NOTES AND MEMORANDA

APPLICATIONS OF PROBABILITIES TO ECONOMICS.—II.

THIS second exemplification of applied Probabilities is like the first,[1] derived from the theory of Monopoly. The feature of that theory with which we are now concerned is the power of the monopolist to discriminate between different species of commodities and customers, not preserving that unity of price which characterises a perfectly competitive market. The subject may fittingly be introduced by a quotation from the earliest, and still, I think, the highest authority on the theory of discrimination, Dupuit. In his epoch-making paper on the measurement of utility Dupuit puts the following case :—

"Waterworks are constructed for the use of a town situated on a hill which had before great difficulty in procuring water. The value of water had been so high that an annual subscription of 50 francs was required to pay for a daily supply of a hectolitre [22 gallons]. . . . But now that pumps have been set up, that amount of water costs only 30 francs. As a consequence, the consumer will now employ water for less pressing, less essential wants. . . . Again, owing to the improvement of the pumps, or by the mere fact of increased consumption, the price is reduced to 20 francs. Our man will now want to have four hectolitres, so as to be able to clean his house every day. Supply him with water at 10 francs per hectolitre, and he will demand ten hectolitres, so as to be able to water his garden. At 5 francs he will demand twenty hectolitres, to maintain a sheet of ornamental water; at 1 franc he will want a hundred hectolitres, to have a fountain constantly playing." [2]

With reference to this illustration, it may be asked : supposing that water for use within the house and water for external use, in the garden or pond, form two categories between which it is possible for a monopolist to discriminate ; is it to be supposed that when the price is lowered from 20 francs to 10 francs, and accordingly water begins to be employed for external uses, the whole of

[1] See ECONOMIC JOURNAL, vol. xx, p. 288, *et seq.*

[2] *Annales des Ponts et Chaussées*, 1844, vol. 2, p. 337.

the additional six hectolitres are employed on external uses or part on (additional) internal uses? The question is not explicitly raised by Dupuit; being indeed not relevant to his context. But I am concerned to postulate for the cases of discrimination with which I deal that a lowered price *is* attended with an increased demand for both of the uses. The species of discrimination which I have in view may be made more conspicuous by noticing its absence from another illustration given by Dupuit:—

"A footbridge is constructed between two populous quarters of a large town at a cost of 150,000 francs. At the rate of 5 centimes per passenger the proceeds prove to be only 5,000 francs [per annum]. The concern is accordingly a failure; the *entrepreneur* who had borrowed the greater part of the 150,000 francs, being unable to pay the interest on this sum, is soon ruined. The bridge is sold to an intelligent man who studies the demand for the use of the bridge, with the object of increasing his own profits. Thus he observes that his bridge connects a quarter of the town in which there are manufacturing works with the quarter in which the workmen live; and that they have, morning and evening, to make a long detour in order to reach their destination. The use of the bridge would greatly shorten the distance which they have to traverse; but a workman could not afford to pay out of his wages as much as ten centimes a day. . . . [Under the circumstances] the proprietor might insert in his tariff a clause to this effect: 'For passengers wearing a cap, blouse, or jacket[1] the toll is reduced to 1 centime.' [He will thus, suppose, gain an additional 3,000 francs from 300,000 new passengers—per working year of 300 days; but he may lose a part of his original profits, 5,000 francs, as] "a certain number of passengers at 5 centimes will, by reason of their attire, benefit by the reduction which was not intended for them." [However] "by new artifices he may succeed in reducing the loss. Thus he may stipulate that the reduction of the toll shall be given only at the hours at which the workshops open and close, or only to workmen showing a certificate[2] of employment."[3]

In this and other passages Dupuit suggests a type of discrimination which may thus be formulated. Considering the demand for the undiscriminated commodity (*e.g.*, passage of the bridge without respect of persons) as made up of the demands for different species between which discrimination is possible; it is (α) conceived that the demand for one species is independent of, uncorrelated with, the demand for another species—Dives will not offer less because the toll is lowered for certificated workmen; (β) it is admissible, if indeed it is not essential, that the demand for each

[1] Casquette, blouse, or veste.
[2] *Livret.*
[3] *Loc. cit.*, 1849, p. 220.

species is practically limited (*e.g.*, the amount of water employed in internal uses will not be materially increased, however low the water-rate falls). A similar conception is entertained by M. Colson, who walks in the way of Dupuit.[1] I recognise that the conception is of great importance for the purposes of both theory and art. But I emphasise it here only to make clear that it is not the case with which I am about to deal. I am indifferent about the attribute (α), and I am not indifferent about (β); I postulate that when price is lowered the amount of each species—as well as of the genus—increases. For example, if there are two species (such as water for internal, and water for external use) whereof the amounts x_1, x_2 are demanded at the prices y_1, y_2, I suppose that (for any assigned value of y_2) x_1 continually increases as y_1 diminishes.[2] The case is quite sufficiently important to reward attention to its properties. In dealing with it, I shall for convenience of enunciation confine my statements mostly to the variety in which only two species are discriminated; but the propositions thus enunciated are readily adapted to any finite number of species.

Concerning the kind of discrimination thus defined, I propose to prove the three following theses :—

1. *Very probably a system of prices can be assigned, such that both the monopolist and his customers may gain by discrimination.* The gain to consumers may well be so great that they are better off than they would have been, other things being equal, under a *régime* of competition.

2. *Probably the prices which the monopolist will fix in order to render his profit a maximum are such that the customers will lose through discrimination;* except when the amount demanded of one species before the discrimination is much less than the amount then demanded of the other.

3. *Probably, if the disturbance of prices caused by discrimination is not considerable, the portion of the monopolist's maximum which is due to the infliction of loss on the customers is inconsiderable.* For a small consideration the (perfectly self-interested) monopolist may be induced to adopt a system of prices such that the customers will not lose through discrimination; for a small addition to that consideration he may be induced to adopt a system of prices such that they will be materially the gainers.

The general presumptions above described as *à priori* are avail-

[1] See, for some account of M. Colson's conception, ECONOMIC JOURNAL, vol. xx, p. 59 *et seq*; and compare below, p. 454.

[2] Thus in the example designated *C* at p. 456 below, each of the component (as well as the compound) demands tails off towards infinity as the price sinks to zero.

able to show that the first proposition is probable. The gain of the monopolist by discrimination depending on the addition to or subtraction from each price, may be likened to the height, say z, of a surface shaped like a hill, varying with co-ordinates x and y, such as the longitude and latitude of any position on the hill. Now, in general one can reach a higher position on a hill when free to move in any direction than when one is restricted to motion along a certain path. In the case before us a limitation of this sort exists prior to discrimination; the monopolist being constrained to charge only one price for the whole class of commodity, or in other words equal prices for the two species. When this limitation is removed, the monopolist will tend to start off in a direction which has been called the *line of preference*;[1] perpendicular to another line on the plane of xy called the *line of indifference*. Likewise the consumer will have his lines of preference and indifference. But from our general knowledge of the relations between buyer and seller, we may presume that the lines pertaining to one party are not coincident with the lines pertaining to the other party. Accordingly the direction in which both parties can move together (from the original position), both being gainers by discrimination, is probably represented by an angle of sensible magnitude; the probability of mutual gain is measured by the ratio of that angle to four right angles.

The probability thus discerned will appear greater if we formulate what is known about the relation of the monopolist to his customers. On Fig. 1 let the addition to, or subtraction from, the price of one species be measured on the axis OX, OX', and likewise the alteration of the other price on the axis of y. Prior to discrimination, the monopolist was constrained to move along a right line, representing the condition that the two prices must be the same, the line TT' making equal angles with the axes. When the monopolist becomes free to move, otherwise than in this line, his line of preference is evidently not in the same quadrant as this line; not in the direction implying that both the variations of price are positive—between OX and OY—nor yet in a direction implying that both variations are negative—between OX' and OY'. For if either of these directions represented the monopolist's preference, he would not, prior to the discrimination, have stopped at O. Not his line of preference, but his line of *indifference* slopes in the same general direction as the original path. In the figure the line TT' does duty both for the path of constraint and the line of indifference; but these loci are not

[1] *Mathematical Psychics*, p. 22, and context.

in general coincident. But the line of preference pertaining to the customers is evidently in the direction between OX' and OY' since the variation most advantageous to the purchasers is the fall of both prices. Accordingly, their line of indifference will slope in the general direction represented by the line tt' in the figure. The interests of the two parties are concurrent for variations of price which are represented by a step in any direction between OT and Ot.

