



Some Isochordic Circles Related to the Triangle

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Source: *The Mathematical Gazette*, Vol. 11, No. 161 (Dec., 1922), pp. 194-199

Published by: Mathematical Association

Stable URL: <http://www.jstor.org/stable/3603026>

Accessed: 11-01-2016 07:13 UTC

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SOME ISOCHORDIC CIRCLES RELATED TO THE TRIANGLE.

TAKING ABC as the triangle of reference, with the usual notation for its sides and angles, let $l=s-a$, $m=s-b$, $n=s-c$. Let I, I_1, I_2, I_3 be the centres of the inscribed and escribed circles $DEF, D_1E_1F_1, D_2E_2F_2, D_3E_3F_3$.

Through A, B, C draw parallels to the opposite sides forming the triangle $A_1B_1C_1$, so that AA_1, BB_1, CC_1 are the medians of the triangle ABC , and there are then ten triangles like ABB_1, ACB_1 , etc., of the same area.

On B_1C_1 as diameter describe a circle cutting BA, CA produced in P and Q , and AB, AC produced if necessary in P_1, Q_1 . Let R, S, R_1, S_1 be similarly found by drawing the circle on diameter C_1A_1 , and T, U, T_1, U_1 by the circle on diameter A_1B_1 .

Now $AF=s-a$ and $AP=a$.

$$\therefore FP=s.$$

Similarly FS, DR, DU, ET, EQ are each equal to s .

Therefore a circle with I as centre and radius $\sqrt{(r^2+s^2)}$ will pass through the six points P, Q, R, S, T, U .

Any circle which, like this one, cuts off equal chords on the sides of the triangle ABC , I have called an *isochordic circle*.

Again, $AE_3=s-b$ and $AT_1=b-c$.

$$\therefore E_3T_1=s-c=n.$$

$$E_3Q=AQ-AE_3=a-s+b=s-c=n.$$

Similarly $S_1F_3, F_3P_1, D_3U_1, D_3R$ each $=n$.

\therefore A circle with centre I_3 and radius $=\sqrt{(r_3^2+n^2)}$ will pass through Q, R, P_1, U_1, T_1, S_1 .

Similarly circles with centres I_1 and I_2 will pass through S, T, Q_1, R_1, U_1, P_1 and U, P, S_1, T_1, R_1, Q_1 ; and their radii will be $\sqrt{(r_1^2+l^2)}, \sqrt{(r_2^2+m^2)}$.

In each case the triangle formed by two radii of a circle and one of its intercepts is equal to the area of ABC . This adds 12 more triangles each of the same area, making 22 in all.

Let S_1T_1, P_1U_1, Q_1R_1 , the common chords of three of these circles, meet at their radical centre V , and let V_1, V_2, V_3 be the other radical centres.

Now $AS=AB+BS=c+b=TC+CA=AT$.

$\therefore ST$ is parallel to the external bisector of angle BAC .

And $AS_1=BS_1-BA=b-c=CA-CT_1=AT_1$.

$\therefore S_1T_1$ is parallel to the internal bisector of angle BAC , and so is perpendicular to ST .

Similarly P_1U_1 is perpendicular to PU and Q_1R_1 to QR . Therefore V is the orthocentre of the triangle $V_1V_2V_3$.

Through A_1 draw YZ parallel to BC to meet AB in Y and AC in Z .

Then

$$(AS/SY)(YA_1/A_1Z)(ZT/TA)=(b+c)/(-b+c)(a/a)/(b-c)/(b+c)=-1.$$

Therefore the three points S, A_1, T are collinear, and A_1 lies on the line V_2V_3 .

Similarly B_1, C_1 lie on V_3V_1 and V_1V_2 .

Incidentally we have made 2 or 6 more triangles each equal to ABC .

Let S_1T_1 meet BC in J .

