

would give out a characteristic radiation. Experiments show that with proper conditions every substance can be so chosen as to give a similar type of radiation. It is important to notice that Bragg and Madsen (Phil. Mag. Oct. 1908) have shown that the character of the  $\beta$  radiation caused by  $\gamma$ -rays is independent of the atom in which it arises, and depends solely on the nature of the  $\gamma$ -rays to which it is due. The present investigation shows that this is also true for the secondary  $\gamma$  radiation.

The quality of the secondary  $\gamma$  radiation shows no sudden change from that of the primary. There is simply a gradual softening the more the secondary radiation is deflected from its original direction. The gradual softening is the same for every radiator. Other investigators have shown that  $\beta$ -rays are scattered in their passage through matter. The scattering of  $\gamma$  rays appears to be analogous to the scattering of  $\beta$ -rays. The primary  $\gamma$ -rays possess a wide range of penetrating power. The softening of the secondary radiation that has been observed is the result of this heterogeneity of the primary rays. The softer radiation is more scattered than the harder radiation; as the radiator is increased in thickness more of the harder gets turned aside, and in consequence we get both the hardening of the primary and of the secondary. The hardening is due in the one case to the cutting out of the softer radiation, and in the case of the secondary to the addition of a more penetrating scattered radiation. There is no evidence of selective absorption. The production of this secondary  $\gamma$  radiation is undoubtedly a scattering effect, as Madsen had concluded from previous experiments.

I desire to thank Professor Rutherford for the use of large quantities of radium and of radium emanation, and also for his suggestions in the course of this work.

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CV. *The Approximate Calculation of Bessel Functions of Imaginary Argument.* By J. W. NICHOLSON, M.A., D.Sc.\*

IN the British Association Report for 1908, some formulæ were given suitable for the rapid tabulation of Bessel functions whose argument is purely imaginary and large, and whose order may be of any magnitude. The same results apply if the order is large, and the argument of any magnitude. A proof was not appended, and the object of the

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\* Communicated by the Author.

present note is to supply a short proof. The corresponding formulæ for functions of real argument have been very completely dealt with in a series of papers in the *Philosophical Magazine* \*. The asymptotic expansions of functions of imaginary argument present only one type instead of the three in the case of real argument, and their treatment can therefore be given briefly. It is most conveniently deduced as a special case of that of the general associated Legendre functions  $P_n^m(\mu)$  and  $Q_n^m(\mu)$ , which has been developed in a recent paper †.

The functions of order  $m$  and argument  $ix$  satisfy the equation

$$\frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} - \left(1 + \frac{m^2}{x^2}\right)y = 0, \quad . . . \quad (1)$$

where  $x$  is itself real, and they are usually defined in the forms

$$\begin{aligned} I_m(x) &= i^{-m} J_m(ix) = \frac{x^m}{2^m \Gamma(\frac{1}{2}) \Gamma(m + \frac{1}{2})} \int_0^\pi \cosh(x \cos \phi) \sin^{2m} \phi \, d\phi \\ &= \frac{x^m}{2^m \Gamma(m+1)} \left\{ 1 + \frac{x^2}{2^2 \cdot 1! m+1} + \frac{x^4}{2^4 \cdot 2! m+1 \cdot m+2} + \dots \right\} \quad (2) \end{aligned}$$

and

$$K_m(x) = \left(\frac{x}{2}\right)^m \frac{\Gamma(\frac{1}{2})}{\Gamma(m + \frac{1}{2})} \int_0^\infty d\phi \sinh^{2m} \phi e^{-x \cosh \phi}, \quad . . \quad (3)$$

the latter function vanishing exponentially when  $x$  is large.

Let  $P_n^m(\mu)$ ,  $Q_n^m(\mu)$  be the general associated Legendre functions of argument  $\mu$ , degree  $n$ , and order  $m$ . A comprehensive definition of these functions for all values of these three quantities has been given by Hobson ‡. They are the functions which, when  $m$  and  $n$  are positive integers, may be expressed in relation to the ordinary zonal harmonics  $P_n(\mu)$ ,  $Q_n(\mu)$  by the equations

$$\begin{aligned} P_n^m(\mu) &= (\mu^2 - 1)^{\frac{1}{2}m} \cdot d^m / d\mu^m \cdot P_n(\mu) \} \\ Q_n^m(\mu) &= (\mu^2 - 1)^{\frac{1}{2}m} \cdot d^m / d\mu^m \cdot Q_n(\mu) \} \quad . . \quad (4) \end{aligned}$$

when  $\mu$  is greater than unity, the only case needed for our purpose. But in the proof contained in this paper, restriction of the order and degree to integer values is not necessary, and the final results derived for the Bessel functions are true for any real value of  $m$ .

\* Dec. 1907; Aug. 1908; July 1909; Feb. 1910.

† Quarterly Journal, April 1910.

