ON THE RECIPROCITY FORMULA FOR THE GAUSS'S SUMS IN THE QUADRATIC FIELD

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SOME account of the Gauss's sums, that is series of the type

$$G\left(\frac{a}{b}\right) = \sum_{s=0}^{|b|-1} e^{\pi i a s^2/b}, \tag{1}$$

where a and b are integers, not necessarily positive, is to be found in text books on the theory of numbers as part of the fundamental elements of the subject. It is well* known that (if b or a is even, or if the summation for s extends to $2|b|-1$)

$$G\left(\frac{a}{b}\right) = \left(\frac{bi}{a}\right)^{\frac{1}{2}} G\left(-\frac{b}{a}\right), \tag{2}$$

where the radical is taken with a positive real part, and that this formula contains implicitly not only the sum of the series (1) but also the ordinary law of quadratic reciprocity.

2. The last twenty-five years, however, have seen the laws of not only quadratic reciprocity, but also of l-ic reciprocity (where l is any prime) for the general algebraic field, investigated with complete success by Hilbert and Furtwängler, in a series of memoirs of the greatest importance in the advancement of mathematical knowledge.† The latter writer, de-

* See Bachmann, *Zahlentheorie*, Vol. 3, p. 160.

† For an interesting résumé of the subject and references, see the paper by Fueter "Die Klassenkörper der komplexen Multiplikation und ihr Einfluss auf die Entwicklung der Zahlentheorie," *Jahresberichte der Deutschen Mathematiker-Vereinigung*, Vol. 20 (1911). Hilbert's chief papers are his well known "Bericht über die Theorie der algeb. Zahlkörper," of which there is a French translation published by A. Hermann, "Über die Theorie des relativ quadratischen Zahlkörpers," *Math. Annalen*, Vol. 51 (1898), and "Über die Theorie der relativ-Abelschen Zahlkörper," *Gött. Nachr.*, 1898, or *Acta Math.*, Vol. 26. Furtwängler's chief papers are in the *Math. Annalen*, Vols. 58, 63, 67, 72, 74, and in the *Gött. Nachr.*, 1911.

veloping and extending the ideas initiated by Hilbert, proved the general law for any algebraic field about ten years ago.

Under these circumstances, it seems rather surprising that the Gauss's sums were not also generalized for an algebraic field at the same time. This, however, was done as follows, only in the last few years, by Prof. Hecke, in a very interesting paper,* reminding us what vast mathematical treasures are still at hand if we could only find them.

Let K be the quadratic field of discriminant $-d$, and suppose $\sqrt{d}$ is taken with a plus sign if d is positive, and with a positive imaginary part if d is negative. If Ω is any number in K, we write

$$S(\Omega) = \Omega+\Omega',$$

where Ω' is the conjugate of Ω. It is easily seen then that $S(\omega/\sqrt{d})$ is a rational integer if ω is an integer in K. If, however, ω is fractional, we remove the common ideal factors from its numerator and denominator, put

$$\omega = A/B,$$

and refer to the ideal B as the denominator of ω. It is now clear that if b is any rational integer divisible by the ideal B, then $bS(\omega/\sqrt{d})$ is an integer. Hence if k is any integer in K, $e^{2\pi i S(\omega k/\sqrt{d})}$ is a b-th root of unity depending upon the residue of k (mod B).

The Gauss's sum for the quadratic field is then defined by

$$G(\omega) = \sum_{\rho} e^{2\pi i S(\rho^2\omega/\sqrt{d})},$$

where ρ takes all the values of any complete set of residues (mod B). Prof. Hecke then proves a number of results very similar to those for the ordinary Gauss's sum, and in particular that

$$G(\omega k) = \left(\frac{k}{B}\right) G(\omega), \tag{3}$$

if the ideal B is prime to 2, and k is an integer in K prime to B and where $\left(\frac{k}{B}\right)$ is the symbol of quadratic reciprocity in the quadratic field K.