Then

$$\begin{aligned} BJ/JC &= (BS_1/S_1A)(AT_1/T_1C) \\ &= b/c \\ &= BA_1/CA_1. \end{aligned}$$

$\therefore A_1J$ is the bisector of the angle BA_1C , and is therefore parallel to AI .

$\therefore A_1JVT_1S_1V_1$ is a straight line.

Similarly B_1V bisects angle ABC and C_1V bisects angle AC_1B . So V is the centre of the inscribed circle of triangle $A_1B_1C_1$, and its radius $=2r$.

$$\begin{aligned}\text{Again } VS_1/S_1P_1 &= \sin S_1P_1U_1/\sin S_1VP_1 \\ &= \sin \frac{1}{2}B/\sin \{180^\circ - \frac{1}{2}(A+B)\} \\ &= \sin \frac{1}{2}B/\cos \frac{1}{2}C.\end{aligned}$$

$$\text{Similarly } VT_1/Q_1T_1 = \sin \frac{1}{2}C/\cos \frac{1}{2}B.$$

$$\begin{aligned}\text{Also } S_1P_1 &= S_1B - BA + AP_1 = 2n, \\ Q_1T_1 &= 2m.\end{aligned}$$

$$\begin{aligned}\therefore VS_1 \cdot VT_1 &= 4mn \tan \frac{1}{2}B \tan \frac{1}{2}C \\ &= 4lmn/s \\ &= 4\Delta^2/s^2 = 4r^2.\end{aligned}$$

Therefore the inscribed circle of triangle $A_1B_1C_1$ is the orthogonal circle of the three circles whose intercepts are $2l, 2m, 2n$.

$$\begin{aligned}\text{Similarly } V_1S_1/PS_1 &= \cos \frac{1}{2}B/\sin \frac{1}{2}B, \\ V_1T_1/Q_1T_1 &= \cos \frac{1}{2}C/\sin \frac{1}{2}C. \\ \therefore V_1S_1 \cdot V_1T_1 &= 4mn \cot \frac{1}{2}B \cot \frac{1}{2}C \\ &= 4mns/l \\ &= 4\Delta^2/l^2 \\ &= 4r_1^2.\end{aligned}$$

And so the escribed circles of the triangle $A_1B_1C_1$ are the orthogonal circles of the other three sets of isochordic circles.

As most or all of these properties were first proved by trilinear coordinates, I will furnish the trilinear equations of the principal points, lines and circles, first of all premising that from the I_1 point of view a and a are both to be considered negative; so that:

$$\begin{aligned}s &= \frac{1}{2}(a+b+c) & \text{becomes } \frac{1}{2}(-a+b+c) \text{ or } l, \\ l &= \frac{1}{2}(-a+b+c) & \text{becomes } \frac{1}{2}(+a+b+c) \text{ or } s, \\ m &= \frac{1}{2}(a-b+c) & \text{becomes } \frac{1}{2}(-a-b+c) = -n, \\ n &= \frac{1}{2}(a+b-c) & \text{becomes } \frac{1}{2}(-a+b-c) = -m,\end{aligned}$$

while Δ remains unaltered.

Thus, from that point of view, the chord intercepted on the sides of the triangle by any of these circles is equal to the perimeter of the triangle, and the radius of the inscribed circle is equal to $2\Delta/\text{perimeter}$.

The coordinates of P, U are given by the equations:

$$\begin{aligned}a/\{b(c+a)\} &= \beta/(-a^2) = \gamma/0 = 2\Delta/abc, \\ a/0 &= \beta/(-c^2) = \gamma/b(c+a) = 2\Delta/(abc),\end{aligned}$$

and the equation of UP is

$$a^2a + b(c+a)\beta + c^2\gamma = 0.$$

Similarly the coordinates of P_1, U_1 are given by the equations

$$\begin{aligned}a/\{b(c-a)\} &= \beta/a^2 = \gamma/0 = 2\Delta/abc, \\ a/0 &= \beta/c^2 = \gamma/\{-b(c-a)\} = 2\Delta/abc.\end{aligned}$$