To obtain an idea of the distance to which they may travel

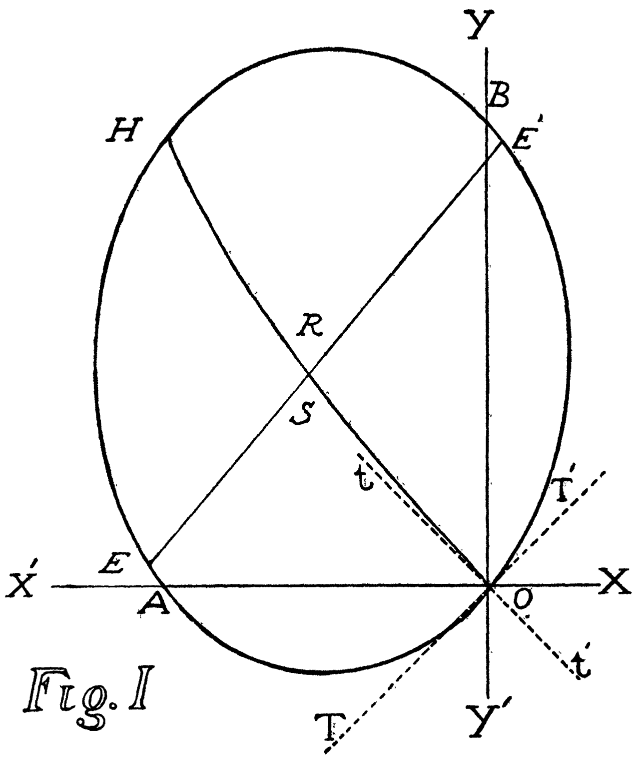


Fig. I

concurrently, we may employ a more elaborate construction; which is also required for the proof of the second and third theses. Let us begin by assigning a particular form to the demand-curves of the customer; and first of all the simplest of all forms, the right line. Let x_1 be the amount of one species of commodity, x_2 that of the other demanded at any price, y; and let $2x = x_1 + x_2$—be the amount of the generic commodity (*e.g.*, water for any purpose) demanded at the price of y. Then by hypothesis x is connected with y by a (linear) relation of the form.

$x = A - By$; where A and B are numerical coefficients. The monopolist's profit, supposing at first that cost of production may be left out of account, $= xy = Ay - By^2$. This will be a maximum when $y = \frac{1}{2}A \div B$ and accordingly $x = \frac{1}{2}A$.[1] If we call this maximum value of x, a, and the corresponding value of y, b, we have $A = 2a$, $B = a \div b$; and accordingly the equation of the (average, generic) demand-curve may be written in the form

$$\frac{x}{a} = 2 - \frac{y}{b}.$$

This line is represented by BA in Fig. 2; on the supposition that $a = b$ (as may always be effected by properly taking the units of commodity and price).

Let us at first suppose (in accordance with the main portion of thesis 2) that x_1 and x_2 are equal at the price which is fixed by the monopolist prior to discrimination. Let us also for the present suppose that there is no correlation[2] between the demands for the two species of commodity. Then the two specific demand-curves (as they may be called, although they are straight lines) will intersect at the point P, which represents the price and *half* of the quantity demanded before the discrimination. The two curves will diverge at that point as represented by the dotted lines in Fig. 2, in such wise that any horizontal line intercepts between the *average* demand-line (AB) and either of the specific demand-lines (*e.g.*, A_1B_1) a length equal to that which it intercepts between the former line (AB) and the other specific demand-line (A_2P produced as far as the point at which the ordinate $= OB_1$) For instance, on the horizontal line through ω, the intercepts $\alpha\alpha_1$ and $\alpha\alpha_2$ are equal. Likewise $AA_1 = AA_2$.

This property may conveniently be represented by the following construction :—

$$\text{Let } x = a(1+\xi)\ ;\ y = b(1+\eta).$$

Then if ξ and η are measured from P along the rectangular axes, the relation of ξ to η is represented by the line AB (provided that $a = b = 1$). In other words, $\xi = -\eta$.

$$\text{Likewise, if } x_1 = a(1+\xi_1),\ y_1 = b(1+\eta_1),$$
$$x_2 = a(1+\xi_2),\ y_2 = b(1+\eta_2).$$

$\xi_1 = -q_1\eta_1$, $\xi_2 = -q_2\eta_2$, where q_1 and q_2 must be so selected that $q_1 + q_2 = 2$; say $q_1 = 1 + \beta$, $q_2 = 1 - \beta$, where β is a proper fraction.[3]

[1] I use the old-fashioned sign of division $\div$ in the text, but in the more technical notes the now generally adopted sloping line, as thus, A/B.

[2] Cp. above, p. 442.

[3] As follows from the condition that the line representing a demand-curve must be inclined negatively to the axis of x.

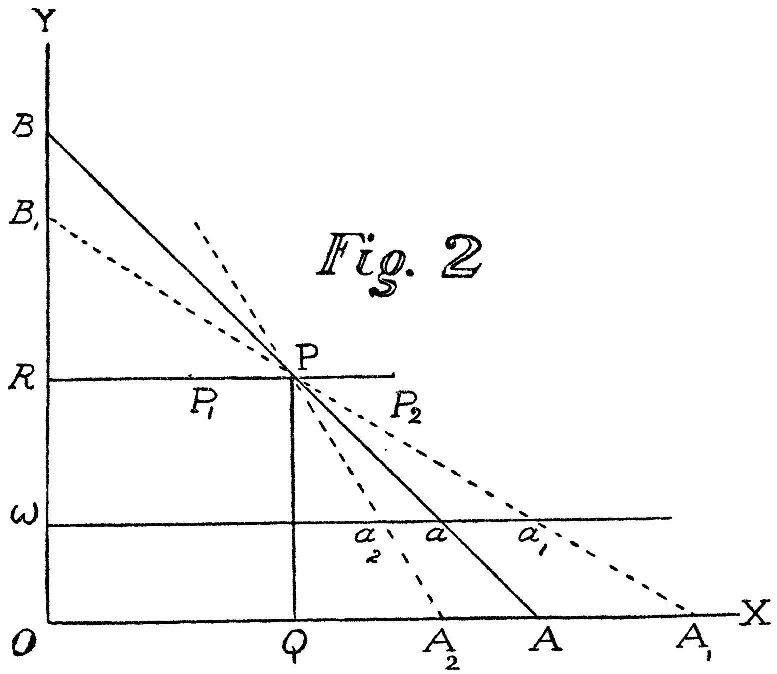
The apprehension of the general theorem may be facilitated by assigning a particular numerical value to β. Let us assign a value which is neither very great, nor very small, namely, $\frac{1}{5}$ (0·2).

$$\xi_1 = -1{\cdot}2\eta_1;\ \xi_2 = -0{\cdot}8\eta_2.$$

Now the gain of the monopolist through discrimination, say R, being the difference between his profit after the discrimination and what it was before

$$= x_1y_1 + x_2y_2 - 2ab.$$
$$= ab(\eta_1 + \xi_1 + \eta_1\xi_1) + ab(\eta_2 + \xi_2 + \eta_2\xi_2).$$

Fig. 2



Substituting for ξ_1 and ξ_2 their respective values in terms of η_1 η_2 we have

$$R = ab(-0{\cdot}2\eta_1 - 1{\cdot}2\eta_1^2 + 0{\cdot}2\eta_2 - 0{\cdot}8\eta_2^2).$$

Thus the relation between (changes η) prices, which afford the same profit, the locus of constant revenue, is given by equating the expression within the brackets to a constant. This locus is an ellipse, which when $R = o$ passes through the origin from which ξ and η may be measured (on rectangular axes). In Fig. 2 O represents this origin, and the curve $OAHB$ is supposed to fulfil the condition

$$1{\cdot}2\eta_1^2 + 0{\cdot}8\eta_2^2 + 0{\cdot}2\eta_1 - 0{\cdot}2\eta_2 = 0.$$

Likewise the locus of constant Consumers' Surplus is found from first principles to be an ellipse with equation

$$\tfrac{1}{2}1{\cdot}2\eta_1^2+\tfrac{1}{2}0{\cdot}8\eta_2^2-\eta_1-\eta_2=constant.$$

In the figure the curve *OSH* represents the case in which the said constant is zero, the locus of null gain to the consumers through discrimination.

