Terms of weight 12 may be determined with the assistance of the same table, for evidently the expression, $b_1\times$ terms of weight $11+b_2\times$ terms of weight $12+\ldots$, contains all terms of weight 12, including duplications. The process may be continued so as to determine all terms of any given order and weight *.

Having determined $p, q\ldots$ in the manner indicated above for any particular value of n, the corresponding coefficient is readily determined by substitution in either of the preceding

* For a precise method of determining order and weight, see a paper by R. A. Harris, " On the Expansion of Sn x," Annals of Mathematics, 1888, p. 87.

formulas. By proceeding in this way, the first 13 terms of the reverse series of (1) are found to be

$$
\begin{aligned}
x = z &+ b_1 z^2 + [b_2 + 2b_1^2] z^3 + [b_3 + 5b_1b_2 + 5b_1^3] z^4 \\
&+ [b_4 + 6b_1b_3 + 3b_2^2 + 21b_1^2b_2 + 14b_1^4] z^5 \\
&+ [b_5 + 7(b_1b_4 + b_2b_3) \\
&\quad + 28(b_1^2b_3 + b_1b_2^2) + 84b_1^3b_2 + 42b_1^5] z^6 \\
&+ [b_6 + 4(2b_1b_5 + 2b_2b_4 + b_3^2) \\
&\quad + 12(3b_1^2b_4 + 6b_1b_2b_3 + b_2^3) \\
&\quad + 60(2b_1^3b_3 + 3b_1^2b_2^2) + 330b_1^4b_2 + 132b_1^6] z^7 \\
&+ [b_7 + 9(b_1b_6 + b_2b_5 + b_3b_4) \\
&\quad + 45(b_1^2b_5 + b_1b_3^2 + b_2^2b_3 + 2b_1b_2b_4) \\
&\quad + 165(b_1^3b_4 + b_1b_2^3 + 3b_1^2b_2b_3) \\
&\quad + 495(b_1^4b_3 + 2b_1^3b_2^2) + 1287b_1^5b_2 + 429b_1^7] z^8 \\
&+ [b_8 + 5(2b_1b_7 + 2b_2b_6 + 2b_3b_5 + b_4^2) \\
&\quad + 55(b_1^2b_6 + b_2^2b_4 + b_2b_3^2 + 2b_1b_2b_5 + 2b_1b_3b_4) \\
&\quad + 55(4b_1^3b_5 + 6b_1^2b_3^2 + 12b_1^2b_2b_4 + 12b_1b_2^2b_3 + b_2^4) \\
&\quad + 715(b_1^4b_4 + 2b_1^2b_2^3 + 4b_1^3b_2b_3) \\
&\quad + 1001(2b_1^5b_3 + 5b_1^4b_2^2) + 5005b_1^6b_2 + 1430b_1^8] z^9 \\
&+ [b_9 + 11(b_1b_8 + b_2b_7 + b_3b_6 + b_4b_5) \\
&\quad + 22(3b_1^2b_7 + 3b_2^2b_5 + 3b_1b_4^2 + 6b_1b_2b_6 + 6b_1b_3b_5 + 6b_2b_3b_4 + b_3^3) \\
&\quad + 286(b_1^3b_6 + b_2^3b_3 + 3b_1^2b_2b_5 + 3b_1^2b_3b_4 + 3b_1b_2^2b_4 + 3b_1b_2b_3^2) \\
&\quad + 1001(b_1^4b_5 + b_1b_2^4 + 2b_1^3b_3^2 + 4b_1^3b_2b_4 + 6b_1^2b_2^2b_3) \\
&\quad + 1001(3b_1^5b_4 + 10b_1^3b_2^3 + 15b_1^4b_2b_3) \\
&\quad + 8008(b_1^6b_3 + 3b_1^5b_2^2) + 19448b_1^7b_2 + 4862b_1^9] z^{10} \\
&+ [b_{10} + 6(2b_1b_9 + 2b_2b_8 + 2b_3b_7 + 2b_4b_6 + b_5^2) \\
&\quad + 78(b_1^2b_8 + b_2^2b_6 + b_2b_4^2 + b_3^2b_4 + 2b_1b_2b_7 + 2b_1b_3b_6 \\
&\qquad + 2b_1b_4b_5 + 2b_2b_3b_5) \\
&\quad + 182(2b_1^3b_7 + 2b_2^3b_4 + 2b_1b_3^3 + 3b_1^2b_4^2 + 3b_2^2b_3^2 \\
&\qquad + 6b_1^2b_2b_6 + 6b_1^2b_3b_5 + 6b_1b_2^2b_5 + 12b_1b_2b_3b_4) \\
&\quad + 273(5b_1^4b_6 + 20b_1^3b_2b_5 + 20b_1^3b_3b_4 + 20b_1b_2^3b_3 \\
&\qquad + 30b_1^2b_2^2b_4 + 30b_1^2b_2b_3^2 + b_2^5) \\
&\quad + 2184(2b_1^5b_5 + 5b_1^4b_3^2 + 5b_1^2b_2^4 + 10b_1^4b_2b_4 + 20b_1^3b_2^2b_3) \\
&\quad + 12376(b_1^6b_4 + 6b_1^5b_2b_3 + 5b_1^4b_2^3) \\
&\quad + 15912(2b_1^7b_3 + 7b_1^6b_2^2) + 75582b_1^8b_2 + 16796b_1^{10}] z^{11}
\end{aligned}
$$

