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Source: *The Mathematical Gazette*, Vol. 9, No. 128 (Mar., 1917), pp. 43-45

Published by: [Mathematical Association](#)

Stable URL: <http://www.jstor.org/stable/3605226>

Accessed: 01-03-2016 04:11 UTC

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EUCLID'S DEFINITION OF PARALLEL STRAIGHT LINES CONSIDERED IN REFERENCE TO THE "LINE AT INFINITY."

By J. L. S. HATTON, M.A.

EUCLID'S definition of two parallel straight lines seems to be the correct one for the absolute ideal of a plane and of a straight line. The conception of a line at infinity, it would seem, should be stated if used in some such words as these. Points may be supposed to exist at such a distance from the points generally considered in a plane that their distances from all such points in the plane may be regarded as equal, *i.e.* the differences of these distances may in general be regarded as negligible. These points, as a first approximation, may, under certain circumstances, be regarded, when considered in relationship to points at a finite distance, as lying on a straight line. Parallel straight lines may be regarded—for most practical purposes—as far as points at a finite distance are concerned, as intersecting at one of these points.

On the other hand, it would seem that the so-called line at infinity is not a straight line nor even a curve. It is a region of the plane or surface considered, which is so distant from the finite region of the plane that, in reference to points in that region, it may be regarded as a straight line or other curve.

The geometrical conception of "the point of infinity on a straight line" may be regarded as based upon the idea that, if A and B are two points at a finite distance apart on a straight line, I is the (or a) point at infinity on the line, if it is on the line and is such that IA may be regarded as equal to IB . If this is the case, a point I' may be taken at a finite distance a from I . Then, if $IA = IB$, it may be assumed that $IA + a = IB + a$. Hence the point at infinity on a straight line is not a single point on the line but an assemblage of points which, on account of their distance and a common property, may for certain purposes be regarded as a single point. The best physical representation of a mathematical point at an infinite distance is a fixed star, say the Polar Star. Yet this is an assemblage of points which is greater than the whole system of points on the earth which it is customary to study.

The geometrical conception of the line at infinity is based on the fact that every straight line at a finite distance in one plane σ , is projected from a point into a line on another plane σ' . In the latter plane σ' there is one straight line v' , the vanishing line, which is not projected into a line at a finite distance by the reverse process in the first plane σ . This line is projected into a curve or region in the first plane σ , which, by analogy, is termed the line at infinity in that plane. By analogy it is doubtless convenient to call this locus or portion of space the line at infinity, and for points at a finite distance this region of space has generally the property of a straight line.

There are two ways of reconciling Euclid's definition of parallel straight lines with the conception of the line at infinity.

It is possible to adhere to Euclid's definition of two parallel straight lines in which case it must be assumed that a pair of points can be found one on each of two parallel straight lines situated at such a distance from the finite portion of the plane considered, that with reference to this portion of the plane they may be regarded as one and the same point. In fact, the distance between these two points is infinitesimal in regard to their distance from points in the finite portion of the plane.

On the other hand, the lines considered may be regarded as not parallel but as indefinitely approaching Euclid's definition of parallelism. Such lines may be regarded as intersecting in a point complying with the conception of the point or points at infinity on a straight line.

Looked at from the point of view of projection, it is seen that points on the vanishing line v' , or infinitely near it in the plane σ' , are projected into points

at an infinite distance in the plane σ . The fact that they are projected to an infinite distance enables those points which are infinitely near in the plane σ' to be projected into points at a finite distance apart in the plane σ . In fact, they form a system of points whose natures and properties are only those of a straight line, because they are so far away. The nature and arrangement of these points depend on quantities and assumptions which, in the case of points at a finite distance, do not require consideration. In fact, by the projection of these points to infinity, small quantities and considerations which could be neglected in the case of points at a finite distance have been changed into finite quantities and considerations of importance.

If parallel straight lines are not regarded as absolutely complying with Euclid's definition of parallelism, but as being straight lines which in the limit approached to Euclid's definition of parallelism—just as we speak of quantities in the calculus which approached the limit zero—and assume that such lines meet at a point at infinity, it is necessary to consider the conditions under which the lines approach to this limit and certain small quantities become important.

Our plane must no longer be regarded as simply a plane. It is something infinitely approaching a plane. Its equation may be written

$$0 = k + ax + by + cz + a'x^2 + b'y^2 + c'z^2 + 2gxz + 2fyz + 2hxy + a''x^3 + b''y^3 + \dots,$$

where $a'b' \dots a''b'' \dots$ are infinitely small compared with k, a, b, c ; in fact, so small that they can be disregarded when points at a finite distance are considered and are only of importance in connexion with points at an infinite distance. These quantities, however, determine the nature of the curve on which the points at infinity may be regarded as being situated.