When the constant in the last written equation is positive, the curve of Consumers' Surplus lies below and to the left of *OSH*. Consider in particular the curve of this family passing through *T*, on the supposition that the point *T* represents the (lower, identical) prices which would prevail, other things being equal, if the *régime* were one of competition not monopoly. It is quite possible that this curve (not shown in the figure) should *cut* the locus of null monopoly profit, the ellipse *OAHB*. There will then be intercepted between these two curves an area any point in which represents a pair of prices which fulfil the secondary part of our first thesis.

The range of variations in price, from *O* the position before discrimination, that are advantageous both to the monopolist and his customers is represented by the space intercepted between the curves *OSH* and *OAH*. The point *H* may be described as the limit of the range and the index of its extent, if it is understood not to mean that the direct path from *O* to *H* can be travelled concurrently, with mutual advantage, by both parties. So the Pillars of Hercules are described by a geographical writer as the limit up to which the navigation of the early Mediterranean peoples extended; though a people situate like the ancient inhabitants of Marseilles could not sail in a straight line to that limit, but must hug a curvilinear shore (that of Spain) comparable with our curve *OSH*.

The index thus defined is found[1] to be the point of which the abscissa (η_1) is $-0{\cdot}1855..$, and the ordinate (η_2), $+0{\cdot}2268..$; corresponding to prices relatively 18·55 per cent. lower and 22·68 per cent. higher than the prices prior to the discrimination. There is thus a considerable range of variation; considerably greater, as will presently appear, than that which corresponds to the monopolist's maximum profit. Thus the first thesis is amply verified.

Going on to the second thesis, we have first to determine the prices which render the monopolist's profit a maximum. They

[1] The calculation is facilitated by the incident that the intersection of the two curves is also the intersection of either of them with the line $\eta_1(1+\tfrac{1}{2}\beta)+\eta_2(1-\tfrac{1}{2}\beta)=0$.

prove to be $\eta_1 = -\frac{1}{12}$, $\eta_2 = +\frac{1}{8}$.[1] We have now to observe how the Consumers' Surplus is affected by the adoption of these prices. Substituting the values of η_1 and η_2 in S, the expression for the consumers' gain by the discrimination, we find the gain to be *negative*, namely—$\frac{1}{32}$. The *sign* of this quantity is all that is required to fulfil the second thesis; but it is interesting to notice that the *amount* of loss is considerably greater[2] than the amount of the monopolist's gain, viz., $\frac{1}{48}$ (that is, a gain of more than 2 per cent. upon his profits before the discrimination).

To verify the third thesis, we have to compare the maximum monopoly revenue, R', as above determined, the *absolute* maximum as it may be called, with the *relative* maximum, which is the greatest possible gain to the monopolist consistent with the condition that there should be no loss to the consumer. The required positions may be explored by means of the theorem[3] that the maximum monopoly revenue *relative* to, or limited by, the condition that the Consumers' Surplus should have any assigned value is realised by a system of prices such that the elasticity[4] is the same for each of the demand-curves. In the simple case before us, the locus of equal elasticity is a right line inclined to the axis x at an angle of which the tangent is $\frac{27}{22}$,[5] and passing through the point R, the line ERE' in Fig. 2. Thus we have only to determine the intersection of this line with the curve of null gain to the consumers. Let θ_1 and θ_2 be the respective differences between the known co-ordinates of R, η'_1, η'_2, and the sought co-ordinates of the point of intersection S, say η''_1 and η''_2. Substituting in the expression for S for η_1, $-\frac{1}{12}+\theta_1$ and for η_2, $+\frac{1}{8}+\theta_2$, and then putting $\theta_2 = \frac{27}{22}\theta_1$, we obtain for θ_1 a quadratic equation of which I find the root to be $-·014067$. The corresponding value for θ_2 is $+0·01726$. Whence we obtain for η''_1, $-0·09740$ and for η''_2, $+0·10774$. Substituting these values for η_1 and η_2 in the general expression for R, I find for the new value of R, R'', as we may call it, 0·02035. This is to be compared with R', the absolute maximum, namely $\frac{1}{48}$, or 0·02083. The difference between R' and R'' is very small, namely, 0·00048; about 2·3 per cent. of R'. That is the proportion of the

[1] Differentiating R with respect both to η_1 and η_2, and observing that the second term of variation is negative.

[2] In absolute quantity.

[3] See NOTE at the end.

[4] The elasticity *proper*, referred to on a preceding page (*ante* p. 290).

[5] In general $\frac{1+\beta}{1-\beta} \Big/ \frac{1+\frac{1}{2}\beta}{1-\frac{1}{2}\beta}$.

monopolist's maximum profit which is dependent on the Consumers' loss—a very small proportion in accordance with our third thesis.

When, other things being the same, we suppose the extent of discrimination as measured by the constant β to be increased, it will be observed that the first and the second theses continue to hold good. But the subordination predicated by the third thesis becomes less and less; though it retains some significance for values of β much greater than that which we have considered—say up to $\frac{1}{2}$. To illustrate the failure of the third thesis (while the first and second are eminently fulfilled) put $\beta=1$. Proceeding as before, I find for R' now 0·8, for $R'-R''$ 0·1205; the latter more than 15 per cent. of the former.

The case which has been considered in which the demand-curves with which we are concerned are straight lines may be regarded as a *Lemma*, which forms a convenient introduction to the far more typical case in which the curves are of the second degree, to wit, parabolas. One obvious difference between the type and the Lemma is the incident that whereas before in the expression for R and S there occurred only *squares* (and first powers) of the variables (η_1 and η_2), there now occur *cubes* of those quantities. But this difference is not from the present point of view the essential one; since the η's are supposed to be so small, or at least so far from great, that their *third* powers may be, I will not say "neglected," but *subordinated*, in comparison with the second powers. It is a more essential circumstance that the coefficient of the second powers in the expression for R now takes on different values, depending on a certain coefficient which is of great significance in the theory of monopoly.[1]

Still facilitating the acceptance of general truth by a particular example, let us suppose that the demand-curve for $2x$ $(=x_1+x_2)$ prior to discrimination is a parabola of the kind sometimes called horizontal; so that x is of the form $A-By^2$ (A and B both positive). If as before we express the coefficients in terms of the values of x and y, for which xy is a maximum, we have

$$\frac{x}{a}=\frac{3}{2}-\frac{1}{2}\left(\frac{y}{b}\right)^2.$$

Whence, if as before $x=a(1+\xi)$, $y=b(1+\eta)$,

$$\xi=-(\eta+\tfrac{1}{2}\eta^2)$$

$$\xi_1=-1\cdot2(\eta_1+\tfrac{1}{2}\eta_1^2)\ ;\ \xi_2=-0\cdot8(\eta_1+\tfrac{1}{2}\eta_1^2)._2$$

[1] The coefficient ω, as to which see the final NOTE. If the equation to the typical parabola is $\xi=-\eta-\lambda\eta^2$, the coefficient of η^2 in R, viz., $-(1+\lambda)$, $=-\frac{1}{2}\omega$.