$$
\begin{aligned}
+ [b_{11} &+ 13(b_1 b_{10} + b_2 b_9 + b_3 b_8 + b_4 b_7 + b_5 b_6) \\
&+ 91(b_1^2 b_9 + b_2^2 b_7 + b_1 b_5^2 + b_3^2 b_5 + b_3 b_4^2 + 2b_1 b_2 b_8 + 2b_1 b_3 b_7 \\
&\qquad + 2b_1 b_4 b_6 + 2b_2 b_3 b_6 + 2b_2 b_4 b_5) \\
&+ 455(b_1^3 b_8 + b_2^3 b_5 + b_2 b_3^3 + 3b_1^2 b_2 b_7 + 3b_1^2 b_3 b_6 + 3b_1 b_2^2 b_6 \\
&\qquad + 3b_1^2 b_4 b_5 + 3b_1 b_2 b_4^2 + 3b_1 b_3^2 b_4 + 3b_2^2 b_3 b_4 + 6b_1 b_2 b_3 b_5) \\
&+ 1820(b_1^4 b_7 + b_2^4 b_3 + 2b_1^3 b_4^2 + 2b_1^2 b_3^3 + 4b_1^3 b_2 b_6 + 4b_1^3 b_3 b_5 \\
&\qquad + 4b_1 b_2^3 b_4 + 6b_1^2 b_2^2 b_5 + 6b_1 b_2^2 b_3^2 + 12b_1^2 b_2 b_3 b_4) \\
&+ 6188(b_1^5 b_6 + b_1 b_2^5 + 5b_1^4 b_2 b_5 + 5b_1^4 b_3 b_4 + 10b_1^3 b_2^2 b_4 \\
&\qquad + 10b_1^3 b_2 b_3^2 + 10b_1^2 b_2^3 b_3) \\
&+ 18564(b_1^6 b_5 + 3b_1^5 b_3^2 + 5b_1^3 b_2^4 + 6b_1^5 b_2 b_4 + 15b_1^4 b_2^2 b_3) \\
&+ 50388(b_1^7 b_4 + 7b_1^6 b_2 b_3 + 7b_1^5 b_2^3) \\
&+ 125970(b_1^8 b_3 + 4b_1^7 b_2^2) + 293930 b_1^9 b_2 + 58786 b_1^{11}] z^{12} \\
+ [b_{12} &+ 7(2b_1 b_{11} + 2b_2 b_{10} + 2b_3 b_9 + 2b_4 b_8 + 2b_5 b_7 + b_6^2) \\
&+ 35(3b_1^2 b_{10} + 3b_2^2 b_8 + 3b_2 b_5^2 + 3b_3^2 b_6 + 6b_1 b_2 b_9 + 6b_1 b_3 b_8 \\
&\qquad + 6b_1 b_4 b_7 + 6b_1 b_5 b_6 + 6b_2 b_3 b_7 + 6b_2 b_4 b_6 + 6b_3 b_4 b_5 + b_4^3) \\