* "Reziprozitätsgesetz und Gauss'sche Summen in quadratischen Zahlkörpern," *Gött. Nachr.*, 1919.

All this, of course, applies to the general algebraic field; and it is all the more surprising that it has not been discovered sooner, when we note that sums involving an exponent similar to $S(\omega)$ had already been considered by Stickleberger* in a paper of exceptional beauty, wherein he generalized some results of Eisenstein, who had proved† some formulæ such as: if p is a prime of the form $7n+2$, then from

$$x^2+7y^2 = p = 7n+2$$

we have
$$x \equiv \tfrac{1}{2}\frac{(3n)!}{n!\,(2n)!} \pmod p, \quad x \equiv 3 \pmod 7.$$

Having defined the Gauss's sum, Prof. Hecke, who had previously discovered a method of associating a theta function‡ with an ideal, an idea through which he has already considerably enriched mathematics, deduced in the case of a real quadratic field, from the transformation formula for the theta function with two variables, the formula

$$G\left(\frac{b}{a}\right) = e^{\frac{1}{4}\pi i(\operatorname{sgn} ab - \operatorname{sgn} a'b')}\, 2\left|\frac{bb_1}{aa_1}\right|^{\frac{1}{2}} \frac{N(A)}{N(B_1)}\, G\left(-\frac{a}{4b}\right), \tag{4}$$

where A is the denominator of b/a, and B_1 the denominator of $-a/4b$. Also $N(A)$ is the norm of the ideal A, while $\operatorname{sgn} ab = \pm 1$ according as ab is positive or negative. He then applies this formula to the proof of the law of quadratic reciprocity in the real quadratic field K.

In a recent paper,§ I gave a very simple method for summing the series (1) in the particular case when $a = 2$. The same method, however, applied to the general series (1) gives at once the reciprocity formula (2), as I noticed when writing that paper, though I did not mention it at the time. In reading Prof. Hecke's paper, I saw at once that my method gives immediately the reciprocity formula for any quadratic field, real or imaginary. This I shall now prove.

Let a function $f(z)$ be defined by

$$(e^{2\pi i\mu z}-1)f(z) = \sum_{\xi,\,\eta} \exp \pi i S\left[(z+\rho)^2\,\omega/\sqrt{d}\right],$$

* "Ueber eine Verallgemeinerung der Kreisteilung," *Math. Annalen*, Vol. 37 (1890).

† *Crelle's Journal*, Vol. 37, or H. J. S. Smith, *Collected Works*, Vol. 1, p. 280.

‡ It seems difficult to realize that as long ago as 1845 Hermite, in his first letter to Jacobi (Hermite, *Œuvres*, t. 1, p. 100), gave a method for associating a definite quadratic form with an algebraic number.

§ "On a Simple Summation of the Series $\sum_{s=0}^{n-1} e^{2s^2\pi i/n}$," *Messenger of Mathematics*, Vol. 48 (1918).

where $\mu = \pm 1$ will be fixed later, and

$$\rho = \xi + \eta\theta,$$

where the numbers (1, θ) form the base of the quadratic field K, so that

$$\theta = \tfrac{1}{2}\sqrt{d} \qquad \text{if} \quad d \equiv 0 \pmod 4,$$

or

$$\theta = \tfrac{1}{2}(-1+\sqrt{d}) \quad \text{if} \quad d \equiv 1 \pmod 4.$$

Also

$$S[(z+\rho)^2\,\Omega] = (z+\rho)^2\,\Omega + (z+\rho_1)^2\,\Omega_1,$$

where ρ_1, Ω_1 are the conjugates of ρ, Ω respectively.