Proceeding as before, we shall now find

$$R = -\cdot 2\eta_1 - 1\cdot 8\eta_1^2 - 0\cdot 6\eta_1^3, +0\cdot 2\eta_2 - 1\cdot 2\eta^2_2 - 0\cdot 4\eta^3_2$$
$$S = -\eta_1 + 0\cdot 6\eta_1^2 + 0\cdot 2\eta_1^3, -\eta_2 + 0\cdot 4\eta_2^2 + 0\cdot 1\dot{3}\eta^3_2.$$

The intersection of these curves forms the limit to the range of prices advantageous to both parties. If we leave out of account the terms in R and S which involve *third* powers of the η's, we may proceed as before to find the co-ordinates H_1 and H_2 of the intersection. They are respectively $0\cdot\dot{1}2\dot{7}$ and $0\cdot\dot{1}4\dot{5}$;[1] of the same order as the true values obtained by taking into account the third powers of the variables, namely, 0·1258 and 0·1438 respectively.

The values of H_1 and H_2 prove to be in this instance, as in the Lemma, considerably greater, roughly speaking about double those of η'_1 and η'_2, the co-ordinates which represent the prices affording maximum profit to the monopolist. For these I find :

By the summary method,

$$\eta'_1 = -0\cdot 0\dot{5},\ \eta'_2 = +0\cdot 08\dot{3};$$

Taking account of the subordinate cubic terms,

$$\eta'_1 = -0\cdot 05719,\ \eta'_2 = 0\cdot 08012.$$

Whether calculated by the true or the approximate method, the values of R', the monopolist's maximum gain by discrimination, and S', the consequent loss to the customers, prove to be much the same; and accordingly the relation between them not materially different. As thus :—

	R'.	$-S'$	$S' \div R'$
Approximate	0·01388	0·02315	1·667
Accurate ...	0·01378	0·02222	1·612

The approximate calculation may evidently be trusted as a verification of the second thesis.

Going on to the third thesis, I find approximately after the manner of the Lemma, for the prices which make the monopolist's profit a maximum subject to the conditions that the customer is not a loser (or gainer),

$$\eta_1'' = \eta_1' - 0\cdot 010044,\ \eta''_2 = \eta'_2 - \tfrac{2\frac{1}{1}}{16} = 0\cdot 010044$$

where η'_1 and η'_2 have the approximate values above found, namely, $0\cdot 0\dot{5}$ and $0\cdot 08\dot{3}$ respectively. Whence it is deducible that the gain which the monopolist must forgo in order not to occasion

[1] The calculation of the co-ordinates is facilitated by the circumstance that the point of intersection between the curves lies on the straight line $8\eta_1 + 7\eta_2 = O$. It happens (in this particular example) that this convenient proposition holds good for the true curves, including the cubic terms, as well as of the curves truncated by the omission of those terms.

loss to his customers is about ·0004, about 3 per cent. of the absolute maximum (above stated). To compare the true result with this approximate one would require a very laborious calculation. The following partial test must suffice. Assign to the ordinate η_2 a value less than that which affords the (true) maximum profit by an amount which the approximate investigation suggests; for example, put $\eta''_2 = 0·07$, less than η'_2 ($=0·08012$) by about 0·01. Now find that abscissa of the curve $S=O$ (roughly as to the general shape of that portion with which we are concerned illustrated by the curve *OSH* in Fig. 2), for which the ordinate 0·07. That abscissa is found to be $-0·06548$. Accordingly, $+0·07$ and $-0·06548$ represent prices for which the consumer's loss is null. But the gain which the monopolist forgoes by the adoption of those prices, say η''_1, η''_2 instead of η'_1, η'_2, is found (by substituting those values in the expression for R) to be a small percentage of R', namely, about 2 per cent. But that percentage, small as it is, exceeds the true percentage which would be obtained by using the true η''_1 and η''_2 instead of the assumed or "trial" values.

The peculiar interest of this example is that it is typical of an immense variety of demand-curves, or *functions* representing x, the amount demanded in terms of y, the price.[1] Very generally,

[1] The essence of the general reasoning may be indicated as follows. In the notation above employed we have for R, the gain of the monopolist through discrimination (cp. note to p. 450 above)

$$(1+\xi_1)(1+\eta_1)-1, \quad +(1+\xi_2)(1+\eta_2)-1\,;$$
$$= -\beta\eta_1-(1+\beta)\tfrac{1}{2}\omega\eta_1^2\cdots, \quad +\beta\eta_1-(1-\beta)\tfrac{1}{2}\omega\eta_2^2\cdots;$$

the *dots* indicating omission of terms involving higher powers. Whence for the prices affording maximum profit we have

$$\eta_1' = -\frac{\beta}{(1+\beta)\omega}, \qquad \eta_2' = \frac{\beta}{(1-\beta)\omega}.$$

Also the gain of the customers by discrimination

$$= -\eta_1 - \int_o^{\eta_1}\xi_1 d\eta_1, \quad -\eta_2 - \int_o^{\eta_2}\xi_2 d\eta_2$$
$$= -\eta_1 + \tfrac{1}{2}(1+\beta)\eta_1^2\cdots, \quad -\eta_2+\tfrac{1}{2}(1-\beta)\eta_2^2\cdots$$

Substituting in the expression for S the above-written values for η'_1 η'_2, we obtain for S'_1 the gain of the customers through discrimination

$$-\frac{2\beta^2}{\omega(1-\beta^2)}\left\{1-\frac{1}{2}\frac{1}{\omega}\right\}$$

Which will be negative in accordance with the second thesis, unless ω is small, $<\frac{1}{2}$.

To prove the third thesis consider R and S as functions of θ_1, θ_2, where $\theta_1 = \eta_1 - \eta'_1$, $\theta_2 = \eta_2 - \eta'_2$. Then the position of relative maximum as above defined must lie on the locus of common tangents to curves of the respective families $R = const.$, $S = const.$; that is

$$\frac{dR}{d\theta_1}\Big/\frac{dR}{d\theta_2} = \frac{dS}{d\theta_1}\Big/\frac{dS}{d\theta_2}.$$

Whence we obtain (R not involving the first powers of the θ's) $\theta_2 = q\theta_1\cdots$, where in virtue of presumptions above enunciated,[1] such a function may be expanded in ascending powers of y of the type

$$A + By + My^2, \ + Ny^3 \ . \ . \ . \qquad ,$$

with a coefficient M of such an order of magnitude in comparison with subsequent coefficients that, y being a small fraction, the first three terms of the expansion afford an approximation to the value of the function that is adequate for purposes like the present one. If a thesis like ours, not demanding numerical precision, is true of this approximation to the function, it is probably also roughly true of the function itself.

Of course, it must be presumed that the functions with which we are concerned are of an ordinary character—not discontinuous or otherwise abnormal. For example, suppose one of our demand-curves to have the following extraordinary form. Ascending from zero price the locus is a vertical line, say as far as P—it is the perpendicular from P on the abscissa—in Fig. 2. From P the locus is a horizontal line, the perpendicular from P on the ordinate. In this peculiar case our first thesis would be fulfilled; all the better, as there is avoided all *dead loss—perte sèche* in M. Colson's phrase—that is loss to the consumers, which is not gain to the monopolist. Also our second thesis would be eminently fulfilled; for it would be in the power of the monopolist now to charge prices (b_1 and b_2) by which not only one group of customers, but both groups, would have a bad bargain : Consumers' Surplus being theoretically zero or practically only just above it. But our third thesis in this peculiar case would fail altogether. Peculiar as it may seem, this example is not essentially different from one which is at least suggested by very high authority—the Dupuit-Colson type referred to on a previous page,[2] if the attributes there designated α and β are supposed predicable in their strictest form. We are presented with the conception of the area within the demand-curve resolvable into a series of separate columns—as it were so many sacks standing upright, each of which the monopolist can deplete down to any point which it pleases him to fix.[3]

q is a coefficient of the order unity. Substitute this value of θ_2 for θ_1 in the equation to zero of S, which is of the form

$$S' - A\theta_1 - B\theta_2 \cdots,$$

where S' is of the order β^2, A and B are of the order unity; we find the required value of θ_1 and therefore θ_2 to be of the order of β^2. But R is of the form $R' - (1+\beta)\omega\theta_1^2 - (1-\beta)\omega\theta_2^2 \cdots$ Therefore $R' - R''$ (the difference between the absolute and the relative maximum profit) is of the order β raised to the *fourth* power.