&+ 140(4b_1^3 b_9 + 4b_2^3 b_6 + 6b_1^2 b_5^2 + 6b_2^2 b_4^2 + 12b_1^2 b_2 b_8 + 12b_1^2 b_3 b_7 \\
&\qquad + 12b_1 b_2^2 b_7 + 12b_1^2 b_4 b_6 + 12b_1 b_3^2 b_5 + 12b_1 b_3 b_4^2 \\
&\qquad + 12b_2^2 b_3 b_5 + 12b_2 b_3^2 b_4 + 24b_1 b_2 b_3 b_6 + 24b_1 b_2 b_4 b_5 + b_3^4) \\
&+ 2380(b_1^4 b_8 + b_2^4 b_4 + 2b_2^3 b_3^2 + 4b_1^3 b_2 b_7 + 4b_1^3 b_3 b_6 + 4b_1^3 b_4 b_5 \\
&\qquad + 4b_1 b_2^3 b_5 + 4b_1 b_2 b_3^3 + 6b_1^2 b_2^2 b_6 + 6b_1^2 b_2 b_4^2 \\
&\qquad + 6b_1^2 b_3^2 b_4 + 12b_1^2 b_2 b_3 b_5 + 12b_1 b_2^2 b_3 b_4) \\
&+ 1428(6b_1^5 b_7 + 15b_1^4 b_4^2 + 20b_1^3 b_3^3 + 30b_1^4 b_2 b_6 + 30b_1^4 b_3 b_5 \\
&\qquad + 30b_1 b_2^4 b_3 + 60b_1^3 b_2^2 b_5 + 60b_1^2 b_2^3 b_4 + 90b_1^2 b_2^2 b_3^2 \\
&\qquad + 120b_1^3 b_2 b_3 b_4 + b_2^6) \\
&+ 27132(b_1^6 b_6 + 3b_1^2 b_2^5 + 6b_1^5 b_2 b_5 + 6b_1^5 b_3 b_4 + 15b_1^4 b_2^2 b_4 \\
&\qquad + 15b_1^4 b_2 b_3^2 + 20b_1^3 b_2^3 b_3) \\
&+ 19380(4b_1^7 b_5 + 14b_1^6 b_3^2 + 28b_1^6 b_2 b_4 + 35b_1^4 b_2^4 + 84b_1^5 b_2^2 b_3) \\
&+ 67830(3b_1^8 b_4 + 24b_1^7 b_2 b_3 + 28b_1^6 b_2^3) \\
&+ 248710(2b_1^9 b_3 + 9b_1^8 b_2^2) + 1144066 b_1^{10} b_2 + 208012 b_1^{12}] z^{13} \\
&+ \ldots\ldots
\end{aligned}
$$