The summation is extended to the values

$$\left.\begin{array}{l} \xi = 0,\ 1,\ 2,\ \dots,\ 2\tau\,|\,bb_1\,|-1 \\ \eta = 0,\ 1,\ 2,\ \dots,\ 2\tau M-1 \end{array}\right\}, \tag{5}$$

where

$$M = \left|\frac{ab_1 - a_1 b}{\sqrt{d}}\right|,$$

b_1 is the conjugate of b, a_1 the conjugate of a, and $\omega = a/b$, $\tau = |\,aa_1bb_1\,|$.

Consider now the integral

$$\int f(z)\,dz$$

taken around the parallelogram $ABCD$ where the parallel sides AD, BC cut the real axis of z at $z = -\tfrac{1}{2}$, $z = \tfrac{1}{2}$, respectively, and are inclined to its positive direction at an acute or obtuse angle, according as $S(\omega/\sqrt{d})$ is positive or negative. The sides DC and AB respectively are at an infinite distance above and below the real axis. The integral around the sides AB, DC obviously* vanishes, since if we put $z = x+iy$,

$$|\,e^{\pi i z^2 S(\omega/\sqrt{d})}\,| = e^{-2\pi xy\, S(\omega/\sqrt{d})},$$

and the direction of the sides DA, BC is such that $xy\, S(\omega/\sqrt{d})$ is positive.* The only singularity of the integrand is a simple pole at $z = 0$. Hence,

* Provided that ω is not rational, for then $S(\omega/\sqrt{d}) = 0$. The results of the paper are trivial in this case.

by Cauchy's theorem,

$$\int_A^D [f(z+1)-f(z)]\,dz = \mu \sum_{\xi,\eta} \exp \pi i\, S(\rho^2\omega/\sqrt{d}). \tag{6}$$

We now take the standard expression for the Gauss's sum in a form slightly different from that used by Prof. Hecke, and write

$$G\left(\frac{a}{b}\right) = \sum_{\rho} \exp \pi i\, S(\rho^2\omega/\sqrt{d}),$$

where ρ runs through a complete set of residues (mod B), and B is the denominator of $\omega/2 = a/2b$. Hence the right-hand side of (6), when we adopt the limits of summation given by (5), can be written as

$$\frac{\tau^2 4\mu M\,|\,bb_1\,|}{N(B)}\, G\left(\frac{a}{b}\right). \tag{6a}$$

The success of my method depends upon the fact that $f(z+1)-f(z)$ is an integral function of z, really a sum of exponentials of the form $\exp(mz^2+nz)$. Hence as the path of integration can be deformed into the real axis of z from either $-\infty$ to ∞ or ∞ to $-\infty$ according as

$$S\left(\frac{\omega}{\sqrt{d}}\right) = \frac{ab_1-a_1b}{bb_1\sqrt{d}}$$

is positive or negative, we can evaluate the left-hand side of (6) which then becomes, except for unimportant factors, a sum which is symmetrical in a/b and $-b/a$.

For we have

$$(e^{2\pi i\mu z}-1)[f(z+1)-f(z)] = \sum_{\eta} \exp \pi i [S(z+2\tau\,|\,bb_1\,|+\eta\theta)^2\,\omega/\sqrt{d}] - \sum_{\eta} \exp[\pi i S(z+\eta\theta)^2\,\omega/\sqrt{d}],$$

where the summation refers to $\eta = 0, 1, \ldots, 2\tau M-1$. The general term on the right-hand side is the product of two factors of which the first is

$$\exp[\pi i S(z+\eta\theta)^2\,\omega/\sqrt{d}],$$

while the second is

$$-1+\exp \pi i\, S[4\tau\,|\,bb_1\,|\,(z+\eta\theta)\,\omega/\sqrt{d}+4\tau^2b^2b_1^2\omega/\sqrt{d}].$$

