AN EXTENSION OF A THEOREM ON OSCILLATING SERIES

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1. Suppose that $$\lambda_1,\ \lambda_2,\ \lambda_3,\ \ldots$$
is an ascending sequence of positive* numbers increasing beyond all limit with n. Let
$$C(\omega) = \sum_{\lambda_n \leqslant \omega} c_n,$$
so that $$C(\omega) = c_1 + c_2 + \ldots + c_p,$$
if $\lambda_p \leqslant \omega < \lambda_{p+1}$; and let
$$C^{\kappa}(\omega) = \sum_{\lambda_n \leqslant \omega} (\omega - \lambda_n)^{\kappa} c_n,$$
κ being any positive number. It will easily be verified that
$$C^{\kappa}(\omega) = \kappa \int_0^{\omega} C(\tau)(\omega - \tau)^{\kappa-1}\, d\tau.$$
Then, if $$\omega^{-\kappa} C^{\kappa}(\omega) \rightarrow C,$$
as $\omega \rightarrow \infty$, we shall say that the series Σc_n is *summable* (R, λ, κ)† *to sum* C.

If $\lambda_n = n$, this definition of the sum of an oscillating series is equivalent to Cesàro's.‡

* λ_1 may be zero.

† That is to say, *by Riesz's means of type* λ *and order* κ. These methods of summation were introduced by M. Riesz in a note in the *Comptes Rendus* of 5 July, 1909. A more systematic account of them, and of their applications to the theory of Dirichlet's series, will be given in the *Cambridge Tract* on "The General Theory of Dirichlet's Series" that Dr. Riesz and I are now preparing in collaboration.

‡ Riesz, *Comptes Rendus*, 12 June, 1911. When I speak of Cesàro's methods of summation I include the methods of non-integral order whose theory has been developed by Knopp and Chapman (Knopp, *Inaugural Dissertation*, Berlin, 1907; Chapman, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 369).

If, in the general expression for $C^\kappa(\omega)$, we take $\kappa = 1$, $\omega = \lambda_p$, we obtain

$$\frac{C^\kappa(\omega)}{\omega} = \frac{(\lambda_2-\lambda_1)C_1+(\lambda_3-\lambda_2)C_2+\ldots+(\lambda_p-\lambda_{p-1})C_{p-1}}{\lambda_p},$$

where
$$C_n = c_1+c_2+\ldots+c_n.$$

Thus, when $\kappa = 1$, Riesz's definition is the natural generalisation of Cesàro's which arises when we attach *weights* to the successive partial sums C_n.

2. My principal object in this note is to prove the following theorem.

THEOREM 1.—*If* Σc_n *is summable* (R, λ, κ), *and*

(1)
$$c_n = O\left(\frac{\lambda_n-\lambda_{n-1}}{\lambda_n}\right),$$

then Σc_n *is convergent. In other words, no series which satisfies the condition* (1) *can be summable by Riesz's methods without being convergent.*

This theorem is the generalisation for Riesz's methods of summation of what Mr. Littlewood and I have called *the general Cesàro-Tauber theorem*, the theorem* which asserts that a series for which $c_n = O(1/n)$ cannot be summable by Cesàro's means without being convergent.

I have already published a proof† of the special case of this theorem in which $\kappa = 1$, assuming, however, that the λ's are subject to the restriction

(2)
$$\lambda_{n+1}/\lambda_n \rightarrow 1.$$

When this last condition is satisfied my theorem, and its extension to general values of κ, may be deduced as a corollary from Mr. Littlewood's extension of Tauber's theorem.‡ This method of procedure is, however, open to several objections. In the first place, Mr. Littlewood's theorem is a more difficult theorem than that which we are using it to prove. Secondly, the proof also depends on another theorem of which no proof has yet been published, viz., the theorem that, *if* Σc_n *is summable* (R, λ, κ), *to sum* C, *and* $\Sigma c_n e^{-\lambda_n x}$ *is convergent for* $x > 0$, *then*

$$\Sigma c_n e^{-\lambda_n x} \rightarrow C,$$

* Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 307.

† *L.c.*, p. 313.

‡ Littlewood, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 434.

as $x \to 0$.* Finally, until a proof is given of Mr. Littlewood's theorem, which shall be free from restrictions on the λ's, we are compelled to adhere to the assumption (2). The presence of this unnecessary restriction is at any rate an æsthetic blemish in the theorem.

