PAPERS

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

ON THE FUNDAMENTAL THEOREM IN THE THEORY OF FUNCTIONS OF A COMPLEX VARIABLE

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[Received December 24th, 1910.—Read January 12th, 1911.]

1. The fundamental theorem in the Theory of Functions of a Complex Variable states, as is well known, that, if dw/dz exists at all points in the interior of a circle of radius c, the function w of the complex variable z is analytic, so that, in particular, all the higher differential coefficients of w exist. There is no difficulty in translating this theorem into one which employs the language of the real variable, and the question has thus been raised as to whether it is not possible to prove the theorem without the use of the complex variable. I understand, indeed, that this very matter was the subject of conversation at a meeting of the Society held early in the present year.

The progress of the Theory of Functions of a Real Variable has been recently very rapid; but, regarded as a science, in the modern sense of the term, it is but a younger brother of the Theory of Functions of a Complex Variable. This has been largely owing to the great simplicity brought about in the latter theory by precisely the theorem of which it is the question. On the other hand, from the point of view of the Theory of Functions of a Real Variable, it is certainly unsatisfactory that a theorem so important, and in any but its most recent formulations long well known, should not yet have been proved except by the use of the complex variable.

I hope, therefore, that the present contribution, though not a complete solution of the problem in all its generality, will be regarded as of interest, more especially in view of the fact that the only restriction introduced is that of the boundedness of dw/dz. In point of generality, therefore, the result arrived at comes between those of Goursat's former and latter papers on the subject. I may add that the nature of the reasoning employed admits obviously of a slightly more general statement than that given, and it is not impossible that further knowledge of the properties of functions of a real variable may admit of the argument used below applying *mutatis mutandis* to the most general case.

As to the importance of the method exposed, two further remarks may be made. The one is that the actual expression of the function is obtained, so that in point of length the present method compares favourably with the classical one, as completed by Goursat. Further, just as even mathematicians of the calibre of Kronecker endeavoured to minimise or even banish the employment of irrational numbers in mathematics, so there have always been, and it may be expected there always will be, mathematicians who regard a theorem as only then completely established when the mode of proof employed avoids the use of the complex variable. For such mathematicians the complex variable is only a powerful method of investigation. In the present case, the theorem with which we are concerned is that which is fundamental in the whole Theory of Functions of a Complex Variable; it consequently opens the door to the use of chains of reasoning by which the real results of that theory are established from first to last by the methods of the real function theory alone.

It may be added that Goursat's extremely elegant and skilful proof is one which, besides involving the complex variable, employs considerations which offer numerous difficulties to the reader. Only a select few can be expected to be able to judge, in the present state of mathematical teaching, of the validity of that proof.

2. It is known that the necessary and sufficient conditions that $w = u+iv$ should possess a differential coefficient dw/dz with respect to the complex variable $z = x+iy$ are that u and v should possess first differentials and that

$$\frac{du}{dx} = \frac{dv}{dy} \quad \text{and} \quad \frac{du}{dy} = -\frac{dv}{dx}.$$

Let, then, $u(x, y)$ and $v(x, y)$ be functions of (x, y) which have the following properties:—

(i) u_x, u_y, v_x, v_y all exist, and u and v possess differentials at all points of a certain circle of radius c and centre the origin.

(ii) $$u_x = v_y, \quad \text{and} \quad u_y = -v_x. \tag{1}$$

(iii) u_x, u_y, v_x, and v_y are bounded functions of (x, y) in the circle considered.

Then, since $u(x, y)$ possesses a differential,

$$\frac{du}{dr} = u_x \frac{dx}{dr} + u_y \frac{dy}{dr} \quad \text{and} \quad \frac{du}{d\theta} = u_x \frac{dx}{d\theta} + u_y \frac{dy}{d\theta};$$

therefore

$$\left.\begin{aligned} \frac{du}{dr} &= u_x \cos\theta + u_y \sin\theta = \frac{1}{r}\frac{dv}{d\theta} \\ \text{and} \qquad \frac{1}{r}\frac{du}{d\theta} &= -u_x \sin\theta + u_y \cos\theta = -\frac{dv}{dr} \end{aligned}\right\}. \tag{2}$$

We may evidently suppose, without loss of generality, that $u = 0$ and $v = 0$ at the origin. It follows also, from (2), that u_r, v_r and u_θ/r, v_θ/r are all bounded functions of (r, θ).