But $$S(4\,|\,bb_1\,|\,\theta\omega/\sqrt{d}) \quad \text{and} \quad S(4b^2b_1^2\,\omega/\sqrt{d})$$

are even integers. Also

$$S(4\tau\,|\,bb_1\,|\,\omega z/\sqrt{d}) = 4\tau\,\frac{|\,bb_1\,|\,(ab_1-a_1b)}{bb_1\sqrt{d}}\,z = 4\mu\tau Mz,$$

if we take $$\mu = \operatorname{sgn}\,[bb_1(ab_1-a_1b)/\sqrt{d}].$$

Hence since $\exp(4\pi i\mu\tau Mz)-1$ is divisible by $\exp(2\pi i\mu z)-1$, we have

$$f(z+1)-f(z) = \sum_{\eta,\,\zeta}\exp\{\pi i S[(z+\eta\theta)^2\,\omega/\sqrt{d}]+2\pi i\mu\zeta z\} = \sum_{\eta,\,\zeta}\exp\pi i V \text{ say,} \tag{7}$$

where η and ζ also take the values $0, 1, 2, \ldots, 2\tau M-1$.

Now it is well known that

$$\int_{-\infty}^{\infty} e^{fz^2+2gz}\,dz = \left(-\frac{\pi}{f}\right)^{\frac{1}{2}} e^{-g^2/f},$$

where the radical is taken with a positive real part. In evaluating the left-hand side of (6), we change z into $z+\frac{1}{2}\eta$ when $2\theta = -1+\sqrt{d}$, and hence we have

$$V = z^2\,\frac{(ab_1-a_1b)}{bb_1\sqrt{d}}+z\,\frac{\eta(ab_1+a_1b)}{bb_1}+\frac{\eta^2(ab_1-a_1b)\sqrt{d}}{4bb_1}+2\mu\zeta z+\nu,$$

where $\nu=\mu\eta\zeta$ or 0 according as $d\equiv 1$ or 0 (mod 4). Hence the integral (6), remembering that the path of integration is deformed into the real axis from $-\infty$ to ∞ or ∞ to $-\infty$, becomes

$$\operatorname{sgn}\left(\frac{ab_1-a_1b}{bb_1\sqrt{d}}\right)\left(\frac{ibb_1\sqrt{d}}{ab_1-a_1b}\right)^{\frac{1}{2}}\sum_{\eta,\,\zeta}(-1)^{\nu}$$
$$\times\exp\left[\pi i\eta^2\,\frac{(ab_1-a_1b)\sqrt{d}}{4bb_1}-\pi ibb_1\sqrt{d}\,\frac{[\mu\zeta+(ab_1+a_1b)\eta/2bb_1]^2}{ab_1-a_1b}\right]$$

and this reduces to

$$\operatorname{sgn}\left(\frac{ab_1-a_1b}{bb_1\sqrt{d}}\right)\left(\frac{ibb_1\sqrt{d}}{ab_1-a_1b}\right)^{\frac{1}{2}}\sum_{\eta,\,\zeta}(-1)^{\nu}$$
$$\times\exp\left(\frac{-\pi i\sqrt{d}}{ab_1-a_1b}\right)[aa_1\eta^2+\mu\eta\zeta(ab_1+a_1b)+bb_1\zeta^2]. \tag{9}$$

Now it is clear, by putting $\zeta+2\tau M$ for ζ, that the summation for ζ (also for η) need only refer to any complete set of residues (mod $2\tau M$), that is to say we can replace ζ in the summation by $\mu\zeta$.

It is then obvious that the sum of the series $\sum\limits_{\eta, \zeta}$ is unaltered if we replace a, b by $-b$, a. Hence noting (6), (6a), and (9), we have at once

$$\frac{\operatorname{sgn}(bb_1)\, bb_1\, G\left(\dfrac{a}{b}\right)}{N(B)} \left(\frac{ab_1 - a_1 b}{ibb_1\sqrt{d}}\right)^{\frac{1}{2}} = \frac{\operatorname{sgn}(aa_1)\, aa_1\, G\left(-\dfrac{b}{a}\right)}{N(A)} \left(\frac{ab_1 - a_1 b}{ibb_1\sqrt{d}}\right)^{\frac{1}{2}},$$

