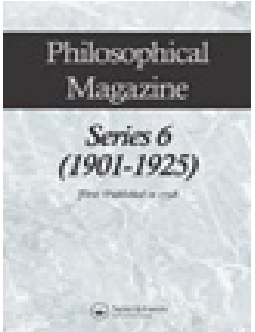

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XII. On the approximate solution of certain problems relating to the potential.—II

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XII. *On the Approximate Solution of certain Problems relating to the Potential.*—II. *By* Lord RAYLEIGH, *O.M., F.R.S.**

THE present paper may be regarded as supplementary to one with the same title published a long while ago †. In two dimensions, if ϕ, ψ be potential and stream-functions, and if (*e. g.*) ψ be zero along the line $y=0$, we may take

$$\phi = \int f\,dx - \frac{y^2}{1\,.\,2} f' + \frac{y^4}{1\,.\,2\,.\,3\,.\,4} f''' - \quad . \quad . \quad (1)$$

$$\psi = yf - \frac{y^3}{1\,.\,2\,.\,3} f'' + \frac{y^5}{1\,.\,2\,.\,3\,.\,4\,.\,5} f^{\text{iv}} -, \quad . \quad (2)$$

f being a function of x so far arbitrary. These values satisfy the general conditions for the potential and stream-functions, and when $y = 0$ make

$$d\phi/dx = f, \qquad \psi = 0.$$

Equation (2) may be regarded as determining the lines of flow (any one of which may be supposed to be the boundary) in terms of f. Conversely, if y be supposed known as a function of x and ψ be constant (say unity), we may find f by successive approximation. Thus

$$f = \frac{1}{y} + \frac{y^2}{6}\frac{d^2}{dx^2}\left(\frac{1}{y}\right) + \frac{y^2}{36}\frac{d^2}{dx^2}\left\{y^2\frac{d^2}{dx^2}\left(\frac{1}{y}\right)\right\} - \frac{y^4}{120}\frac{d^4}{dx^4}\left(\frac{1}{y}\right). \quad (3)$$

We may use these equations to investigate the stream-lines for which ψ has a value intermediate between 0 and 1. If η denote the corresponding value of y, we have to eliminate f between

$$1 = yf - \frac{y^3}{6} f'' + \frac{y^5}{120} f^{\text{iv}} - \ldots,$$

$$\psi = \eta f - \frac{\eta^3}{6} f'' + \frac{\eta^5}{120} f^{\text{iv}} - \ldots;$$

whence

$$\eta = \psi y + \frac{f''}{6}(y\eta^3 - \eta y^3) - \frac{f^{\text{iv}}}{120}(y\eta^5 - \eta y^5),$$

* Communicated by the Author.

† Proc. Lond. Math. Soc. vii. p. 75 (1876); Scientific Papers, vol. i. p. 272.

or by use of (3)

$$\eta=\psi y+\frac{y^4(\psi^3-\psi)}{6}\frac{d^2}{dx^2}\left(\frac{1}{y}\right)+\frac{y^7(\psi^2-1)(3\psi^2-\psi)}{36}\left\{\frac{d^2}{dx^2}\left(\frac{1}{y}\right)\right\}^2$$

$$+\frac{y^4(\psi^3-\psi)}{36}\frac{d^2}{dx^2}\left\{y^2\frac{d^2}{dx^2}\left(\frac{1}{y}\right)\right\}-\frac{y^6(\psi^5-\psi)}{120}\frac{d^4}{dx^4}\left(\frac{1}{y}\right). \quad (4)$$

The evanescence of ψ when $y=0$ may arise from this axis being itself a boundary, or from the second boundary being a symmetrical curve situated upon the other side of the axis. In the former paper expressions for the "resistance" and "conductivity" were developed.