In order to obtain a complete check, the numerical coefficients in the above series have been computed twice with Professor McMahon's formula and once with formula (8). Use has also been made of the partial check obtained by putting

$$b_1 = b_2 = \dots = b_n = \dots = -1,$$

for in this case

$$x = y - y^2 + y^3 - \dots,$$

as is otherwise made evident by writing the original and the reverted equations in the respective forms,

$$y = (x)(1-x)^{-1} \quad \text{and} \quad x = y(1+y)^{-1}.$$

This result suffices to establish a theorem in regard to the coefficients of terms of all orders and of a given weight, viz.: the sum of the numerical coefficients of the terms of even order is greater or less by unity than the sum of the numerical coefficients of the terms of odd order according as the weight is even or odd.

There are a number of special series reducible to the form (1) and therefore capable of reversion in the usual manner. Such are for example equations containing an absolute term c. It is then sufficient to replace y in (1) by $y_1 = y - c$. If the power series contains even powers only, it may be written

$$y = a_0 X^2 + a_1 X^4 + a_1 X^6 + \dots,$$

and this series is reduced to the form (1) by the substitution $x = X^2$. If the coefficients of the first successive powers of x vanish,

$$y = c + a_{m-1} x^m + a_m x^{m+1} + \dots,$$

and the required transformation is

$$y_1 = (y - c) \div a_{m-1} = x^m (1 + \alpha_1 x + \alpha_2 x^2 + \dots)$$
$$y_1^{1/m} = x + \beta_1 x^2 + \beta_2 x^3 + \dots .$$

There are m reversions in this case corresponding to the m roots of y_1.

Other series containing zero coefficients are reversed by substitution in the complete expansion. Thus, if the series contains odd powers only,

$$b_1 = b_3 = b_5 = \dots = 0,$$

and the reversed series is

$$\begin{aligned} x = z + b_2 z^3 + [b_4 + 3b_2^2] z^5 + [b_6 + 8b_2 b_4 + 12 b_2^3] z^7 \\ + [b_8 + 10 b_2 b_6 + 5 b_4^2 + 55 b_2^2 b_4 + 55 b_2^4] z^9 \\ + [b_{10} + 12 b_2 b_8 + 12 b_4 b_6 + 78 b_2^2 b_6 \\ + 78 b_2 b_4^2 + 364 b_2^3 b_4 + 273 b_2^5] z^{11} \\ + [b_{12} + 14 b_2 b_{10} + 14 b_4 b_8 + 7 b_6^2 \\ + 105 b_2^2 b_8 + 210 b_2 b_4 b_6 + 35 b_4^3 \\ + 560 b_2^3 b_6 + 840 b_2^2 b_4^2 + 2380 b_2^4 b_4 \\ + 1428 b_2^6] z^{13} \\ + \dots\dots . \end{aligned}$$

Again, if the series proceeds by alternate odd powers beginning with the first,

$$b_1 = b_2 = b_3 = b_5 = b_6 = b_7 = b_9 = \dots = 0,$$

and the preceding series reduces to

$$x = z + b_4 z^5 + (b_8 + 5b_4^2) z^9 + (b_{12} + 14 b_4 b_8 + 35 b_4^3) z^{13} + \dots .$$

An important case arises when the number of coefficients which do not vanish is finite. The reversed series is then an expression in terms of an infinite series for one real root of a polynomial of the nth degree *. The first terms in the solution of the quadratic, cubic, and biquadratic are given below.

(*a*) Solution of quadratic.

$$y = a_0 x + a_1 x^2$$

$$z = x - b_1 x^2$$

$$\begin{aligned} x = z + b_1 z^2 + 2b_1^2 z^3 + 5 b_1^3 z^4 + 14 b_1^4 z^5 + 42 b_1^5 z^6 + 132 b_1^6 z^7 \\ + 429 b_1^7 z^8 + 1430 b_1^8 z^9 + 4862 b_1^9 z^{10} + 16796 b_1^{10} z^{11} \\ + 58786 b_1^{11} z^{12} + 208012 b_1^{12} z^{13} + \end{aligned}$$

* Joseph B. Mott, "On the Solution of Equations," The Analyst, vol. ix. 1882, p. 104. Merriman and Woodward, 'Higher Mathematics,' p. 27.

(*b*) Solution of cubic.

$$(1)\quad y=a_0x+a_2x^3$$
$$z=x-b_2x^3$$

$$x=z+b_2z^3+3b_2^2z^5+12b_2^3z^7+55b_2^4z^9+273b_2^5z^{11}+1428b_2^6z^{13}+\ \ldots .$$

$$(2)\quad y=a_0x+a_1x^2+a_2x^3$$
$$z=x-b_1x^2-b_2x^3$$

$$\begin{aligned}x=z&+b_1z^2+[b_2+2b_1^2]z^3+[5b_1b_2+5b_1^3]z^4\\&+[3b_2^2+21b_1^2b_2+14b_1^4]z^5\\&+[28b_1b_2^2+84b_1^3b_2+42b_1^5]z^6\\&+[12b_2^3+180b_1^2b_2^2+330b_1^4b_2+132b_1^6]z^7\\&+[165b_1b_2^3+990b_1^3b_2^2+1287b_1^5b_2+429b_1^7]z^8\\&+\ \ldots\ldots .\end{aligned}$$