[1] *Ante*, p. 285. [2] Above, p. 443.

[3] As I interpret, there is supposed to be reached a stage of analysis at which the ordinary properties of a demand-curve break down ; much as the soap-bubble breaks

To return to probable matter, if the discrimination is not so complete as to suspend the ordinary properties of demand-curves, the theory above propounded may be considered as evident *à priori* in our sense of the term. Accordingly it does not stand in need of specific verification. Nevertheless, as even in mathematics seeing is believing, as the temperament of Didymus is prevalent among those whom I wish to persuade, I have thought it worth while to verify my theory by showing that it holds good for several different laws of demand. For this purpose I select four functions which are in very common use throughout applied mathematics.[1] There is first (A) the one most used and most useful of all, to evaluate which requires only the operations of arithmetic up to and including *Involution*; in short, the parabola—the common parabola, if no higher power than the second occurs. An example of this law has already been given. But it may be well to consider a second example of a variety less favourable to our (third) thesis.[2] Next (B) we shall place a function which requires *Evolution* so far as the extraction of the *square root*. Next comes

when the tenuity of the film approaches the dimensions of the constituent molecules. The distinguished economists who entertain this conception are aware of the impossibility of perfectly realising it in practice (cp. Dupuit, *Annales des Ponts et Chaussées*, 1842, vol. i, p. 222; Colson, *Cours*, vol. vi, p. 38. Cp. p. 227, par. 2).

[1] The following table exhibits the functions which are employed in two forms: the first referred to the zero of commodity and the zero of price as origin, and abbreviated by putting x for x/a, where x is any amount of commodity and a is that amount of which the sale affords maximum profit to the monopolist, and likewise putting y for y/b (cp. above, p. 446, pars. 1, 3). For the secondary form of the functions the point of which the co-ordinates are x=1 y=1 is taken as the origin and the co-ordinates are respectively

$$\xi \equiv \mathrm{x} - 1 (\equiv (x-a)/a)$$
$$\eta \equiv \mathrm{y} - 1 (\equiv (y-b)/b).$$

There is added in a third column a coefficient corresponding to M in the immediate context (to ω in the final NOTE), a coefficient which must be positive and is presumably not a very small fraction.

	x.	ξ.	$\frac{1}{2}\omega$.
A	$\frac{1}{4}(3-\mathrm{y})^2$	$-\eta + \frac{1}{4}\eta^2$	$\frac{3}{4}$
B	$\sqrt{3-2\mathrm{y}}$	$\sqrt{1-2\eta} - 1$	$\frac{3}{2}$
C	$-\log \mathrm{y}/e$	$-\log(1+\eta)$	$\frac{1}{2}$
D	$e^{1-\mathrm{y}}$	$e^{-\eta} - 1$	$\frac{1}{2}$

[2] Cp. the final NOTE.

(C) a function which is of wide application in physics, and even in economics has been frequently employed,[1] the *logarithm*. Then follows (D) the nearly related function, which is sometimes called the *anti-logarithm*.[2] I have experimented on these functions in the following uniform manner. I take a curve of the kind under consideration to represent the *average* law of demand, the *half* of the amount demanded at any assigned price, of both species of the commodity. To represent the demands separated by discrimination, I suppose this curve to be thus disturbed, or strained. To the value of x at any price, y, there is added the quantity $\beta\ (x-a)$ to constitute x_1, the demand for one species at that price; and from the value of x there is subtracted the quantity $\beta\ (x-a)$; where, as before, β is a (not large) proper fraction, a is the amount of commodity sold and β the price which affords maximum profit to the monopolist prior to discrimination. (The enunciation applies primarily to the tract of curve for which x is larger than a; for the tract beyond that point we may read $a-x$ for $x-a$, and interchange the words "addition" and "subtraction.") The fraction β is in each case determined so as to render the increase of the price that is raised equal to $12\frac{1}{2}$ per cent. of the original price.[3] I now determine an index of the range of prices that are mutually advantageous—those Pillars of Hercules, up to which, as explained with reference to our Lemma, the two parties can travel concurrently. Only it is not always convenient to find the actual position of the Straits; it suffices to find a point, as it were, on the African shore, as in example A, or even as in the other examples, a rock at some distance from that shore, on the Mediterranean side of the Straits. The limits so understood are given in the first column of the subjoined table. I then determine the prices which make the monopolist's profit a maximum, the (money-measure of) loss to the customers by the adoption of those prices, and compare the amount of that loss to the amount of the monopolist's profit when maximised. The percentage given by that comparison forms the entry in the second column. Further, I find a pair of prices which, while not causing any loss to the customers, yet require the monopolist to forgo only a very small quantity of his (possible maximum) profits. The amount thus forgone, as a percentage of the total profit

[1] To represent the law of diminishing returns and the law of diminishing utility.

[2] The Napierian logarithm, being the ordinary logarithm multiplied by the constant 0·434. . .

[3] It might have been somewhat more elegant, but it would have been considerably more troublesome, to *assign* the coefficient β (as in the treatment of the Lemma) and thence *compute both* the changes of price.

obtained by discrimination, forms the entry in the third column.

Though I have expended much labour on these calculations, yet, as they are long and delicate, I can hardly hope to have entirely avoided mistakes. Especially the decimals in the Table here following and the final and penultimate places of the decimal in the Table of Materials given in the Notes, are open to suspicion. But I am sure that the computation is quite accurate enough to verify propositions in Probabilities.

Table[1] showing verifications of the three theses :—

(1) Changes of price advantageous to both parties; per cent. of the price before discrimination.

Law of Demand.	1 −	1 +	2	3
A	18·5	22·3	201	2·5
B	18	22	98	2·5
C	18	22	200·5	1·15
D	20	24	309	1·7

[1] The subjoined table presents the materials from which the table in the text is constructed, namely,

β, the coefficient of discrimination ;

$(-H_1, +H_2)$ changes of price advantageous to both parties ;

$(-\eta'_1, \eta'_2)$ prices rendering the monopolist's gain by discrimination a maximum ;

R', the monopolist's gain by discrimination when a maximum ;

$-S'$, the loss to the customers by discrimination when the monopolist's profit is a maximum ;

$(-\eta''_1, \eta''_2)$, prices in the neighbourhood of $(-\eta'_1, \eta'_2)$ at which the customers are neither gainers nor losers ;

R'', the monopolist's gain by discrimination when the prices are $(-\eta''_1, \eta''_2)$.

The prices are relative to the prices before discrimination ; the gains (and losses) are relative to the monopolist's profit before discrimination.

Designation of function.	β.	$-H_1$.	H_2.	$-\eta'_1$.	η_2.	$-S'$.	R'.	$-\eta''_1$.	η''_2.	R''.
A	0·149502	0·1856	0·2228	0·08324	0·125	0·03109	0·01547	0·0952	0·105	0·01508
B	0·30217	0·18	0·22	0·08356	0·125	0·03139	0·03213	0·095	0·105	0·03134
C	0·0953199	0·18	0·22	0·090918	0·125	0·0227	0·01132	0·099	0·11	0·01119
D	0·099383	0·2	0·24	0·08316	0·125	0·03117	0·01007	0·1032	0·115	0·0099

(2) Loss to the customers by discrimination when the monopolist's gain thereby is a maximum; per cent. of the monopolist's maximum profit.

(3) Percentage of maximum profit resigned by the monopolist to avoid loss to the customers by discrimination.

The table in the text is thus formed out of the materials.

Column 1 shows H_1 and H_2 each multiplied by 100.

Column 2 shows $-S'/R'$, multiplied by 100.

Column 3 shows $(R'-R'')/R'$, multiplied by 100.

