Singular Solutions of Differential Equations of the Second Order
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SINGULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

An account of singular solutions of differential equations of the first order is given in every text-book on differential equations. The theory of singular solutions of differential equations of higher order is not usually given in elementary treatises. It is discussed by Goursat (*American Journal of Mathematics*, xi., 1889, pp. 329-372), who bases his discussion on the theory of singular solutions of simultaneous equations of the first order. From the didactic point of view a more elementary treatment may be acceptable, and an attempt is here made to indicate how singular solutions of a given equation of the second order may be found by simple means.

As in the case of equations of the first order we may deduce the singular solutions from the given differential equation $\phi(x, y, y_1, y_2)=0$, or from its complete primitive $f(x, y, a, b)=0$. (In this $y_1 \equiv \dfrac{dy}{dx}$, $y_2 \equiv \dfrac{d^2y}{dx^2}$, and a, b are arbitrary constants.)

It will be noted that a singular solution contains one arbitrary constant in general.

(I.) *A singular solution, if any, may be found as follows:*
Eliminate x and y from

$$\left.\begin{array}{c} f(x, y, a, b)=0, \quad \dfrac{\partial f}{\partial a}da+\dfrac{\partial f}{\partial b}db=0, \\ \left(\dfrac{\partial f}{\partial x}\cdot\dfrac{\partial^2 f}{\partial y\,\partial a}-\dfrac{\partial f}{\partial y}\cdot\dfrac{\partial^2 f}{\partial x\,\partial a}\right)da+\left(\dfrac{\partial f}{\partial x}\,\dfrac{\partial^2 f}{\partial y\,\partial b}-\dfrac{\partial f}{\partial y}\cdot\dfrac{\partial^2 f}{\partial x\,\partial b}\right)db=0; \end{array}\right\}\text{.........(i)}$$

integrate the resulting differential equation of the first order in a *and* b, *thus expressing* b *as a function of* a *and an arbitrary constant; then the envelope of the singly infinite family of curves $f(x, y, a, b)=0$, now obtained, includes the singular solutions of $\phi(x, y, y_1, y_2)=0$.*

Suppose a singular solution represented by a curve c. Then through any point P of c an infinite number of curves of the family $f(x, y, a, b)=0$ passes. Choose b so that $f(x, y, a, b)=0$ touches c at P. If this is done for every point P of c, we express b as a function of a, and $f(x, y, a, b)=0$ becomes a singly infinite family whose envelope is c. Now at P, $f(x, y, a, b)=0$ and c have the same x, y, and y_1. But they both satisfy $\phi(x, y, y_1, y_2)=0$. Hence they have the same y_2 also, and therefore the singly infinite family $f(x, y, a, b)=0$ osculates its envelope at each point of contact.

Now, if $F(x, y, t)=0$ osculates its envelope at P, we readily prove that $F=0$ and $\dfrac{\partial F}{\partial t}=0$ touch at P, *i.e.*

$$F=0, \quad \frac{\partial F}{\partial t}=0, \quad \frac{\partial F}{\partial x}\cdot\frac{\partial^2 F}{\partial y\,\partial t}=\frac{\partial F}{\partial y}\cdot\frac{\partial^2 F}{\partial x\,\partial t}$$

at P. Applying this to $f(x, y, a, b)=0$, where b is a function of a, we have the result stated above.

A case of practical value is that in which $f(x, y, a, b)\equiv F(x+\xi, y+\eta, \zeta)$, where ζ, η, ζ are functions of a and b, and F is a homogeneous function of

$$x+\xi, \ y+\eta, \ \zeta.$$

The equation in a, b is then $F(d\xi, d\eta, d\zeta)=0$, (where, of course,

$$d\xi=\frac{\partial \xi}{\partial a}da+\frac{\partial \xi}{\partial b}db, \text{ etc.}).$$

For, if $F_1 \equiv \frac{\partial F}{\partial(x+\xi)}$, $F_{23} \equiv \frac{\partial^2 F}{\partial(y+\eta)\partial\zeta}$, etc., we have to eliminate x, y from

$$F=0, \quad F_1 d\xi + F_2 d\eta + F_3 d\zeta = 0,$$

and
$$F_1(F_{21}d\xi + F_{22}d\eta + F_{23}d\zeta) = F_2(F_{11}d\xi + F_{12}d\eta + F_{13}d\zeta);$$

and by Euler's theorem on homogeneous functions the last two equations give $(x+\xi):(y+\eta):\zeta = d\xi : d\eta : d\zeta$; whence $F(x+\xi,\, y+\eta,\, \zeta)=0$ gives

$$F(d\xi,\, d\eta,\, d\zeta)=0.$$

(II.) *A singular solution, if any, may be found as follows:*
Eliminate y_2 *from*

$$\phi(x, y, y_1, y_2)=0, \quad \frac{\partial\phi}{\partial y_2}=0. \quad \text{(ii)}$$

Then the integral of the resulting differential equation of the first order includes the singular solution.

Suppose that in (I.) the curve c osculates two consecutive curves 1 and 2 of the singly infinite family $f(x, y, a, b)=0$ at Q and R.

Suppose that 1 and 2 meet at P, which is consecutive to Q and R. The curvature of 1 at P is in the limit equal to its curvature at Q, *i.e.* to the curvature of c at Q, *i.e.* in the limit to the curvature of c at R, *i.e.* to the curvature of 2 at R, *i.e.* in the limit to the curvature of 2 at P.

Suppose, to fix our ideas, that $\phi(x, y, y_1, y_2)=0$ is an algebraic equation of degree n in y_2. Then, in general, for given x, y, y_1 we get n different values for y_2. But we have just shown that if x, y, y_1 have the values given by the curve c at P, two of the n values of y_2 coincide; and hence $\frac{\partial\phi}{\partial y_2}=0$.