‡ Phil. Trans. 1896 A. p. 443 *et seq.*

With these definitions, a well-known formula due to Heine shows that

$$I_m(x) = \text{Lt}_{n=\infty} n^{-m} P_n^m \left( \cosh \frac{x}{n} \right), \quad . \quad . \quad (5)$$

and a companion formula may be readily derived as follows:—  
When  $\mu$  is greater than unity, and  $m + \frac{1}{2}$ ,  $n - m + 1$  are positive, Hobson\* has shown that

$$Q_n^m(\mu) = \frac{e^{im\pi}}{2^m} \frac{\varpi(n+m)}{\varpi(n-m)} \frac{\varpi(-\frac{1}{2})}{\varpi(m-\frac{1}{2})} (\mu^2 - 1)^{\frac{1}{2}m} \int_0^\infty \frac{\sinh^{2m} w dw}{(\mu + \sqrt{\mu^2 - 1} \cdot \cosh w)^{n+m+1}} \quad . \quad . \quad (6)$$

where  $\varpi(s)$  is Gauss' function, identical with  $\Gamma(s+1)$ , or if  $s$  be an integer, with  $s!$

Write  $\mu = \cosh x/n$ , where  $n$  tends towards infinity. Then  $(\mu^2 - 1)^{\frac{1}{2}m}$  tends to the value  $(x/n)^m$ , and  $\varpi(n+m)/\varpi(n-m)$  to the value  $n^{2m}$ .

Thus

$$\text{Lt}_{n=\infty} n^{-m} Q_n^m(\mu) = \frac{e^{im\pi}}{2^m} \frac{\varpi(-\frac{1}{2})}{\varpi(m-\frac{1}{2})} x^m \int_0^\infty \frac{\sinh^{2m} w dw}{\text{Lt}_{n=\infty} \left( 1 + \frac{x}{n} \cosh w \right)^{n+m+1}}$$

But

$$\text{Lt}_{n=\infty} \left( 1 + \frac{x}{n} \cosh w \right)^{n+m+1} = e^{-x \cosh w},$$

and therefore we deduce by the definition (3),

$$K_m(x) = \text{Lt}_{n=\infty} e^{-im\pi} n^{-m} Q_n^m \left( \cosh \frac{x}{n} \right), \quad . \quad . \quad (7)$$

which is the required companion formula to (5).

#### Asymptotic expansions.

It is now possible to derive the asymptotic expansions of the Bessel functions  $I_m(x)$  and  $K_m(x)$  from those of the Legendre functions. The latter will be quoted from the writer's paper †, for the case of argument greater than unity.

Writing

$$P_n^m(\mu) - e^{-im\pi} Q_n^m(\mu) \frac{\sin(m+n)\pi}{\pi \cos n\pi} = \left\{ \frac{\varpi(n+m)}{\pi \varpi(n-m)(\mu^2 - 1)} \right\}^{\frac{1}{2}} T^{\frac{1}{2}} e^t$$

$$Q_n^m(\mu) = e^{im\pi} \left\{ \frac{\pi \varpi(n+m)}{\varpi(n-m)(\mu^2 - 1)} \right\}^{\frac{1}{2}} T^{\frac{1}{2}} e^{-t}. \quad . \quad . \quad (8)$$

\* *L. c. ante*, p. 500.

† Quarterly Journal, April 1910, pp. 250-252.

Then if  $k = m/n + \frac{1}{2}$ ,  $\nu^2 = \mu^2 - 1$ ,

$$T = (\nu^2/2n+1)\{(\nu^2+k^2)^{-\frac{1}{2}} + \lambda_3(\nu^2+k^2)^{-\frac{3}{2}} + \lambda_5(\nu^2+k^2)^{-\frac{5}{2}} + \dots\}$$

$$\frac{t}{n+\frac{1}{2}} = \left\{1 + \mu_1 \frac{d}{k dk} - \frac{\mu_2}{1} \left(\frac{d}{k dk}\right)^2 + \frac{\mu_3}{1 \cdot 3} \left(\frac{d}{k dk}\right)^3 \dots\right\} \times$$

$$\times \left[ \log \left\{ \mu + \sqrt{\mu^2 + k^2 - 1} \right\} - \frac{1}{2} k \log \frac{\sqrt{\mu^2 + k^2 - 1} + k\mu}{\sqrt{\mu^2 + k^2 - 1} - k\mu} \right. \\ \left. + \frac{1}{2} k \log \frac{1-k}{1+k} \right], \quad . \quad . \quad . \quad (9)$$

where  $k$  is less than unity, and the coefficients of types  $\lambda_r$ ,  $\mu_r$  are given by

$$\lambda_1 = 1, \quad \lambda_3 = -\frac{1}{8}(4k^2-1)/(n+\frac{1}{2})^2-1^2,$$

$$4^2\{(n+\frac{1}{2})^2-2^2\}\lambda_5 = -6k^2(2-3k^2)+3(4k^2-1)(28k^2-9)/8\{(n+\frac{1}{2})^2-1^2\},$$