The coordinates of V and V_1 are given by the equations

$$\begin{aligned}aa/l &= b\beta/m = c\gamma/n = 2\Delta/s, \\ aa/s &= b\beta/(-n) = c\gamma/(-m) = 2\Delta/l.\end{aligned}$$

The equations of the circles $PQRSTU, PS, T_1R_1Q_1U$ are

$$abc\Sigma a\beta\gamma + \Sigma aa\Sigma a^2(b+c)\alpha = 0$$

$$\text{and } abc\Sigma a\beta\gamma = \Sigma aa\{a^2(b+c)\alpha + b^2(a-c)\beta + c^2(a-b)\gamma\}.$$

The equations of the orthogonal circles are

$$abc\Sigma a\beta\gamma = \Sigma aa\Sigma a(b-c)^2a$$

and $abc\Sigma a\beta\gamma = (\Sigma aa)\{a(b-c)^2a + (c+a)^2b\beta + (a+b)^2c\gamma\}$.

The circle with centre A and radius a has the equation :

$$abc\Sigma a\beta\gamma - (c^2b/3 + b^2c\gamma)\Sigma aa + a^2(\Sigma aa)^2 = 0.$$

The circles $V_1V_2V_3$, V_2V_3V have equations

$$abc\Sigma a\beta\gamma + \Sigma aa\Sigma(4s-a)a^2a = 0,$$

$$abc\Sigma a\beta\gamma + \Sigma aa\{- (4l+a)a^2a + (4l-b)b^2\beta + (4l-c)c^2\gamma\} = 0.$$

The nine-point circle of the triangle $V_1V_2V_3$:

$$abc\Sigma a\beta\gamma + \Sigma aa\Sigma a^3a = 0,$$

the inscribed circle of the triangle ABC

$$abc\Sigma a\beta\gamma - \Sigma aa\Sigma l^2aa = 0,$$

the nine-point circle of the triangle ABC

$$2\Sigma a\beta\gamma - \Sigma aa\Sigma a \cos A = 0,$$

and the circumscribed circle, $\Sigma a\beta\gamma = 0$, complete a series of four generations of allied circles.

It is plain from the proposition that has been proved above that the orthogonal circles of the inscribed and escribed circles of the triangle ABC are the four perimetro-chordic circles of the triangle formed by joining the middle points of BC , CA and AB . Calling the points where these cut the sides of that triangle P_3 , Q_3 , R_3 , S_3 , T_3 , U_3 , we find that the coordinates are given by the equations

$$P_3, \quad a/(-b) = \beta/(a+c) = \gamma/b = 2\Delta/(2bc);$$

$$Q_3, \quad a/(-c) = \beta/c = \gamma/(a+b) = 2\Delta/(2bc).$$

Assuming the equation to be

$$\Sigma a^2bc\beta\gamma = \Sigma aa\Sigma Aaa,$$

and determining it to pass through the points P_3 , R_3 , T_3 , we obtain the equation

$$\begin{aligned} 4A\{a^2(b+c) + a(b^2+bc+c^2) + bc(b+c)\} \\ = -a^4(b+c) - a^3(b^2+bc+c^2) + a^2(b^3-2b^2c-2bc^2+c^3) \\ + a(b^4-b^3c-bc^3+c^4) + b^4c-b^3c^2-b^2c^3+bc^4, \end{aligned}$$

which, in spite of its unpromising appearance, is equivalent to

$$4A = -a^2 + (b-c)^2 = -4mn.$$

So the equation becomes

$$abc\Sigma a\beta\gamma + \Sigma aa\Sigma mnaa = 0.$$

And one of its fellows is

$$abc\Sigma a\beta\gamma + \Sigma aa(mnaa - msb\beta - nsc\gamma) = 0,$$

and the intercepted chords on the sides of that triangle are s and l .

All circles concentric with the inscribed and escribed circles of a triangle are isochordic. They can, however, be grouped in three classes.