The idea of obtaining a direct and general proof of the theorem, which should be free from any restriction on the λ's, was suggested to me by Dr. Riesz. Dr. Riesz himself indicated to me the general lines of a proof in the case $\kappa = 1$. The form of this proof was different from that which I adopted in my previous paper. I find, however, that the line of argument which I then followed can be adapted so as to lead to the desired result, and I have followed it here, as the preliminary transformations on which it is based seem to be of some interest in themselves, and can be applied for other purposes.

3. I proceed now to the proof of the theorem. I write

$$b_n = \lambda_n c_n,$$

and use $B(\omega)$, $B^{\kappa}(\omega)$ to denote the sums formed from the b's in the way in which $C(\omega)$, $C^{\kappa}(\omega)$ were formed from the c's. It is evident that there is no loss of generality in supposing κ to be integral. Further, we may suppose our series to be *real*.†

We have

$$\frac{C^{\kappa}(\omega)}{\omega^{\kappa}} - \frac{C^{\kappa+1}(\omega)}{\omega^{\kappa+1}} = \sum_{\lambda_n \leqslant \omega} \left(1 - \frac{\lambda_n}{\omega}\right)^{\kappa} \left\{1 - \left(1 - \frac{\lambda_n}{\omega}\right)\right\} a_n$$

$$= \frac{B^{\kappa}(\omega)}{\omega^{\kappa+1}}.$$

Hence (*a*) *the necessary and sufficient condition that the series* Σc_n, *if known to be summable* $(R, \lambda, \kappa+1)$, *should be summable* (R, λ, κ), *is that*

$$B^{\kappa}(\omega) = o(\omega^{\kappa+1}).$$

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* For the case $\kappa = 1$, see Hardy, *l.c.*, pp. 311 *et seq.* Dr. Riesz has found a proof of a theorem a good deal more general than that stated in the text, which will be published in the *Cambridge Tract* already referred to.

The steps of the deduction referred to in the text are as follows. From (1) we infer the convergence of $\Sigma c_n e^{-\lambda_n x}$ for $\kappa > 0$. From the summability of Σc_n, and the theorem stated in the text, we infer the existence of the limit as $x \to 0$. Finally, from (1), (2) and Mr. Littlewood's theorem, we infer the convergence of Σc_n.

† Cf. Hardy, *l.c.*, p. 303.

Again, $$\frac{d}{d\omega}\left\{\frac{C^{\kappa+1}(\omega)}{\omega^{\kappa+1}}\right\} = \frac{\kappa+1}{\omega^2}\sum_{\lambda_n\leqslant\omega}\left(1-\frac{\lambda_n}{\omega}\right)^{\kappa}\lambda_n a_n$$

$$= (\kappa+1)\frac{B^{\kappa}(\omega)}{\omega^{\kappa+2}}.$$

Hence (β) *the necessary and sufficient condition that the series* Σc_n *should be summable* (R, λ, $\kappa+1$) *is that the integral*

$$\int^{\infty}\frac{B^{\kappa}(\omega)}{\omega^{\kappa+2}}\,d\omega$$

should be convergent.

I shall now show that, when the c's satisfy condition (1) of the theorem, (β) implies (α). From this it will obviously follow that the summability of the series implies its convergence.

We have, in the first place,

(1) $$|c_n| < K(\lambda_n-\lambda_{n-1})/\lambda_n.$$

Now, let us suppose that (α) is not true, and that *e.g.*, the upper limit of $\omega^{-\kappa-1}B^{\kappa}(\omega)$ is positive. Then there is a positive constant H, such that

(2) $$B^{\kappa}(\omega) > H\omega^{\kappa+1},$$

for values of ω exceeding all limit.

Suppose $\xi > \omega$. Then

(3) $$B^{\kappa}(\xi)-B^{\kappa}(\omega) = \kappa\int_{\omega}^{\xi}B^{\kappa-1}(u)\,du.$$

Also $$|B^{\kappa-1}(u)| = \left|\sum_{\lambda_n\leqslant u}(u-\lambda_n)^{\kappa-1}b_n\right|$$

$$< K\sum_{\lambda_n\leqslant u}(u-\lambda_n)^{\kappa-1}(\lambda_n-\lambda_{n-1})$$

$$< K\int_0^u (u-w)^{\kappa-1}\,dw = \frac{K}{\kappa}u^{\kappa}.$$

Hence

(4) $$|B^{\kappa}(\xi)-B^{\kappa}(\omega)| < K\int_{\omega}^{\xi}u^{\kappa}\,du = \frac{K}{\kappa+1}(\xi^{\kappa+1}-\omega^{\kappa+1}) < \frac{K}{\kappa+1}(\xi-\omega)\xi^{\kappa}.$$