Of course, any solution found by the methods (I.) or (II.) must be tested to see whether it really satisfies the differential equation; just as in the case of differential equations of the first order.

For instance, in (II.) we get, besides the singular solution, the locus of a point on $f(x, y, a, b)=0$, at which two branches of this curve touch and have the same curvature there ("cusp of the second species"). The reader will find a discussion of this point in Goursat, *loc. cit.*

Ex. 1. $y_2^2 = 1 - y_1^2$ with complete primitive $y + \cos(x-a) - b = 0$.

Method I.—Equations (i) give, on eliminating x and y, $db = \pm da$, or $b = \pm a + K$. The required singular solution, if any, is therefore the envelope of $y + \cos(x-a) \mp a - K = 0$, *i.e.* $y = \pm x + k$; which is found on testing to be really a singular solution.

Method II.—Equations (ii) give $y_1 = \pm 1$, leading to $y = \pm x + k$, as before.

Ex. 2. $y_1 = y_2^2 + xy_2$ with complete primitive $y = ax^2 + 4a^2x + b$.

Method I.—Equations (i) give, on eliminating x and y, $db = 16a^2 da$, or $b = \frac{16}{3}a^3 + K$. The singular solution is the envelope of $y = ax^2 + 4a^2x + \frac{16}{3}a^3 + K$ or $y = -\frac{1}{12}x^3 + k$.

Method II.—Equations (ii) give, on eliminating y_2, $y_1 = -\frac{1}{4}x^2$ or

$$y = -\tfrac{1}{12}x^3 + k,$$

as before.

Ex. 3. Find the singular solutions, if any, of

$$cy_2^2 = (1+y_1^2)^3, \quad c^2y_2^2 + y_1^2 = 2yy_2,$$
$$y^2(yy_1 - xyy_2 + xy_1^2) = (yy_2 - y_1^2)^2.$$

Ex. 4. Find the singular solutions of the differential equation whose complete primitive is

$$x^2 + y^2 - 2a(2+3b^2)x + 4ab^3y - 3a^2b^4 = 0.$$

We may write this equation $(x+\xi)^2+(y+\eta)^2-\zeta^2=0$, where

$$\xi=-a(2+3b^2),\quad \eta=2ab^3,\quad \zeta=2a(1+b^2)^{\frac{3}{2}};$$

which is a homogeneous equation in $x+\xi$, $y+\eta$, ζ.

The required singular solution is therefore included in the envelope of the given circles when a and b are connected by the relation

$$(bda^2+4adadb)b^3=0,$$

$$d\xi^2+d\eta^2-d\zeta^2=0 \quad\text{or}\quad (b\,da^2+4a\,da\,db)b^3=0,$$

i.e. $a=$constant, or $ab^4=$constant, or $b=0$.

The envelope in the former case is $y^2=4ax$, as is geometrically evident, for the given circle osculates this parabola at $(ab^2, 2ab)$.

Ex. 5. Find the singular solutions of the differential equation whose complete primitive is

$$b^3(x^2+y^2)-(3b^4+1)ax-(b^4+3)ab^2y+3(b^4+1)a^2b=0,$$

$$xy-a\sin b(\sin^2 b+3\cos^2 b)x-a\cos b(3\sin^2 b+\cos^2 b)y+3a^2\sin b\cos b=0,$$

$$y^2+2a\cos^3 b\,x-2a\sin^3 b\,y+a^2(\sin^2 b-2\cos^2 b)=0.$$

The reader will readily apply similar methods to the finding of singular solutions of differential equations of higher orders. H. Hilton.

BERKELEY AND NEWTON.

In his interesting article on Newton in the July *Gazette* Dr. Rouse Ball had space only for a slight reference to Newton's views on what we should now call the Foundations of the Differential Calculus. The subject "involves," as Dr. Rouse Ball has remarked, "some awkward questions of philosophy which before Weierstrass's researches were usually slurred over." Some of the logical difficulties incident to Newton's method of approach were clearly seen and pointed out by a contemporary, the celebrated Bishop Berkeley, in a tract which drew much attention at the time, and was followed by a considerable controversy.

George Berkeley, who was born in 1685, when Newton was writing the *Principia*, was appointed to the bishopric of Cloyne in March, 1734, and the same month saw the publication of his tract *The Analyst.* Newton had died, full of years and honour, seven years before, and the long period of his complete dominance over British mathematicians had already begun. It is therefore not surprising that the *Analyst*, which attacked the very foundations of the great doctrine of Fluxions in the most unceremonious manner, brought down a mathematical storm on its author's head.

The scope of Berkeley's tract may be judged from its full title, which runs as follows: "The Analyst; or, a discourse addressed to an infidel Mathematician. Wherein it is examined whether the object, principles, and inferences of the modern analysis are more distinctly conceived, or more evidently deduced, than religious mysteries and points of Faith. 'First cast the beam out of thine own eye; and then shalt thou see clearly to cast out the mote out of thy brother's eye.'" Although no names were mentioned, it is supposed that Halley was the "infidel Mathematician" glanced at. Of the apologetic aspect of the work we need only say that the piquant flank movement thus foreshadowed was a task well adapted to Berkeley's genius, and that he did not fail to make full use of each point maintained. His purely mathematical arguments are, however, so interesting and so little known in detail that no apology need be prefixed to a brief account of them.

We have at the beginning a clear and fair account of Newton's doctrine of Fluxions. "Lines are supposed to be generated by the motion of points, planes by the motion of lines, and solids by the motion of planes. And whereas