(*c*) Solution of biquadratic.

$$(1)\quad y=a_0x+a_3x^4.$$
$$z=x-b_3x^4.$$

$$x=z+b_3z^4+4b_3^2z^7+22b_3^3z^{10}+140b_3^4z^{13}+\ \ldots .$$

$$(2)\quad y=a_0x+a_2x^3+a_3x^4.$$
$$z=x-b_2x^3-b_3x^4.$$

$$\begin{aligned}x=z&+b_2z^3+b_3z^4+3b_2^2z^5+7b_2b_3z^6\\&+[4b_3^2+12b_2^3]z^7+45b_2^2b_3z^8+[55b_2b_3^2+55b_2^4]z^9\\&+[22b_3^3+286b_2^3b_3]z^{10}+[546b_2^2b_3^2+273b_2^5]z^{11}\\&+[455b_2b_3^3+1820b_2^4b_3]z^{12}+[140b_3^4+4760b_2^3b_3^2+1428b_2^6]z^{13}+\ \ldots .\end{aligned}$$

$$(3)\quad y=a_0x+a_1x^2+a_2x^3+a_3x^4.$$
$$z=x-b_1x^2-b_2x^3-b_3x^4.$$

$$\begin{aligned}x=z&+b_1z^2+[b_2+2b_1^2]z^3+[b_3+5b_1b_2+5b_1^3]z^4.\\&+[6b_1b_3+3b_2^2+21b_1^2b_2+14b_1^4]z^5\\&+[7b_2b_3+28(b_1^2b_3+b_1b_2^2)+84b_1^3b_2+42b_1^5]z^6\\&+[8b_2b_4+4b_3^2+72b_1b_2b_3+12b_2^3+120b_1^3b_3\\&\quad+180b_1^2b_2^2+330b_1^4b_2+132b_1^6]z^7\\&+[45b_1b_3^2+45b_2^2b_3+165b_1b_2^3+495b_1^2b_2b_3\\&\quad+495b_1^4b_3+990b_1^3b_2^2+1287b_1^5b_2+429b_1^7]z^8\\&+\ \ldots\ldots\end{aligned}$$

The preceding expressions are not always sufficiently convergent* to be useful in practice. They may be made convergent, however, by substituting an approximate value of x in the original equation. It thus becomes possible to obtain all of the real roots of any polynomial. After one or more of the roots $\alpha_1, \alpha_2, \alpha_3, \ldots$ have been obtained, use may be made of the relations†

$$\begin{aligned} a_0 &= -(\alpha_1+\alpha_2+\alpha_3+\ \ldots\ \alpha_n) \\ a_1 &= \ \ (\alpha_1\alpha_2+\alpha_1\alpha_3+\ \ldots\ \alpha_{n-1}\alpha_n) \\ a_2 &= -(\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_3\alpha_4+\ \ldots\ a_{n-2}\alpha_{n-1}\alpha_n) \\ &\cdots\cdots\cdots\cdots \\ a_n &(-1)^n(\alpha_1\alpha_2\alpha_3\ \ldots\ \alpha_{n-1}\alpha_n), \end{aligned}$$

in the evaluation of the remaining roots. A method for determining all of the real and imaginary roots from a single series has been given by McClintock‡.