where A is the denominator of $-b/2a$. Since

$$\left(\frac{ab_1 - a_1 b}{ibb_1\sqrt{d}}\right)^{\frac{1}{2}} = \left| \left(\frac{ab_1 - a_1 b}{bb_1\sqrt{d}}\right) \right|^{\frac{1}{2}} e^{-\frac{1}{4}\pi i \operatorname{sgn} [(bb_1\sqrt{d})/(ab_1 - a_1 b)]},$$

as the left-hand radical is taken with a positive real part, we have

$$|bb_1|^{\frac{1}{2}}\, G\left(\frac{a}{b}\right) e^{-\frac{1}{4}\pi i \operatorname{sgn} [(bb_1\sqrt{d})/(ab_1 - a_1 b)]} / N(B)$$

$$= |aa_1|^{\frac{1}{2}}\, G\left(-\frac{b}{a}\right) e^{-\frac{1}{4}\pi i \operatorname{sgn} [(aa_1\sqrt{d})/(ab_1 - a_1 b)]} / N(A), \qquad (10)$$

for the final result.

In the case of an imaginary field, aa_1 and bb_1 are both positive, and we have

$$|bb_1|^{\frac{1}{2}}\, G\left(\frac{a}{b}\right) \Big/ N(B) = |aa_1|^{\frac{1}{2}}\, G\left(-\frac{b}{a}\right) \Big/ N(A). \qquad (11)$$

In the case of a real field, if aa_1 and bb_1 have the same sign

$$|bb_1|^{\frac{1}{2}}\, G\left(\frac{a}{b}\right) \Big/ N(B) = |aa_1|^{\frac{1}{2}}\, G\left(-\frac{b}{a}\right) \Big/ N(A). \qquad (12)$$

If, however, aa_1 and bb_1 have opposite signs so that

$$\operatorname{sgn}(aa_1 bb_1) = -1,$$

that is

$$\operatorname{sgn}(ab_1) = -\operatorname{sgn}(a_1 b),$$

and hence

$$\operatorname{sgn}\left(\frac{bb_1\sqrt{d}}{ab_1 - a_1 b}\right) = \operatorname{sgn}\left(\frac{bb_1\sqrt{d}}{ab_1}\right) = \operatorname{sgn}(ab),$$

$$\operatorname{sgn}\left(\frac{aa_1\sqrt{d}}{ab_1 - a_1 b}\right) = \operatorname{sgn}\left(\frac{aa_1\sqrt{d}}{ab_1}\right) = \operatorname{sgn}(a_1 b_1),$$

we have

$$|bb_1|^{\frac{1}{2}}\, G\left(\frac{a}{b}\right) \Big/ N(B) = e^{\frac{1}{4}\pi i (\operatorname{sgn} ab - \operatorname{sgn} a_1 b_1)}\, |aa_1|^{\frac{1}{2}}\, G\left(-\frac{b}{a}\right) \Big/ N(A). \qquad (13)$$

This formula, which also includes (12), is equivalent to Prof. Hecke's formula (4).

I need hardly remark that we can prove the law of quadratic reciprocity in the imaginary field just as Prof. Hecke has done in the case of the real field from (13). The details are now rather simpler (as is known to be the case in the general investigations of Hilbert and Furtwängler) because of the absence of the factor $\exp[\frac{1}{4}\pi i(\operatorname{sgn} ab - \operatorname{sgn} a_1 b_1)]$. Thus if a and b are two co-prime numbers of odd norms (*i.e.* aa_1 and bb_1 both odd), and if one of them is a primary number, that is a quadratic residue (mod 4), then

$$\left(\frac{a}{b}\right) = \left(\frac{b}{a}\right),$$

where $\left(\frac{a}{b}\right)$ is the symbol of quadratic reciprocity in the imaginary field.