According to the ordinary conception of the line at infinity, all systems of straight lines which intersect at a point on the line at infinity form parallel systems of straight lines. Since, according to the usual assumption, the line at infinity passes through all these points, it is a line of each of these systems of parallel lines. Therefore all straight lines are parallel to the line at infinity. This paradox holds as long as the line at infinity is regarded as a straight line of the same nature as a straight line at a finite distance. If, however, the line at infinity is regarded as a region of space, as it should be, this paradox disappears. All straight lines at a finite distance, which intersect in a point at infinity, are parallel. Through this point at infinity, different straight lines at infinity can be drawn, but these lines do not necessarily belong to the system of parallel lines, for the essence of parallelism is that the point of intersection, or of approximate intersection, is at an infinite distance from the region of the plane considered. Hence, it follows that the line of infinity has no direction. It is a region of the plane in which lines can be drawn in any direction. There are also points in it which may or may not lie on given straight lines which are entirely at infinity.

Consider the straight line whose equation is

$$\frac{x}{ak} + \frac{y}{bk} = 1, \dots\dots\dots(1)$$

and the points whose coordinates are hk and lk . $\dots\dots\dots(2)$

If k be supposed to become infinitely large, the line becomes a line at infinity, and the point a point at infinity.

The general condition that the point (2) should be on the straight line (1) is

$$\frac{hk}{ak} + \frac{lk}{bk} = 1$$

or
$$\frac{h}{a} + \frac{l}{b} = 1.$$

This is the condition that the particular point at infinity (2) should lie on the particular line at infinity (1). For points at a finite distance, the point (2)

may be regarded as lying on the line (1) when k is infinite, because they are both so distant from the finite portion of the plane.

The equation of a line at infinity should not be written as $0 \cdot x + 0 \cdot y + c = 0$, but as $akx + bky + c = 0$, where k is infinitely small. Thus the equation of a line at infinity is

$$ax + by + \frac{c}{k} = 0.$$

Hence a line at infinity may have any direction according to the ratio of a to b . By giving different values to this ratio, the lines at infinity which are parallel to a given system of parallel lines at a finite distance are obtained.

Our conception of a parabola, which is usually supposed to touch the line at infinity, must be modified. A parabola is a curve which enters and leaves the region at infinity at two points which, with regard to points at a finite distance, may be regarded as coincident; although they may be, and usually are, at a finite distance apart. Hence the inconsistency of supposing that two parabolas whose axes are at right angles touch the same straight line at infinity is done away with.

JOHN L. S. HATTON.

Obituary.

CHARLES SAMUEL JACKSON.

CHARLES SAMUEL JACKSON was born at Winchester on the 6th of December, 1867. His father, George Jackson, was a Yorkshireman, and is described as having made his way, without initial advantages, by sheer force of character and great natural ability. His mother is still alive. Charlie, or, to use the "title" rather than "nickname" of later years, "Slide-Rule Jackson," was one of a family of three. It is probable that from both parents he inherited the gifts and attributes which won him distinction, and the affection of all who knew him.

The boy was sent to a Preparatory School at Worthing, and from thence to Uppingham. On the death of George Jackson the family moved to Bedford, Charlie entering the Grammar School in 1881. He rose to be Captain of the School, won a leaving Exhibition, and a Scholarship at Trinity College, Cambridge. He was Eighth Wrangler in 1889, and obtained in the next year a First Class in the Law Tripos.

Of the teachers at Bedford who had exercised the greatest influence upon his intellectual development he used more particularly to refer to W. S. Phillips and E. B. Buck. His frequent letters from Cambridge to the latter were fairly divided between mathematical problems and subtle questions of Law.

For some time he hovered between these two attractions. He read in chambers with Chancellor (now Sir Lewis) Dibdin, K.C., devilling for him during periods of special pressure. But his interest in Mathematics was not allowed to suffer eclipse after he was called to the Bar.

Two Cambridge men write :

"Jackson as an undergraduate was very much the same as we knew him in later life. He thoroughly enjoyed Mathematics and worked hard, but at the same time entered into and appreciated life at college and its opportunities of forming friendships. He and I attended the same lectures in college and were private pupils of Mr. Webb's, of St. John's College, the class consisting of about half-a-dozen people who came out high in their Tripos. One day Webb told us, not altogether seriously, that we were enjoying the May term and not doing enough