Hence, by the simplest case of Lebesgue's theorem,*

$$u - u_0 = u(x, y) - u(r, 0) = \int_0^\theta u_\theta \, d\theta,$$

and we may integrate both sides of this equation with respect to r after dividing by r.

Again, since $1/r . du/d\theta$ is a bounded function of (r, θ),

$$\int_0^r dr \int_0^\theta \frac{1}{r} u_\theta \, d\theta = \int_0^\theta d\theta \int_0^r \frac{1}{r} u_\theta \, dr,$$

whence $$\int_0^r \frac{1}{r} (u - u_0) \, dr = -\int_0^\theta v \, d\theta. \tag{3}$$

Thus the left-hand side is an integral when regarded as a function of θ, so that we may expand it in a Fourier series.

Similarly $$\int_0^r \frac{1}{r} (v - v_0) \, dr = \int_0^\theta u \, d\theta. \tag{4}$$

We have then

$$\int_0^r \frac{1}{r} (u - u_0) \, dr = \tfrac{1}{2} A_0 + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta). \tag{5}$$

Also, since u_θ is bounded, u is an integral with respect to θ, so that

$$u = \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta),$$

* For a proof of this theorem without the use of Cantor's numbers, see W. H. Young and Grace Chisholm Young, "On the Existence of a Differential Coefficient," Appendix II, 1910, *Proc. London Math. Soc.* (to appear).

where the constant term has been omitted, since, by (4), the integral of u with respect to θ is periodic. Here

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta \left(\cos n\theta \int_0^r \frac{(u-u_0)}{r}\, dr\right), \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} u \cos n\theta \, d\theta.$$

Now for all modes of variation of (r, θ), as we approach the origin, it follows from the Theorem of the Mean for functions of two variables,* bearing in mind that u is zero at the origin, that

$$\frac{u}{r} = \frac{x}{r} u_{x'} + \frac{y}{r} u_{y'},$$

where (x', y') is a certain point between $(0, 0)$ and (x, y), whence it follows that u/r is a bounded function of (r, θ) in the whole circle. The same is, of course, true of u_0/r, since it is the result of putting $\theta = 0$ in u/r.

Hence in the expression for A_n we may change the order of integration, which gives us at once

$$A_n = \int_0^r \frac{a_n}{r}\, dr.$$

Similarly
$$B_n = \int_0^r \frac{b_n}{r}\, dr. \tag{6}$$

We remark also that a_n/r, and similarly b_n/r, since it is the result of a single integration with respect to θ of a bounded function of (r, θ), is certainly a bounded function of r up to and including the point $r = 0$.

Now v is an integral with respect to θ. Hence we may write

$$v = \sum_{n=1}^{\infty} (a'_n \cos n\theta + b'_n \sin n\theta),$$

where, as before, the constant term has been omitted, since, by (3), the integral of v with respect to θ is periodic. Hence, by the theory of Fourier series, we have

$$\int_0^\theta v\, d\theta = \tfrac{1}{2}A'_0 + \sum_{n=1}^{\infty} (a'_n \sin n\theta - b'_n \cos n\theta)/n,$$

whence, by equations (3), (5), and (6), we see that

$$\int_0^r \frac{a_n}{r}\, dr = \frac{b'_n}{n} \quad \text{and} \quad \int_0^r \frac{b_n}{r}\, dr = -\frac{a'_n}{n}. \tag{7}$$

* See, for instance, § 22 of my tract, *On the Fundamental Theorems of the Differential Calculus*, Cambridge University Press, 1910, where Case (i) of the Theorem of the Mean, § 15, is used. Using Case (ii), the result is in the form here required.