(1) After the set we have already considered, we can take those whose intercepts are the same multiple of $2s$, $2l$, $2m$, $2n$; in other words those in which a chord as base with the corresponding centre as vertex give equal triangles.

(2) Those sets in which all the chords made by the circles of every set are equal.

(3) Those in which the chords subtend equal angles at the several centres, and in which the triangles formed by two radii and a chord are similar.

(1) If we consider the circle which makes an intercept XY ($=2\mu s$) on BC with centre I ,

$$BY = \mu s + m,$$

$$CY = \mu s - n,$$

and the coordinates of Y are given by the equations

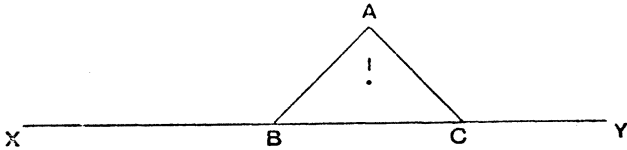
$$\beta/(-c(\mu s - n)) = \gamma/b(\mu s + m) = a/0 = 2\Delta/abc,$$

and the coordinates of X are given by the equations

$$\beta/c(\mu s + n) = \gamma/(-b(\mu s - m)) = a/0 = 2\Delta/abc.$$

\therefore The equation of the circle is :

$$\Sigma\{a^2\alpha^2(\mu^2s^2 - l^2) + 2bc\beta\gamma(\mu^2s^2 + mn)\} = 0.$$



Similarly changing the sign of a and α , the equation of the circle with centre I_1 and radius $2\mu l$ is seen to be

$$a^2\alpha^2(\mu^2l^2 - s^2) + b^2\beta^2(\mu^2l^2 - n^2) + c^2\gamma^2(\mu^2l^2 - m^2) + 2bc\beta\gamma(\mu^2l^2 + mn) - 2ca\gamma\alpha(\mu^2l^2 - ms) - 2aba\beta(\mu^2l^2 - ns) = 0.$$

These equations can be put into the form

$$abc\Sigma a\beta\gamma + \Sigma aa(\alpha a(\mu^2s^2 - l^2)) = 0$$

and $abc\Sigma a\beta\gamma + (\Sigma aa)(aa(\mu^2l^2 - s^2) + b\beta(\mu^2l^2 - n^2) + c\gamma(\mu^2l^2 - m^2)) = 0.$

Similarly the circle whose centre is I_2 and radius $2\mu m$ is

$$abc\Sigma a\beta\gamma + (\Sigma aa)(aa(\mu^2m^2 - n^2) + b\beta(\mu^2m^2 - s^2) + c\gamma(\mu^2m^2 - l^2)) = 0.$$

Hence the common chords of the circles of this series whose centres are I_1 and I_3 is

$$aa(\mu^2(l^2 - m^2) - s^2 + n^2) + b\beta(\mu^2(l^2 - m^2) + s^2 - n^2) + c\gamma(\mu^2 + 1)(l^2 - m^2) = 0,$$

and the common chord of the circles whose centres are I_2 and I_3 is

$$aa(\mu^2 + 1)(m^2 - n^2) + b\beta(\mu^2(m^2 - n^2) - s^2 + l^2) + c\gamma(\mu^2(m^2 - n^2) + s^2 - l^2) = 0.$$

So the coordinates of one radical centre are given by the equations

$$aa/(\mu^2(2l - a) + 2l + a) = b\beta/(\mu^2(2m - b) + 2m + b) = c\gamma/(\mu^2(2n - c) + 2n + c) = 2\Delta/4s,$$

and the locus of these radical centres is

$$aa(m - n) + b\beta(n - l) + c\gamma(l - m) = 0$$

or

$$aa(b - c) + b\beta(c - a) + c\gamma(a - b) = 0.$$

Similarly the locus of the radical centre for the series, taking the three circles whose centres are I_2, I_3, I_1 , is

$$aa(b - c) + b\beta(c + a) - c\gamma(a + b) = 0.$$

These are the straight lines joining I, I_1, I_2, I_3 to the centre of gravity of the triangle ABC .