Is this multiplication of tests like using several triangles of different shapes in order to prove one of Euclid's propositions relating to triangles in general? Or, rather, have we made a contribution towards ascertaining by induction, less roughly than is given by *à priori* evidence, a limit up to which for purposes like ours fractions may be treated as *small*?[1]

Having secured this central position, we can now easily extend the territory subject to our laws; removing limitations by which it has hitherto been circumscribed.

So far we have supposed that prior to discrimination the two categories of consumers were equally important to the monopolist, the amount demanded by each at the single price being the same. Now let us recall this assumption; and, beginning with the Lemma, suppose that at the price b ($=PQ$ in Fig. 2) the amount demanded by one group of consumers is $a(1+\alpha)$, while the amount of the other species demanded is a $(1-\alpha)$. The first demand corresponds to RP_2 in the figure if α is positive, the second to RP_1 ($P_1P=PP_2=a\alpha=\alpha$, if $a=1$). If the specific demand-curves consisted respectively of the lines joining B to P_1 and P_2 (and produced) there would be no discrimination; the two new prices would be identical with the old price, b. But we are to suppose that the dotted lines—not now passing both through P, but one through P_1, another through P_2—are so inclined as to cause a dissilience of prices when the constraining condition that there should be only one price for the whole class is removed. Is it now probable that the consumers as a whole will suffer by the

[1] Compare Mr. Bickerdike's observation with reference to his theory of "incipient taxes" (ECONOMIC JOURNAL, 1907, p. 101), "Rather strong assumptions have to be made as to the elasticity of foreign demand and supply if the rate of the tax affording maximum advantage is to come below ten per cent."

As I understand (cp. ECONOMIC JOURNAL XVIII, p. 399 *et seq.*), the quantity with which the writer is concerned, the net gain to the home country consequent upon a small customs-duty, takes the form $Lx-Mx^2\cdots$; where L is proportionate to the amount of commodity taxed, x is the rate of taxation per unit of commodity; M is such a coefficient as the M described in our text. Or as L must be considered as varying with $x=$ say $L'-Nx\cdots$, we may write the quantity under consideration $L'-M'x^2$ ($M'=M+N$). The value for which this expression is a *maximum* (approximately $\frac{1}{2}L'/M'$) is probably much smaller than the limit up to which the expression is positive.

monopolist's using his power of discrimination so as to make his profits a maximum?

Common-sense will perhaps prejudge this question; pointing to instances in which a railway manager may afford a special rate to exceptional classes of travellers (excursionists and so forth). If the general scale of rates is not disturbed by the favour granted to the occasional passengers, if the one species is advantaged and the other is not affected, there must result advantage to the class as a whole.

Doubtless, I reply, in the extreme case of inequality where the demand of the class favoured by discrimination was so small prior to the discrimination as not sensibly to affect the rates fixed for other classes; for instance, the demand of the workmen for the use of the foot-bridge in the second of the illustrations above cited from Dupuit.[1] But we are not now considering extreme[2] cases, but cases in which α—the measure of inequality—is a proper fraction and primarily at least a small one. For instance, in the first of Dupuit's illustrations, suppose (what was, perhaps, not his meaning) that of the ten hectolitres of water which are demanded when the (single, undiscriminated) price is 10 francs per hectolitre per annum, six are required for *internal* use and four for *external* use;[3] and that both demands expand when price falls. In such a case are the consumers as a whole likely to suffer by discrimination? The answer given by mathematics to a question in the theory of Monopoly is often not that which is expected by common-sense.

As before, let us put ξ_1, ξ_2 for the proportional or relative changes in demand respectively consequent on the relative changes of price η_1 and η_2. Then we may write

$$\xi_1 = -\alpha - (1 - \alpha + \beta)\eta_1,$$
$$\xi_2 = +\alpha - (1 + \alpha - \beta)\eta_1;$$

simpliciter in the case of the Lemma, or with the addition of terms involving second powers of the η's to fit the more general type. Forming the general expressions for the Monopoly Revenue and the Consumers' Surplus we find that, as long as α and β remain small fractions, the triple thesis is fulfilled nearly as well as when we dealt with β only. Now, likewise, as either of the coefficients becomes large, the second thesis, that the monopolist tends to fix a set of prices prejudicial to the customers, ceases to be qualified by the third thesis, that his interest in their detriment is small.[4] The second hypothesis retains some

[1] Cp. above, p. 442. [2] Cp. ante p. 284 *et seq.* [3] Above, p. 441.

[4] See observations on the Lemma above, p. 450, par. 2.

probability even when the coefficients are considerable; in the absence of knowledge that the forms of the functions with which we have to deal—the higher powers of the variables which now come into play—are unfavourable to the thesis. We are, of course, here, as throughout, contemplating the money-measure of Consumers' Surplus; not taking into account that the consumers on a small scale may be the poor and needy.

If it is given that α is very large (nearly unity) then the exception[1] enounced in connection with Thesis 2 occurs. But if nothing is given about the coefficients, then we may still affirm the thesis in a certain *à priori* sense.[2] No doubt this kind of probability is not so useful as that which obtains when it is given that conditions favourable to the theses, such as the smallness of both α and β, are realised in the particular case with which we have to deal.

These considerations are readily extended to the general case.[3]

A further extension of our laws is effected by removing the condition that the commodities in which the monopolist deals should be, like the mineral waters in Cournot's classical illustration, unaffected by cost of production. First, let us make the simplest supposition, that there is a uniform cost of production for all articles of the class considered without regard to the species into which it may be discriminated, or to the total amount produced. This simple case may be represented by measuring in Fig. 2 the net price on which the monopolist's profits are calculated, no longer from the abscissa, but from a horizontal line at a distance from the abscissa, say $O\omega$, which represents the cost of production per unit.[4] The position of maximum profit (prior to discrimination) will now be given by bisecting $\omega\alpha$ and $\omega\beta$, instead of OA and OB. The units of the system being the same as before, the price and amount will not now be each unity. Or if we take the new price and the new amount as the units (in which lengths on

[1] This exception deserved to be specified on account of its importance in practice; the attribute by which it is defined—the ratio between the amounts demanded before discrimination—being commonly capable of identification. Theoretically, other exceptions have a right to be enounced; for instance, the case when β is (known to be) large, or γ (below note 3), or ω small (final NOTE).

[2] Compare the argument employed in the ECONOMIC JOURNAL, 1908, p. 551.

[3] In general there may be any number of coefficients of discrimination in addition to the α and β which have been introduced. Prior to discrimination let

$$\xi(=x/(a-1)= -\eta+\lambda\eta^2+\mu\eta^3+\nu\eta^4\cdots$$

After discrimination

$$\xi_1=\pm\alpha-(1\pm\beta)\eta_1+(1\pm\gamma)\lambda\eta_1^2+(1\pm\delta)\mu\eta_1^3\cdots;$$
$$\xi_2=\mp\alpha-(1\mp\beta)\eta_2+(1\mp\gamma)\lambda\eta_2^2+(1\mp\delta)\mu\eta_2^3\cdots.$$

M. Colson employs largely an equivalent construction.

the axes are respectively measured), the demand-curve referred to the new position of maximum as origin is no longer $\xi = -\eta$, but $\xi = -q\eta$, where q is a coefficient greater than unity.[1] The essential character of the reasoning is not altered by the modification of the data. Nor is that character altered when, instead of $k_1x_1 + k_2x_2$, representing the total cost of producing the quantities of the specific commodities x_1 and x_2, at the respective rates per unit k_1 and k_2, we have to add a term such as $\pm l_1x_1^2 \pm l_2x_2^2$;[2] where the positive sign corresponds to the *law of diminishing returns*, the negative sign to the *law of increasing returns*; nor when we add a term such as $-l_{12}xy$,[3] corresponding to *joint cost*.[4]

The reader will observe what a subsidiary *rôle* is here assigned to *joint cost*; which some distinguished writers on Railway Economics seem to emphasise as the principal cause of discrimination. Joint cost is no doubt favourable to discrimination; but there is a more essential condition, unity of management, monopoly.[5]

A further extension is effected by removing the condition that the specific demands should be uncorrelated. The character of the reasoning is not essentially altered by this alteration in the data. The principal difference in the result may thus be expressed. Whereas previously the amount of profits which the monopolist must forgo in order that the customers should not lose or should even gain by discrimination was (approximately) a quantity of the form $A\theta_1^2 + B\theta_2^2$, where A and B are coefficients of the order unity (roughly speaking), θ_1 and θ_2 are of the order β^2 (β being a small, or rather not large, fraction); now there is added

[1] If k is the cost per unit in the new notation according to which the value of x and the value of y which afford a maximum under the new circumstances are now taken as units; then q may be deduced from the condition that $(1+\xi)(1+\eta) - k(1+\xi), = (1-k) + \eta - q\eta - q\eta^2 + kq\eta$, should be a maximum when $\xi = o$, $\eta = o$. Whence it is deducible that $q = 1/(1-k)$.