We will now suppose that $\psi=0$ along a *circle* of radius a, in substitution for the axis of x. Taking polar coordinates $a+r$ and θ, we have as the general equation

$$(a+r)^2\frac{d^2\psi}{dr^2}+(a+r)\frac{d\psi}{dr}+\frac{d^2\psi}{d\theta^2}=0. \quad . \quad . \quad (5)$$

Assuming

$$\psi=R_1r+R_2r^2+R_3r^3+\ldots, \quad . \quad . \quad . \quad . \quad (6)$$

where R_1, R_2, &c., are functions of θ, we find on substitution in (5)

$$\left.\begin{aligned}2a^2R_2+aR_1&=0,\\ 6a^2R_3+6aR_2+R_1+R_1''&=0;\end{aligned}\right\} \quad . \quad . \quad (7)$$

so that

$$\psi=R_1r-\frac{R_1r^2}{2a}+\frac{(2R_1-R_1'')r^3}{6a^2} \quad . \quad . \quad . \quad (8)$$

is the form corresponding to (2) above.

If $\psi=1$, (8) yields

$$R_1=\frac{1}{r}+\frac{1}{2a}-\frac{r}{12a^2}+\frac{r^2}{6a^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right), \quad . \quad . \quad (9)$$

expressing R_1 as a function of θ, when r is known as such. To interpolate a curve for which ρ takes the place of r, we have to eliminate R_1 between

$$1=R_1r-\frac{R_1r^2}{2a}+\frac{(2R_1-R_1'')r^2}{6a^2},$$

$$\psi=R_1\rho-\frac{R_1\rho^2}{2a}+\frac{(2R_1-R_1'')\rho^2}{6a^2}.$$

Thus

$$\rho = r\psi - \frac{R_1}{2a}(\rho r^2 - r\rho^2) + \frac{2R_1 - R_1''}{6a^2}(\rho r^3 - r\rho^3),$$

and by successive approximation with use of (9)

$$\rho = r\psi + \frac{r^2}{a}\frac{\psi(\psi-1)}{1.2} + \frac{r^3}{a^2}\frac{\psi(\psi-1)(\psi-2)}{1.2.3}$$

$$+ \frac{r^4}{a^2}\frac{\psi(\psi^2-1)}{6}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The significance of the first three terms is brought out if we suppose that r is constant (α), so that the last term vanishes. In this case the exact solution is

$$\log\frac{a+\rho}{a} = \psi\log\frac{a+\alpha}{a}, \quad . \quad . \quad . \quad (11)$$

whence

$$\frac{\rho}{a} = \left(\frac{a+\alpha}{a}\right)^\psi - 1 = \psi\frac{\alpha}{a} + \frac{\psi(\psi-1)}{1.2}\frac{\alpha^2}{a^2}$$

$$+ \frac{\psi(\psi-1)(\psi-2)}{1.2.3}\frac{\alpha^3}{a^3} + \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

in agreement with (10).

In the above investigation ψ is supposed to be zero exactly upon the circle of radius a. If the circle whose centre is taken as origin of coordinates be merely the circle of curvature of the curve $\psi = 0$ at the point ($\theta = 0$) under consideration, ψ will not vanish exactly upon it, but only when r has the approximate value $c\theta^3$, c being a constant. In (6) an initial term R_0 must be introduced, whose approximate value is $-c\theta^3 R_1$. But since R_0'' vanishes with θ, equation (7) and its consequences remain undisturbed and (10) is still available as a formula of interpolation. In all these cases, the success of the approximation depends of course upon the degree of slowness with which y, or r varies.

Another form of the problem arises when what is given is not a pair of neighbouring curves along each of which (*e. g.*) the stream-function is constant, but *one* such curve together with the variation of potential along it. It is then required to construct a neighbouring stream-line and to determine the distribution of potential upon it, from which again a fresh departure may be made if desired. For this purpose we regard the rectangular coordinates x, y as functions of ξ (potential) and η (stream-function), so that

$$x + iy = f(\xi + i\eta), \quad . \quad . \quad . \quad . \quad (13)$$

in which we are supposed to know $f(\xi)$ corresponding to $\eta = 0$, *i. e.*, x and y are there known functions of ξ. Take a point on $\eta = 0$, at which without loss of generality ξ may be supposed also to vanish, and form the expressions for x and y in the neighbourhood. From

$$x + iy = A_0 + iB_0 + (A_1 + iB_1)(\xi + i\eta) + (A_2 + iB_2)(\xi + i\eta)^2 + \ldots,$$