Similarly, using (4) instead of (3), we see that

$$\int_0^r \frac{a'_n}{r}\,dr = -\frac{b_n}{n} \quad \text{and} \quad \int_0^r \frac{b'_n}{r}\,dr = \frac{a_n}{n}. \tag{8}$$

From equations (7) and (8) we see that a_n, b_n, a'_n, and b'_n are all continuous functions of r, and therefore, except possibly for the value zero of r, this remains the case when they are divided by r. Hence the integrands of the integrals in (7) and (8), except possibly for the value $r = 0$, are the differential coefficients with respect to r of their integrals. From (8) it therefore follows that

$$a'_n = -r\frac{d}{dr}\left(\frac{b_n}{n}\right). \tag{9}$$

Similarly

$$b_n = -r\frac{d}{dr}\left(\frac{a'_n}{n}\right). \tag{10}$$

From (9) and (10) we have $n^2 b_n = r\frac{b}{dr}\,r\frac{d}{dr}\,b_n,$

which has been proved to hold for all values of r except zero. Hence, writing

$$t = \log r,$$

we see that, for all values of t corresponding to positive values of $r < c$, that is, for all finite values of t less than $\log c$,

$$n^2 b_n = \frac{d^2}{dt^2}\,b_n,$$

which gives

$$b_n = K_n r^n + L_n r^{-n}.$$

Here we may at once put $L_n = 0$, since b_n/r is bounded up to and including $r = 0$. Thus

$$b_n = K_n r^n.$$

Hence, by (7),

$$a'_n = -K_n r^n.$$

In precisely the same way we see that b'_n and a_n are equal with the same sign, and have a value of the same form as b_n and a'_n.

We may, therefore, with a convenient change in the notation, write

$$\left.\begin{aligned} u &= \sum_{n=1}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta), \\ v &= \sum_{n=1}^{\infty} r^n(-b_n \cos n\theta + a_n \sin n\theta), \end{aligned}\right\} \tag{I}$$

these equations certainly holding for all points in the interior of the circle of radius c, the origin being obviously included.

Now, regarded as functions of r, u and v are here expressed as power series; hence all their differential coefficients with respect to r exist. Hence it also follows that all their differential coefficients with respect to θ exist.

To see this, let
$$\left.\begin{aligned} &\sum_{n=1}^{\infty} r^n(A_n \cos n\theta + B_n \sin n\theta) \\ \text{and}\quad &\sum_{n=1}^{\infty} r^n(-B_n \cos n\theta + A_n \sin n\theta) \end{aligned}\right\} \qquad \text{(II)}$$
be the series got by performing the operation $(r\,d/dr)^p$ term-by-term on the above series (I) representing u and v respectively. Then, since both these series converge for $\theta = 0$, the series $A_n r^n$ and $B_n r^n$ both converge, and therefore, by the theory of power series, the series $|A_n| r^n$ and $|B_n| r^n$ both converge, and therefore the series $(|A_n| + |B_n|) r^n$ also converges, for all positive values of r less than c. Hence, by Weierstrass's test, it follows that the series (II) converge uniformly for all values of the ensemble (r, θ) inside any circle of radius less than c.

But, apart from sign, these series (II) are the same as the series got by differentiating the series (I) p times with respect to θ. Hence, by the theory of term-by-term differentiation of series, these latter series, being uniformly convergent series of functions of θ, represent the p-th differential coefficients of the two series (I).

Thus all the differential coefficients of u and v with respect to r and θ, and therefore also with respect to x and y, exist. Moreover, we have obtained analytical expressions for u and v which can be differentiated term-by-term as often as we please, namely, the series (I).

Finally, $u + iv$ is, of course, a power series in $x + iy$. Hence, translating into the language of the complex variable, we have the theorem stated in the first article, with the restriction there noted.