Following are some examples:

$$(1)\ y=\cosh \mathrm{X}-1=\frac{\mathrm{X}^2}{2\,!}+\frac{\mathrm{X}^4}{4\,!}+\frac{\mathrm{X}^6}{6\,!}+\ldots$$

$$=\frac{x}{2\,!}+\frac{x^2}{4\,!}+\frac{x^3}{6\,!}+\ldots.$$

$$z=2y\ ;\ b_1=-\frac{1}{12}\ ;\ b_2=-\frac{1}{360}\ ;\ \ldots$$

$$\mathrm{X}^2=x=z-\frac{1}{12}z^2+\frac{1}{90}z^3-\frac{1}{560}z^4+\frac{1}{3150}z^5-\ldots$$

$$=2y-\frac{1}{3}y^2+\frac{4}{45}y^3-\frac{1}{35}y^4+\frac{16}{1575}y^5-\ldots.$$

$$(2)\ y=\frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}\,dx=x-\frac{x^3}{1\,!\,3}+\frac{x^5}{2\,!\,5}-\frac{x^7}{3\,!\,7}+\ldots$$

$$z=\frac{\sqrt{\pi}}{2}y\ ;\ b_2=\frac{1}{3}\ ;\ b_4=-\frac{1}{10}\ ;\ b_6=\frac{1}{42}\ ;\ \ldots$$

$$x=z+\frac{1}{3}z^3+\frac{7}{30}z^5+\frac{127}{630}z^7+\frac{4369}{22680}z^9+\frac{34807}{178200}z^{11}+\ldots$$

$$=0{\cdot}8862\ 2693y+0{\cdot}2320\ 1367y^3+0{\cdot}1275\ 5618y^5$$
$$+0{\cdot}0865\ 5213y^7+0{\cdot}0649\ 5962y^9+0{\cdot}0517\ 3128y^{11}+\ldots.$$

* For a proof that the reverse series converges in the same domain as the original series, see Harkness and Morley's 'Treatise on the Theory of Functions,' p. 116.

† Burnside and Panton's 'Theory of Equations,' Chapter III.

‡ Bulletin Am. Math. Soc. i. 1894, p. 3; Am. Jour. of Math. xvii. 1895, pp. 89–110. Merriman and Woodward's 'Higher Mathematics,' p. 27.

(3) $x^3-2x^2+20x=2$

$$\frac{1}{10}=x-\frac{1}{10}x^2+\frac{1}{20}x^3$$

$$z=\frac{1}{10};\ b_1=\frac{1}{10};\ b_2=-\frac{1}{20}$$

$$x=0{\cdot}1+0{\cdot}001-0{\cdot}00003-0{\cdot}000002=0{\cdot}100968.$$

(4) $x^4=17$

$$x=x_1+2$$

$$x_1^4+8x_1^3+24x_1^2+32x_1=1$$

$$\frac{1}{32}=x_1+\frac{3}{4}x_1^2+\frac{1}{4}x_1^3+\frac{1}{32}x_1^4$$

$$z=\frac{1}{32};\ b_1=-\frac{3}{4};\ b_2=-\frac{1}{4};\ b_3=-\frac{1}{32}.$$

$$x_1=0{\cdot}0312500-0{\cdot}0007324+0{\cdot}0000266-0{\cdot}0000011$$

$$=0{\cdot}030543 \qquad x=2{\cdot}030543.$$

Substituting the original letters y, a_0, a_1, a_2, ... , and arranging the terms with respect to the successive powers of the a's, the general equation is*—

$$x=\frac{y}{a_0}-\frac{a_1}{a_0}\frac{y^2}{a_0^2}+\left[2\frac{a_1^2}{a_0^2}-\frac{a_2}{a_0}\right]\frac{y^3}{a_0^3}+\left[-5\frac{a_1^3}{a_0^3}+5\frac{a_1}{a_0}\frac{a_2}{a_0}-\frac{a_3}{a_0}\right]\frac{y^4}{a_0^4}$$

$$+14\frac{a_1^4}{a_0^4}-21\frac{a_1^2}{a_0^2}\frac{a_2}{a_0}+6\frac{a_1}{a_0}\frac{a_3}{a_0}+3\frac{a_2^2}{a_0^2}-\frac{a_4}{a_0}\Bigg]\frac{y^5}{a_0^5}$$

$$+\Bigg[-42\frac{a_1^5}{a_0^5}+84\frac{a_1^3}{a_0^3}\frac{a_2}{a_0}-28\frac{a_1^2 a_3}{a_0^2 a_0}$$

