

Further considerations on the theory of the rotation of the principal planets and on the growth of the minor globes which have finally become satellites.

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In the paper on the origin of the lunar terrestrial system by capture, A. N. 4343 attention has been called to the fact that under the conditions existing in nature it is impossible for bodies of small mass to acquire rapid rotation. We shall now examine this question in more detail, so as to make clear the conditions which may lead to rapid rotation, and vice versa. If we adopt the Laplacian law of density (cf. A. N. 3992, Eq. 21), we have

$$\sigma = \sigma_0 \frac{\sin(qx)}{qx} = \frac{L \sin\left(q \frac{r}{a}\right)}{\frac{r}{a}} \quad (1)$$

where for the earth $\sigma_0 = 11.215$, water $= 1$; $q = 144^\circ 53' 55''.2 = 2.528959$ radians; $x = \frac{r}{a}$, a being the earth's radius, and r the radius of any shell; $L = 4.43463$; and we shall find for the mass

$$M = 4\pi \int_0^a \sigma r^2 dr = 4\pi a^3 L \int_0^a \frac{r}{a} \sin\left(q \frac{r}{a}\right) d\frac{r}{a} = \frac{4\pi a^3 L}{q^2} [\sin q - q \cos q]. \quad (2)$$

If I represents the earth's moment of inertia for the Laplacian law, when the figure is considered spherical, we shall have

$$I = \int r^2 \cos \Phi dm = \int_0^{2\pi} \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \int_0^a \sigma r^4 \cos^3 \Phi dr d\Phi d\theta. \quad (3)$$

If we use the law indicated in (1) for σ , we shall get

$$I = \int_0^{2\pi} \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \int_0^a \frac{L \sin\left(q \frac{r}{a}\right)}{\frac{r}{a}} r^4 \cos^3 \Phi dr d\Phi d\theta = \frac{8\pi a^5 L}{3q^4} (3(q^2 - 2) \sin q - q(q^2 - 6) \cos q). \quad (4)$$

Introducing the result indicated in equation (2), this becomes

$$I = \frac{2}{3q^2} \left(\frac{3(q^2 - 2) \sin q - q(q^2 - 6) \cos q}{\sin q - q \cos q} \right) M a^2. \quad (5)$$

In homogeneous spheres the value of I is found to be $0.4 M a^2$, but for Laplace's law as applied to the earth, the value is found to be less; namely,

$$I = 0.331278 M a^2. \quad (6)$$

We see therefore that the fraction 0.331278 is determined by the expression

$$k = \frac{2}{3q^2} \left(\frac{3(q^2 - 2) \sin q - q(q^2 - 6) \cos q}{\sin q - q \cos q} \right). \quad (7)$$

Now the moment of momentum of axial rotation H is the product of the moment of inertia by the angular velocity:

$$H = I \omega = k \omega M a^2. \quad (8)$$

And this equation enables us to recognize the different factors which enter into the expression for the moment of momentum of axial rotation. Let the mass and the mean radius be fixed; then it is evident that H will be large only when I and ω are large. Or if k also be fixed by Laplace's law, and taken to be 0.331278 in the case of the earth, then the value of H will depend simply on ω . Thus H will depend wholly on the impulse by which rotation is established and the angular velocity developed.

If k be not fixed, but variable in any manner, then, with constant mass and mean radius, H will depend on k

and ω conjointly. To make k a maximum the density has to be a maximum at the surface of the sphere; but this condition is dynamically unstable, and such arrangements neither arise in nature, nor would they long endure if started by artificial means. In the observed nebulae the density increases fairly rapidly toward the centre, and the same law obviously holds among the stars and planets (cf. A. N. 4053).

If the arrangement of the internal density followed the law for a monatomic gas, as outlined in A. N. 4053, and there applied to the sun, major planets and fixed stars, we should have

$$\sigma = \sigma_0 \left(\frac{1}{x^2} \frac{d\mu}{dx} \right) = \sigma_0 (1 - \alpha_1 x^2 + \alpha_2 x^4 - \alpha_3 x^6 + \alpha_4 x^8 \dots). \quad (9)$$

The coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \dots$ in this series are given in equation (5), A. N. 4053, p. 327-328. It is shown in

$$M \left(\frac{m_i r_i}{M + m_i} \right)^2 \sqrt{1 - e_i^2} \Omega_i + m_i \left(\frac{M r_i}{M + m_i} \right)^2 \sqrt{1 - e_i^2} \Omega_i = \frac{M m_i}{M + m_i} \sqrt{1 - e_i^2} r_i^2 \Omega_i. \quad (12)$$

And all the satellites revolving within the planet's control will contribute to the moment of momentum of axial rotation

$$\sum_{i=0}^{i=i} \left(\frac{M m_i}{M + m_i} \right) \sqrt{1 - e_i^2} r_i^2 \Omega_i = M \sum_{i=0}^{i=i} \left(\frac{m_i}{M + m_i} \right) \sqrt{1 - e_i^2} r_i^2 \Omega_i. \quad (13)$$

This expression is large when $\sum_{i=0}^{i=i} m_i$ is large compared to M ;

so that if the planet has a large swarm of satellites, with a total mass which is considerable, each moving at its appropriate distance r_i , and with angular velocity Ω_i , then the bringing of them all down upon the planet by the influence of the resisting medium will very materially augment the moment of momentum of axial rotation. The planet's central attraction at any distance r is increased as follows:

$$g(1 + \gamma) = \frac{M + \sum_{i=0}^{i=i} m_i}{r^2} = \frac{M}{r^2} \left(1 + \frac{\sum_{i=0}^{i=i} m_i}{M} \right) \quad (14)$$

or

$$\gamma = \frac{1}{g} \frac{\sum_{i=0}^{i=i} m_i}{r^2}.$$

But as the increase in the moment of momentum by (13) is more rapid than that of gravity, since each mass m_i is multiplied by $r_i^2 \Omega_i$, and therefore the larger $r_i^2 \Omega_i$ the larger the product $\sum_{i=0}^{i=i} m_i r_i^2 \Omega_i$ is, the eccentricity e_i

being so small as to be disregarded in this discussion, we perceive that the moment of momentum and the oblateness will frequently increase with the growth of the central mass. Therefore the larger planets, on the whole, have the most rapid rotations, and have thus been rendered quite oblate; while all the smaller planets, such as the Earth and Mars, have slower rotation and smaller oblateness. To produce

A. N. 4053, p. 342, that the radius of inertia of a monatomic sphere is 0.45; so that for a monatomic sphere

$$I = (0.45)^2 M a^2. \quad (10)$$

Accordingly for a monatomic globe we should have

$$H = (0.202483) \omega M a^2. \quad (11)$$

So that k is about one-fifth, and with this modification the above reasoning would still hold true.

Finally, in a nebula essentially devoid of hydrostatic pressure, the satellites, prior to their absorption into the principal planets, are revolving as free planetary bodies; and their moment of momentum of orbital motion is by collision added to that of the planet's rotation about its axis. Any satellite contributes an element of orbital moment of momentum given by the expression

large moment of momentum of axial rotation there must be a large central mass, so as to give a strong central force,

and a large amount of matter $\sum_{i=0}^{i=i} m_i$ added to the planet from the vortex circulating about it. This gives large moment of momentum about the axis of rotation.

The expression for the moment of momentum of axial rotation is $H = \omega \cdot k M a^2 = \omega I$; so that the increase of mass by the addition of satellites affects M , a , and k , as well as ω . But relatively the largest change is in ω ; for as the mass grows the central attraction grows in proportion, but the added moment of momentum is

$$\sum_{i=0}^{i=i} m_i r_i^2 \Omega_i,$$

and thus augmented by the factors depending on the increased radius and angular velocity, which are themselves enlarged by the increase of the central mass.

These are the general conditions of the problem, without regard to how the vortex is started about the planet; but it may be noticed also that in small bodies the hour-glass shaped space connecting with the sun's sphere of control is so small and narrow that but few particles enter it, and what few do enter will experience a more nearly equal division between retrograde and direct revolutions about the planet. It thus appears that on the whole the larger masses have a tendency to augment the moment of momentum of axial rotation and oblateness, which is greater than in small masses.