[2] The "l's," as well as the "k's," being positive.

[3] l_{12} being positive. The proposition is, of course, equally true when this coefficient is negative; that is, in the less frequently specified case of rival production (cp. ECONOMIC JOURNAL, vol. vii, p. 54, par. 1).

[4] Nor when higher powers of the variables occur; with the usual assumptions as to the magnitude of their coefficients.

[5] It may be objected that discrimination arises without monopoly in the case of large establishments; for instance, when an hotelkeeper discriminates between wines of different species, though his profits are subject to competition with other hotelkeepers. But I submit that he can practise discrimination just because he enjoys a certain degree of monopoly. If the wines were sold separately by open competition, if there was on the spot a sherry-market and a port-market, the prices paid by the customers would each of them—instead of as now on an average, summed up in the hotel-bill—conform to the cost of production.

to this expression a new term of the same order as the others, $C\theta_1\theta_2$.[1]

I need not point out in detail that most of the propositions above predicated of the Lemma and the simple type are true of the generalised conditions. Enough has been said to show that these propositions hold good through a wide range of circumstances, with as much truth as can be expected of a theory which belongs at once to Mathematical Economics and to the Calculus of Probabilities. Indeed, I am disposed to claim for the theory a greater degree of practical importance than can generally be ascribed to those branches of study.

Mathematical economics serve generally to present comprehensive views as to the interdependence of variable quantities, rather than to solve particular problems; as Professor Pareto has

[1] Let us begin with the simple case of linear laws of demand, amounts of the two species demanded before discrimination equal, and no cost of production. Let x_1, y_1 denote the quantities demanded after discrimination at the prices y_1, y_2; and let $\mathrm{x}_1 = x_1/a$, $\mathrm{y}_1 = y_1/b$, $\mathrm{x}_2 = x_2/a$, $\mathrm{y}_2 = y_2/b$, where b is the price before discrimination and a, *half* the quantity demanded at that price. We have now x_1, and likewise x_2, a linear function of both the y's. As thus:

$$\mathrm{x}_1 = p_1 - q_1\mathrm{y}_1 - r_1\mathrm{y}_2\,;\ \mathrm{x}_2 = p_2 - q_2\mathrm{y}_2 - r_2\mathrm{y}_1.$$

These coefficients are subject to certain conditions. The expressions for y_1 in terms of x_1 and x_2, and likewise for y_2, must be such that

$$\mathrm{y}_1 = \frac{dU}{d\mathrm{x}_1},\ \mathrm{y}_2 = \frac{dU}{d\mathrm{x}_2};$$

where U is a function of x_1, x_2 representing (the money-value of) the utility obtained from the consumption of the quantities of commodity x_1, x_2. Whence it is deducible that $r_1 = r_2 =$, say, ρ. If now we put $\mathrm{x}_1 = 1 + \xi_1$ $\mathrm{x}_2 = 1 + \xi_2$ $y_1 = 1 + \eta_1$ $y_2 = 1 + \eta_2$, we have

$$\xi_1 = -q_1\eta_1 - \rho\eta$$
$$\xi_2 = -\rho\eta_1 - q_2\eta_2.$$

Whence

$$R, = \mathrm{x}_1\mathrm{y}_1 - 1 + \mathrm{x}_2\mathrm{y}_2 - 1,$$
$$= \eta_1(1 - q_1 - \rho) + \eta_2(1 - q_2 - \rho) + \xi_1\eta_1 + \xi_2\eta_2.$$

Now when η_1 is constrained to be equal to η_2, that is before discrimination when there is only one price for both species of commodity, say, $1 + \eta$, we know by construction that R is a maximum when $\eta = 0$. Equating the coefficient of η to zero, we have $2 - (q_1 + q_2) - 2\rho = 0$; $q_1 + q_2 = 2 - 2\rho$; say, $q_1 = 1 + \beta - \rho$, $q_2 = 1 - \beta - \rho$.

Substituting the equivalents for q_1 and q_2 in the above-written expression for ξ_1 ξ_2, we have now to determine the values of η_1 and η_2 for which R is a maximum. They prove to be $\eta'_1 = -\frac{1}{2}\beta(1 - \beta)/\Delta$, $\eta'_2 = +\frac{1}{2}\beta(1 + \beta)/\Delta$; where $\Delta = 1 - \beta^2 - 2\rho$. (The corresponding values for ξ are as in the simple case, $\xi'_1 = \frac{1}{2}\beta$, $\xi'_2 = -\frac{1}{2}\beta$.) These values are now to be substituted in the expression for S, that is the difference between the consumer's surplus as it is after discrimination and as it was before, that is the difference between $U - \mathrm{x}_1\mathrm{y}_1 - \mathrm{x}_2\mathrm{y}_2$, as it is when we put for the x's and y's their values in term of the ξ's and η's ($\mathrm{x}_1 = 1 + \xi_1$, etc.), and what it becomes when we put for each of x's and y's *unity*. Substituting and reducing we have the expression for S in terms of the ξ's and η's. Substitute in this expression the above-written values of η'_1 and η'_2 (ξ'_1 and ξ'_2), and there results as before the value of S' proportional to the *second* power of β. The reasoning may thence be pursued on the lines traced in the note to the typical case (above, p. 452).

recently pointed out in this JOURNAL.[1] But I submit that there is an exception to this general limitation; that mathematics play a more direct part in the theory of monopoly. What if an exception should be formed by the application of the preceding theorems to one of the doctrines propounded by Professor Pareto himself—not certainly a particular problem, yet a general view which purports to be of direct practical significance. I refer to his argument directed against Socialism, that at best it would not essentially alter the distribution and production of wealth. "Economic goods will be distributed according to the rules which we have discovered in studying a *régime* of competition. . . ." "Prices reappear," or "will at most change their name."[2] But we have seen that a regulated discrimination of prices, such as might conceivably be practised by a Socialist Directory, but is not possible in a *régime* of competition, tends to increase the sum-total of utility. A conception still less familiar to popular Socialism is suggested by what may be called the *external case* of our theory, that which is presented when "monopolist" is interpreted to mean sole *buyer*. The suggestion is that to discriminate between labourers on grounds other than efficiency—not always to pay the same wages for the same amount of work done—might diminish the "dead loss" of Producers' Surplus which the contrary policy involves.[3]

But if this advantage is either of a negligible order in relation to the stupendous consequences of a Socialist revolution, or is over-balanced by the liability to enormous abuses; may we not hope for a less precarious application to a more familiar kind of monopoly, the control of railways and generally public works?[4] That hope is justified by experience. For the mathematical principles on which our reasoning is mainly based are actually applied under the skilful direction of M. Colson to the railway policy of France. Such is the proposition that a small reduction of price, so small as to cause a very small sacrifice of profit to the monopolist, is likely to be attended with considerable relief to the customers. Our third thesis but superadds to this received proposition the following one:—In the case of discrimination (in certain not unusual circumstances) the relief to the customers afforded by a small sacrifice of the monopolist's profits is likely to be so considerable that they will be gainers, or at least not losers, by the introduction of discrimination.

[1] In his appreciative tribute to the memory of Walras, March, 1910.