If we make the centre of gravity the radical centre, we find $\mu^2 = -\frac{1}{3}$.

If we make I the radical centre, we find $\mu^2 = -1$.

The locus of the intersection of the circles of this series whose centres are I and I_1 is the circle

$$2(b + c)\Sigma a^2bc\beta\gamma - \Sigma aa\{(b + c)(a^2 + (b + c)^2)aa + (a^2 + b^2 - c^2)b^2\beta + (a^2 - b^2 + c^2)c^2\gamma\} = 0,$$

and the centre of this circle is

$$aa/\{-(b + c)(3a^2 + (b + c)^2)\} = b\beta/\{b(a^2 + 3(b + c)^2)\} = c\gamma/\{c(a^2 + 3(b + c)^2)\} = 2\Delta/\{2(b + c)((b + c)^2 - a^2)\}.$$

This circle meets BC , where $b^2\beta = c^2\gamma$ and where $\beta \cos C = \gamma \cos B$.

And it meets AB and AC where the external bisector of angle A'

$$(b+c)aa + b^2\beta + c^2\gamma = 0$$

cuts them, and also where the parallel straight line

$$(a^2 + (b+c)^2)aa + (a^2 + b^2 - c^2)b\beta + (a^2 - b^2 + c^2)c\gamma = 0$$

cuts them. These lines are both parallel to the external bisector of angle A , the latter through the point

$$a/0 = b\beta / \cot B = c\gamma / (-\cot C).$$

The locus of the intersection of the circles whose centres are I_2 and I_3 is the circle

$$2(b-c)\Sigma a^2bc\beta\gamma + \Sigma aa\{(-b+c)(a^2 + (b-c)^2)aa - (a^2 + b^2 - c^2)b^2\beta + (a^2 - b^2 + c^2)c^2\gamma\} = 0,$$

which is the same as the previous equation with signs of c and γ changed, so its centre is

$$aa/\{(-b+c)(3a^2 + (b-c)^2)\} = b\beta/\{b(a^2 + 3(b-c)^2)\} = 2\Delta/\{2(b-c)/(b-c)^2 - a^2\},$$

and it meets BC , where $b^2\beta = -c^2\gamma$ and where $\beta \cos C = -\gamma \cos B$.

(2) An isochordic set of isochordic circles, where $2d$ is the length of the chord, will have the equations

$$abc\Sigma a\beta\gamma + \Sigma aa\Sigma(a(d^2 - l^2)) = 0, \\ abc\Sigma a\beta\gamma + \Sigma aa(aa(d^2 - s^2) + b\beta(d^2 - n^2) + c\gamma(d^2 - m^2)) = 0.$$

The common chord of those with centres I_1 and I_2 will be the same for all values of d , i.e.

$$aa(n^2 - s^2) + b\beta(s^2 - n^2) + c\gamma(l^2 - m^2) = 0$$

or

$$aa(a+b) - b\beta(a+b) + c\gamma(a-b) = 0,$$

which is the straight line parallel to the internal bisector of angle C through the middle point of AB .

And the radical centre of the circles with centres I_1, I_2, I_3 is given by the equations

$$aa/(b+c) = b\beta/(c+a) = c\gamma/(a+b) = 2\Delta/4s = \frac{1}{2}r.$$

So every isochordic set of isochordic circles has the same set of radical centres as the inscribed and escribed circles.

The equation of the straight line joining the radical centres of the circles whose centres are I, I_2, I_3 and I, I_1, I_3 , i.e. the points

$$aa/(b+c) = b\beta/(c-a) = c\gamma/(b-a) = 2\Delta/4l = \frac{1}{2}r_1$$

and

$$aa/(-b+c) = b\beta/(c+a) = c\gamma/(a-b) = 2\Delta/4m = \frac{1}{2}r_2,$$

is

$$aa(a-b) - b\beta(a-b) + c\gamma(a+b) = 0.$$

This is the straight line parallel to the external bisector of angle C through the middle point of AB , and the distance between the radical centres is divided at that point in the ratio $m : l$.