The planet Saturn has a decidedly larger closed space about it than Jupiter, because although of smaller mass it

is at greater distance; and hence as measured by the resulting oblateness, has effectively the most rapid rotation. But this also depends on the density, and the conditions for giving large oblateness in the case of Saturn are most favorable. Uranus and Neptune also have large closed spaces about them, but their masses are smaller, the density greater, and the resulting oblateness is therefore no doubt smaller than in the case of Saturn. The oblateness of Uranus, however, appears to exceed that of Jupiter (cf. A. N. 3992). It is doubtless true that some matter comes under the control of the planets without passing through the neck of the hour-glass space defined by the surfaces of zero velocity extending around the sun; but the amount thus gathered from miscellaneous sources probably is small, and not an important element in the theory of satellites and of planetary rotation.

From the considerations already adduced in the paper on the Dynamical Theory of Satellites, it is evident that detachment of masses by rapid rotation, if it occurs at all, must be exceedingly difficult to bring about. If the descending stream of matter was supplied to the rotating spheroid in a certain way, which, however, will seldom arise in nature, because it would all have to be directed against the periphery of the rotating mass, so as to give maximum angular velocity of rotation, a process of partial fission might in the course of time develop. Yet even if this augmentation of velocity should come about, it is more than probable that the matter subsequently detached, by acceleration of rotation, would be in the form of a swarm of particles, and would have great difficulty in collecting together into one mass. All the well-known objections to Laplace's theory of ring formation and condensation could be urged here with full effect. If the separation was in the form of a lump or nucleus, it might survive, provided the action of the resisting medium did not again bring about its precipitation upon the central mass, which, however, would be almost certain to follow. It is evident, therefore, that, while the separation of masses by accelerated rotation is not impossible, this process requires such very special conditions that it seldom takes place in Nature.

If we consider, for example, the case of the Earth and Moon, where the primordial central mass would have had to acquire an enormously rapid rotation, in less than $2^h 50^m$, it will become evident that there are in Nature no forces which could produce such very rapid rotation. That is, there is no regular process at work, which could produce such an effect. A grazing collision of two already existing globes, if properly aimed with suitable velocities, might give rise to one common mass spinning so rapidly that scattered portions of it would be detached, and after separation circulate around the residual central mass. But collisions of solid globes are so rare, owing to their very small size compared to the large vacant spaces in which they move, and so nearly impossible to effect under the conditions ordinarily existing in cosmical systems dominated by central forces, that this extreme hypothesis has no interest. More-

over, even if the primordial mass were disrupted in this way, the scattered fragments could never get together to form a single globe like the moon.

It appears that the only place in which such globes can be formed is in the midst of a diffused nebula, where the nuclei are not disrupted or prevented from growing by the strong attractive forces of neighboring masses. The absence of great attractive centres allows the smaller nuclei to grow, both because their spheres of influence are more extended than when near large masses, and because the large masses have not yet swallowed up all of the surrounding nebulosity; so that the smaller masses have a good chance to grow by accretion.

It is very evident that the origin of the planets and satellites dates back to the earliest nebular stage, and that the embryo of a body such as our Moon was at one time on the very outskirts of the system, where Neptune now revolves, or even beyond. Matter equivalent to 27 million such globes was swallowed up in laying the foundation of the sun; and the fact that moons or planets of rather small size have been captured by all the principal planets from the Earth to Neptune, indicates how widely diffused such globes were in the condensing nebula which originally formed the solar system. These globes have grown somewhat in later times, by the gathering up of cosmical dust, but their main growth was attained in the nebular stage of the system, which has now quite disappeared.

These considerations afford us some conception of the immeasurable ages which have elapsed since the foundations of the solar system were laid in a whirlpool nebula. The total duration of time involved is certainly to be reckoned in billions of years. The extent of the system has grown less with the lapse of ages, and the orbits have grown rounder as well as smaller. And since there are good reasons to believe that even now unseen planets will be found to circulate at least three times as far away as Neptune, we see how vast must have been the extent of the solar nebula when that primordial cosmical vortex was just starting. It may easily have extended to 1000 times the distance of the Earth from the Sun.

Considerations of this kind explain the great length of time involved in the development of cosmical systems. For in such a tenuous nebula the process of transformation is slow, because the resistance of the diffuse nebulosity is slight, and a dominant central sun has not yet developed. This line of thought also enables us to understand the vast extent of the spiral nebulae, and the insensible character of their rotatory movements. Unless the central mass is enormous, these gigantic cosmical vortices must necessarily revolve with extreme slowness. Therefore it is not probable that motion can be detected in less than centuries, and in many instances the period required to disclose a whirling movement is more likely to be reckoned in thousands of years.