Let $$\Omega = (1+\rho)\,\omega,$$

where ρ is a positive constant to be chosen later. Then, for $\omega < \xi < \Omega$, we have, by (2) and (4),

$$B^\kappa(\xi) \geqslant B^\kappa(\omega) - |B^\kappa(\xi) - B^\kappa(\omega)| > H\omega^{\kappa+1} - \frac{K}{\kappa+1}(\xi-\omega)\xi^\kappa$$

$$> \left\{H - \frac{K\rho(1+\rho)^\kappa}{\kappa+1}\right\}\omega^{\kappa+1}.$$

But, when H and K are given, we can evidently choose ρ so that

$$\frac{K\rho(1+\rho)^\kappa}{\kappa+1} < \tfrac{1}{2}H.$$

Then $$B^\kappa(\xi) > \tfrac{1}{2}H\omega^{\kappa+1},$$

for $$\omega < \xi < \Omega,$$

and so $$\int_\omega^\Omega \frac{B^\kappa(\xi)}{\xi^{\kappa+2}}\,d\xi > \tfrac{1}{2}H(\Omega-\omega)\frac{\omega^{\kappa+1}}{\Omega^{\kappa+2}} = \frac{H\rho}{2(1+\rho)^{\kappa+2}}.$$

That this should hold for values of ω exceeding all limit is a contradiction of the hypothesis (β). The theorem is therefore established.

4. I add some remarks relating to the case in which the increase of the λ's is *rapid and fairly regular*.

Suppose that $$\lambda_{n+1}/\lambda_n \geqslant 1+\delta > 1,$$

where δ is a constant. Then the condition (1) of the theorem reduces to

$$c_n = O(1).$$

But, in this case, *more* is true than is asserted by the theorem. It is, in fact, true that *no* series can be summable (R, λ, κ) unless it is convergent; in other words, for such forms of λ_n, Riesz's methods are completely trivial; they sum convergent series and convergent series only. This can be deduced from another of Riesz's theorems, viz., that* *if* Σc_n *is summable* (R, λ, κ) *to sum* C, *then*

$$C_n - C = o\left(\frac{\lambda_{n+1}}{\lambda_{n+1}-\lambda_n}\right)^\kappa.$$

This conclusion, so far as it goes, bears out Mr. Littlewood's conjec-

* A proof of this theorem also will be given in the *Tract*.

ture* that, for such "regular high indices" λ_n, the existence of the limit

$$\lim_{\kappa \to 0} \Sigma c_n e^{-\lambda_n x},$$

always involves the convergence of Σc_n.

5. Landau has generalised the "general Cesàro-Tauber" theorem by substituting for the hypothesis

$$c_n = O(1/n),$$

the less exacting hypothesis $c_n = O_L(1/n)$,

that is to say, $c_n > -K/n$.†

There is, of course, a corresponding extension of Theorem 1, which runs as follows.

THEOREM 2.—*A series Σc_n, for which*

(1) $$c_n = O_L\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right)$$

cannot be summable by Riesz's means unless it is convergent.

The proof of this theorem requires merely a slight modification of the proof of Theorem 1. We have

(1) $$c_n > -K(\lambda_n - \lambda_{n-1})/\lambda_n.$$

If

(2) $$B^\kappa(\omega) > H\omega^{\kappa+1}$$

for values of ω exceeding all limit, we take

$$\Omega = (1+\rho)\omega,$$

and show, by substantially the same argument‡ as in § 3, that

$$B^\kappa(\xi) > \tfrac{1}{2}H\omega^{\kappa+1} \quad (\omega < \xi < \Omega).$$

We thus prove, as in § 3, that

$$\overline{\lim_{\omega \to \infty}} \frac{B^\kappa(\omega)}{\omega^{\kappa+1}} = 0.$$

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* Littlewood, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 446.

† *Prac Matematyczno-Fizycznych*, Vol. 21, p. 97. In Landau's form of the theorem c_n must obviously be supposed to be real, or it must be asserted explicitly that both the real and the imaginary parts of c_n satisfy his condition.

‡ The only difference is that we use algebraical inequalities instead of inequalities between absolute values.

In order to prove that the lower limit of indetermination is also zero, that is to say, that the inequality

$$B^{\kappa}(\omega) < -H\omega^{\kappa+1} \quad (H > 0),$$

cannot hold for values of ω exceeding all limit, we need only modify the argument to the extent of considering an interval

$$(1-\rho)\,\omega = \Omega < \xi < \omega,$$

instead of an interval $\omega < \xi < \Omega = (1+\rho)\,\omega$.*

* *Either* interval would have served our purposes in proving Theorem 1.