[2] *Cours d'Économie Politique*, p. 1014 and context.

[3] Above, p. 453, *et passim*.

[4] In the sense of the term in which it is employed by M. Colson.

It is true that these propositions are but probable; liable to failure in particular cases. But we are not altogether dependent on the more precarious kind of *à priori* probability, that which is exemplified by the predication of our second thesis[1] in the absence of data as to the extent and elasticity of demand. Such data would often be available sufficiently to show what case we had to deal with. The sun of full knowledge may illuminate part of our course. There may be enough of that daylight to enable us at least to select the proper path; which may then be pursued in safety by the starlight of Probabilities.

NOTE.—*On certain coefficients.* The first differential coefficient of a monopolist's profit have an interesting relation to the elasticity of his customers' demand. The former coefficient may be written, in our notation, when there is no cost of production,

$$x+y\frac{dx}{dy}=x\left(1+\frac{y}{x}\frac{dx}{dy}\right)=x(1-e);$$

where e is the elasticity as defined by Professor Marshall (*Principles of Economics*). When cost of production is taken into account the expression becomes $x(1-c-e)$; where c is the cost per unit.

This proposition may be employed to prove the theorem above enounced (p. 449), that when a monopolist discriminates between different species of custom, subject to the conditions that the subtraction from (or addition to) the benefit of his customers as a whole should be nil, or have any other assigned value, the elasticity of demand is the same for the different species which are discriminated (cost of production being null or constant). For consider the Consumers' Surplus, say W, as the difference between the money-measure of the utility resulting from the consumption of the commodities, and the purchase-money thereof, we have, in the case of two species of commodity,

$$W=\int_0^{x_1} ydx_1+\int_0^{x_2} ydx_2-x_1y_2-x_2y_2, \;=\; -\int_0^{y_1} x_1dy_1-\int_0^{y_2} x_2dy_2.$$

Likewise the profit of the monopolist, say V, is, in the absence of cost of production, $x_1y_1+x_2y_2$. Now the quantity which the monopolist aims at maximing is $V+\lambda W$; where λ is the indeterminate coefficient proper to problems of relative maximum. We have accordingly

$$\frac{d}{dx_1}(V+\lambda W)=o\,;\quad \frac{d}{dx_2}(V+\lambda W)=o\,;$$

whence $e_1=1-\lambda=e_2$. The proposition continues to hold good when the cost of production per unit is a constant other than zero, but loses its simplicity when the cost (per unit) involves the variables. It may be remarked that the property of equal elasticities is also characteristic of another kind of discrimination which may seem particularly suitable for a State Monopoly to practise, namely, that regulation of prices which has for its object the maximum benefit to the purchasers as a whole, consistent with the retention by the monopolist of a fixed profit—a fixed amount, or a fixed percentage of the output, that is of the cost of production, supposing cost to be constant.

[1] Above, p. 459.

The affinity between elasticity and the increment of monopoly profits extends to the second order of differentials. Putting $V = xy$ we have $\frac{1}{x}\frac{dV}{dy}$ (or is it more elegant to write $\frac{y}{V}\frac{dV}{dy}$?) $= 1 - e$ (for any value of the variable). Accordingly, $\frac{1}{x}\frac{d^2V}{dy^2} = -\frac{de}{dy}$, at the point of maximum, since then $\frac{dV}{dy} = 0$. This coefficient, or rather its negative, namely $\frac{de}{dy}$, is identical with our ω which, as will have been observed, plays an important *rôle* in the theory of monopoly. (*Mutatis mutandis* when cost of production enters.) The coefficient ω is necessarily positive and presumably not (often) very small. The smaller it is, the sooner, as we continue to increase the degree of discrimination, the extent to which prices are varied, in the limit reached at which our third thesis breaks down. Thus in the second of the two parabolas above instanced, a smaller value of η'_2 (β and $-\eta'$) will cause the third term of the expansion to become comparable with the second in the case of the second parabola, for which $\omega = \frac{3}{2}$, than in the case of the first parabola, for which $\omega = 3$.

The smallness of ω likewise causes trouble in the exemplification of the theorem that the tax on one of two monopolised articles for which the demand is correlated, may result in the *fall* of *both* prices. The range within which fractions of maximum monopoly profit may present this remarkable property is restricted by the conditions that $\omega_1, \omega_2, \omega_1 \omega_2 - \rho_2$ should each be positive. The last of these conditions was not attended to in the example given in the text (ante p. 298). There is there instanced a fall of both prices which is more profitable to the (taxed) monopolist than the maintenance or increase of the original prices. But in order to make sure that there is not a still more profitable position, at which one at least of the prices is higher than originally, there is required some additional postulate as to the form of the surface representing the monopolist's profit. We should be quite within our rights in making such an assumption —the rather as *some* modification of the simple algebraic functions adopted must be supposed ultimately to set in (ante p. 303, par. 2). But as remarked in a note, when it was too late to alter the text without more urgent reason—a better example would have been formed by a lower rate of taxation (p. 298). For instance, in the case supposed, let there be imposed a tax of 1*s*., more exactly 1*s*. $\frac{3}{8}$*d*. per first-class ticket (theoretically ·05157 £1, that being the tax which causes an increase of third-class passengers from 200,000 to exactly 214,000—for convenience of calculation, I *started* with the addition to the number of passengers, not with the rate of the tax). Then, I find, the monopolist's profits will be at a (genuine) maximum when the first-class fare (originally £1) is reduced by over $3\frac{3}{4}$*d*. and the third-class fare (originally 10*s*.) by about the same figure.

One more coefficient calls for one more remark: elasticity, in the popular sense, that is $F'(p)$ in Cournot's notation, $\frac{dx}{dy}$ in ours. The sort of reader who is content with this usage may be apt to think that the distinction (ante p. 290) which we have emphasised between elasticity and the increment thereof is a refinement of no great practical importance, that what is true of the increment is true enough of the quantity supposed to increase. It may be well, therefore, to point out that between the increment (first differential coefficient) of a variable and the variable itself, there may be all the difference that there is between the velocity at which a body is moving and the distance through which it has moved. Contrast the following propositions:—(1) The higher the speed of a motor car the greater is the danger of accidents; (2) the longer the distance (from any fixed point) that a motor car has travelled (at whatever rate), the greater is the danger of accidents. The former presumption could doubtless be verified by the statistics of accidents. Governments are well advised in making regulations based on this presumption. But what should we think of an expert who advised Government to discourage motorists from travelling beyond a certain distance from, say, New York, in order to prevent accidents? That advice would be of a piece with the theory which predicates of elasticity what is true of the increment of elasticity. No doubt it may be a proof of great natural ability to approach and half discern the truth in such a matter without the aid of the appropriate mathematical conceptions.

F. Y. EDGEWORTH

MUST INVENTIONS REDUCE THE RATE OF INTEREST?

IN his profound work on the *Rate of Interest*, writing of the effect of inventions, Professor Irving Fisher says:—"The effect in raising interest lasts only so long as the resulting income-stream is sufficiently distorted in time and shape to be of a decidedly ascending type . . . . later, however, there will come a time when the income-stream ceases to ascend, when all the necessary investment has been completed, when no further exploitation is possible, and when it is only necessary to keep up the newly constructed capital at a constant level. When this period is reached, the after-effect of the invention will be felt. Society will then have a larger income-stream than before, but no longer an ascending one. A mere increase in the *size* of the income-stream, while its shape remains constant, has the effect, as we have seen, not of increasing, but of somewhat decreasing the rate of time-preference. Consequently the after-effect of all inventions and discoveries is not to increase but to decrease the rate of interest."[1] I shall argue that inventions and discoveries need not cause a diminution of the rate of interest ultimately.

The doctrine that time preference falls as income rises, that is to say, that the proportion of income saved at a given rate of interest rises as income rises, is derived from the proposition that when people are very poor they either cannot save or are strongly tempted to act irrationally and save insufficiently. Were it not for the disturbing effects of poverty Professor Irving Fisher would hold that time preference is

[1] *The Rate of Interest*, pp. 203–204.