$$+7\frac{a_1}{a_0}\left(-4\frac{a_2^2}{a_0^2}+\frac{a_4}{a_0}\right)+7\frac{a_2}{a_0}\frac{a_3}{a_0}-\frac{a_5}{a_0}\Bigg]\frac{y^6}{a_0^6}$$

$$+\Bigg[132\frac{a_1^6}{a_0^6}-330\frac{a_1^4}{a_0^4}\frac{a_2}{a_0}+120\frac{a_1^3}{a_0^3}\frac{a_3}{a_0}$$

$$+36\frac{a_1^2}{a_0^2}\left(5\frac{a_2^2}{a_0^2}-\frac{a_4}{a_0}\right)+8\frac{a_1}{a_0}\left(-9\frac{a_2}{a_0}\frac{a_3}{a_0}+\frac{a_5}{a_0}\right)$$

$$+4\frac{a_2}{a_0}\left(-3\frac{a_2^2}{a_0^2}+2\frac{a_4}{a_0}\right)+4\frac{a_3^2}{a_0^2}-\frac{a_6}{a_0}\Bigg]\frac{y^7}{a_0^7}$$

* See also "Inverse Interpolation by Means of a Reversed Series," this Journal, May 1908.

$$+\Big[-429\frac{a_1^7}{a_0^7}+1287\frac{a_1^5}{a_0^5}\frac{a_2}{a_0}-495\frac{a_1^4 a_3}{a_0^4 a_0}$$
$$+165\frac{a_1^3}{a_0^3}\Big(-6\frac{a_2^2}{a_0^2}+\frac{a_4}{a_0}\Big)+45\frac{a_1^2}{a_0^2}\Big(11\frac{a_2}{a_0}\frac{a_3}{a_0}-\frac{a_5}{a_0}\Big)$$
$$+\ 15\frac{a_1}{a_0}\Big(11\frac{a_2^3}{a_0^3}-3\frac{a_3^2}{a_0^2}\Big)+9\frac{a_1}{a_0}\Big(-10\frac{a_2}{a_0}\frac{a_4}{a_0}+\frac{a_6}{a_0}\Big)$$
$$+\ 9\frac{a_2}{a_0}\Big(-5\frac{a_2\,a_3}{a_0\,a_0}+\frac{a_5}{a_0}\Big)+9\frac{a_3}{a_0}\frac{a_4}{a_0}-\frac{a_7}{a_0}\Big]\frac{y^8}{a_0^8}$$
$$+\ \ldots\ldots$$

The last equation is quite sufficient for relations involving experimental data. In many other cases, however, such for example as the construction of mathematical tables, it is convenient to have the equation in the more extended form, and thereby save the labour of making many tedious transformations, or of devising new methods for the development of the inverse (or anti) functions. The second of the preceding problems is a case of this kind. Another important application consists in being able to estimate, without reversion, the error due to neglecting terms of a certain order in the reverse series. For example, the error due to neglecting the term in z^{12} in problem (1) is of the order of magnitude of $58786\times 12^{-11}\times(2y)^{12}$, or roughly $3\times 10^{-4}y^{12}$, while the same error for problem (3) is approximately $6\times 10^{-6}\times y^{12}$. Since in the first case y may exceed unity, the series will not always suffice for computation, but in the second case $y=0{\cdot}1$ and inspection shows that the remaining terms in the coefficient of z^{12} will not greatly exceed $58786\times b^{11}$, consequently the error arising from the omission of z^{12} in this series probably does not exceed 10^{-16}.

U.S. Geological Survey, Washington, D.C.
October 1909.

XXXVIII. *Friction in Gases at Low Pressures.* *By* J. L. Hogg *.

UNDER the title "Friction and Force due to Transpiration as dependent on Pressure in Gases," there was published † some time ago an account of some experiments made to determine the relation between the friction of a gas and the pressure in it, and also the relation between the

* Communicated by Professor Trowbridge.
† Proc. Am. Acad. pp. 42–46 (1906).