(3) When the triangles formed by two radii and a chord are similar :

In this case each intercepted chord is equal to 2μ times the radius of the corresponding inscribed or escribed circle, and the radius of the isochordic circle is $\sqrt{(\mu^2 + 1)}$ times such radius. Under these conditions the centres of similitude remain fixed for all the sets, and are the angular points of the triangle of reference, and the points where the internal and external bisectors of the angles A, B, C meet the opposite side and the circles of the different sets meet on the circle described on the line joining the two corresponding centres of similitude as diameters.

The equation of the circle whose centre is I is

$$\Sigma\{a^2\alpha^2(\mu^2r^2 - l^2) + 2bc\beta\gamma(\mu^2r^2 + mn)\} = 0.$$

And the equation of the circles which are the loci of the intersections of the circles whose centres are I and I_1 , and I_2 and I_3 , are

$$(b+c)\Sigma\alpha\beta\gamma - \Sigma\alpha a(c\beta \cos B + b\gamma \cos C) = 0$$

and

$$(b-c)\Sigma\alpha\beta\gamma + \Sigma\alpha a(c\beta \cos B - b\gamma \cos C) = 0$$

respectively, which are the circles on the internal and external bisectors of angle A as diameters.

The radical centre of the three circles whose centres are I_1, I_2, I_3 is given by the equations

$$\begin{aligned} \alpha a / \{ \mu^2 s(-lm^2 - ln^2 + m^2n + mn^2) + lmn(2l + m + n) \} \\ = b\beta / \{ \mu^2 s(-mn^2 - ml^2 + n^2l + nl^2) + lmn(l + 2m + n) \} \\ = c\gamma / \{ \mu^2 s(-nl^2 - nm^2 + l^2m + lm^2) + lmn(l + m + 2n) \} \\ = 2\Delta / (4lmns) = 1/2\Delta, \end{aligned}$$

and the locus of these radical centres is

$$(b-c)a \cos A + (c-a)\beta \cos B + (a-b)\gamma \cos C = 0.$$

W. W. TAYLOR.

146. (A youth) may have a strong genius for mathematics without being able to comprehend a demonstration of Euclid; because his mind conceives in a peculiar manner, and is so intent upon contemplating the object in one particular point of view, that it cannot perceive it in any other. We have known an instance of a boy, who, while his master complained that he had not capacity to comprehend the properties of a right-angled triangle, had actually, in private, by the power of his genius, formed a mathematical system of his own, discovered a series of curious theorems, and even applied his deductions to practical machines of surprising construction. — Goldsmith, p. 439, *Essays* xii.

147. I believe I was the first to recommend in parliament a uniformity of weights and measures, especially a decimal system, which eventually got the sovereign for the guinea. I was the mover of the new Board of Longitude, and of all chronometer-improvement measures. For many years I was on the Council of the Royal Society. I always tried to get the Northern powers to have a common thermometric scale, and got it talked of at the Congress of Vienna, and would probably have succeeded but for the escape of Buonaparte from Elba.—Croker to John Murray (*Letters*, 1851, iii. 243).

148. Learned he was in med'c'nal lore,
For by his side a pouch he wore
Replete with strange hermetic powder,
That wounds nine miles point-blank would solder;
By skilful chemist with great cost
Extracted from a rotten post. . . .

Arctophylax (Hudibras, Part. I. c. ii. 11, 223).

149. In mathematics he was greater
Than Tycho Brahe or Erra Pater;
For he, by geometric scale,
Could take the size of pots of ale;
Resolve by sines and tangents straight
If bread or butter wanted weight;
And wisely tell what hour o' th' day
The clock does strike, by Algebra. . . .

Hudibras, Part I. canto i. 11, 119.