we derive

$$x = A_0 + A_1\xi - B_1\eta + A_2(\xi^2 - \eta^2) - 2B_2\xi\eta + A_3(\xi^3 - 3\xi\eta^2) - B_3(3\xi^2\eta - \eta^3) + A_4(\xi^4 - 6\xi^2\eta^2 + \eta^4) - 4B_4(\xi^3\eta - \xi\eta^3) + \ldots,$$

$$y = B_0 + B_1\xi + A_1\eta + 2A_2\xi\eta + B_2(\xi^2 - \eta^2) + A_3(3\xi^2\eta - \eta^3) + B_3(\xi^3 - 3\xi\eta^2) + 4A_4(\xi^3\eta - \xi\eta^3) + B_4(\xi^4 - 6\xi^2\eta^2 + \eta^4) + \ldots.$$

When $\eta = 0$,

$$x = A_0 + A_1\xi + A_2\xi^2 + A_3\xi^3 + A_4\xi^4 + \ldots$$

$$y = B_0 + B_1\xi + B_2\xi^2 + B_3\xi^3 + B_4\xi^4 + \ldots$$

Since x and y are known as functions of ξ when $\eta = 0$, these equations determine the A's and the B's, and the general values of x and y follow. When $\xi = 0$, but η undergoes an increment,

$$x = A_0 - B_1\eta - A_2\eta^2 + B_3\eta^3 + A_4\eta^4 - \ldots, \quad (14)$$

$$y = B_0 + A_1\eta - B_2\eta^2 - A_3\eta^3 + B_4\eta^4 + \ldots, \quad (15)$$

in which we may suppose $\eta = 1$.

The A's and B's are readily determined if we know the values of x and y for $\eta = 0$ and for equidistant values of ξ, say $\xi = 0$, $\xi = \pm 1$, $\xi = \pm 2$. Thus, if the values of x be called x_0, x_{-1}, x_1, x_2, x_{-2}, we find

$$A_0 = x_0, \qquad \text{and}$$

$$A_1 = \frac{2}{3}(x_1 - x_{-1}) - \frac{1}{12}(x_2 - x_{-2}), \quad A_3 = \frac{x_2 - x_{-2}}{12} - \frac{x_1 - x_{-1}}{6},$$

$$A_2 = \frac{2(x_1 + x_{-1} - 2x_0)}{3} - \frac{x_2 + x_{-2} - 2x_0}{24},$$

$$A_4 = \frac{x_2 + x_{-2} - 2x_0}{24} - \frac{x_1 + x_{-1} - 2x_0}{6}.$$

The B's are deduced from the A's by merely writing y for x throughout. Thus from (14) when $\xi = 0$, $\eta = 1$,

$$x = x_0 - \frac{5}{6}(x_1 + x_{-1} - 2x_0) + \frac{1}{12}(x_2 + x_{-2} - 2x_0)$$
$$- \frac{5}{6}(y_1 - y_{-1}) + \frac{1}{6}(y_2 - y_{-2}). \quad . \quad . \quad . \quad (16)$$

Similarly

$$y = y_0 - \frac{5}{6}(y_1 + y_{-1} - 2y_0) + \frac{1}{12}(y_2 + y_{-2} - 2y_0)$$
$$+ \frac{5}{6}(x_1 - x_{-1}) - \frac{1}{6}(x_2 - x_{-2}). \quad . \quad . \quad . \quad (17)$$

By these formulæ a point is found upon a new stream-line ($\eta = 1$) corresponding to a given value of ξ. And there would be no difficulty in carrying the approximation further if desired.

As an example of the kind of problem to which these results might be applied, suppose that by observation or otherwise we know the form of the upper stream-line constituting part of the free surface when liquid falls steadily over a two-dimensional weir. Since the velocity is known at every point of the free surface, we are in a position to determine ξ along this stream-line, and thus to apply the formulæ so as to find interior stream-lines in succession.

Again (with interchange of ξ and η) we could find what forms are admissible for the second coating of a two-dimensional condenser, in order that the charge upon the first coating, given in size and shape, may have a given value at every point.