and in general

$$k^4(k^2-1)(r-2)(r-4)(r-6)\lambda_{r-6} + k^2(2-3k^2)(r-2)(r-3)(r-4)\lambda_{r-4} \\ + (r+2)\{k^2+3k^2(r-2)^2-(r-2)^2\}\lambda_{r-2} \\ + (r-1)\{4(n+\frac{1}{2})^2-(r-1)^2\}\lambda_r = 0, \quad . \quad (10)$$

whereas the  $\mu$ 's are defined by the identical relation

$$1 + \mu_1\sigma + \mu_2\sigma^2 + \dots = (1 + \lambda_3\sigma + \lambda_5\sigma^3 + \dots)^{-1}. \quad (11)$$

We proceed to the limit when  $n$  is infinite and  $m$  finite, so that  $k=0$ . In this case,

$$\text{Lt}_{n=\infty} n^2\lambda_3 = \frac{1}{8},$$

$$\text{Lt}_{n=\infty} n^4\lambda_5 = \frac{1}{16}\left(\frac{27}{8}-12m^2\right),$$

and so on. In fact, these limits are the coefficients which in the notation of previous papers dealing with the Bessel functions of real argument, were denoted by

$$-\lambda_2, \quad \lambda_4, \quad -\lambda_6, \quad \dots$$

with  $m$  taking the place of  $n$ . Similarly, in the formula for  $t$ ,  $n^2\mu_1$ ,  $n^4\mu_2$ ,  $\dots$  must be replaced in the limit by

$$-\mu_2, \quad \mu_4, \quad -\mu_6, \quad \dots$$

where the  $\mu$ 's are now the coefficients of earlier papers\*.

\* *Vide e.g.* Phil. Mag. Feb. 1910, p. 240.

In terms of the old notation, therefore, on reduction, the limiting values become

$$t = \left\{ 1 - 2\mu_2 \frac{d}{dm^2} - 2^2 \mu_4 \left( \frac{d}{dm^2} \right)^2 - \dots \right\} \times \\ \times \left\{ \sqrt{x^2 + m^2} - \frac{1}{2} m \log \frac{\sqrt{x^2 + m^2} + m}{\sqrt{x^2 + m^2} - m} \right\} \\ T = x \left\{ (x^2 + m^2)^{-\frac{1}{2}} - \lambda_2 (x^2 + m^2)^{-\frac{3}{2}} + \dots \right\} \quad (12)$$

where the coefficients are defined by

$$\lambda_2 = -\frac{1}{8}, \quad \lambda_4 = \frac{1}{2^7} (27 - 96m^2), \\ \lambda_6 = \frac{1}{2^{10}} (4640m^2 - 1125 - 640m^4),$$

$$4(s+3)\lambda_{s+3} + (s+2)^3\lambda_{s+1} + 2m^2s \cdot (s+1)(s+2)\lambda_{s-1} \\ + m^4s \cdot (s^2-4)\lambda_{s-3} = 0, \quad (13)$$

and the identity

$$1 + \mu_2 x + \mu_4 x^2 + \dots = (1 + \lambda_2 x + \lambda_4 x^2 + \dots)^{-1}. \quad (14)$$

The limiting forms of the substitutions (8) become

$$\text{Lt}_{n=\infty} n^{-m} P_n^m \left( \cosh \frac{x}{n} \right) = \text{Lt}_{n=\infty} \left( \frac{T}{2\pi x} \right)^{\frac{1}{2}} e^t, \\ \text{Lt}_{n=\infty} n^{-m} e^{im\pi} Q_n^m \left( \cosh \frac{x}{n} \right) = \text{Lt}_{n=\infty} \left( \frac{\pi T}{2x} \right)^{\frac{1}{2}} e^{-t},$$

where  $Q_n^m$  has now been rejected in the first substitution, as proportional by the second to  $e^{-t}$  which is very small, for only moderate values of  $x^2 + m^2$ , in comparison with  $e^t$ .

Finally, therefore, by the use of (5) and (7), we obtain the results

$$\left. \begin{aligned} I_m(x) &= \left( \frac{T}{2\pi x} \right)^{\frac{1}{2}} e^t \\ K_m(x) &= \left( \frac{\pi T}{2x} \right)^{\frac{1}{2}} e^{-t} \end{aligned} \right\}, \quad (15)$$

where

$$T = x \left\{ (x^2 + m^2)^{-\frac{1}{2}} - \lambda_2 (x^2 + m^2)^{-\frac{3}{2}} + \lambda_4 (x^2 + m^2)^{-\frac{5}{2}} - \dots \right\}, \\ t = \left\{ 1 - \mu_2 \frac{d}{mdm} - \mu_4 \left( \frac{d}{mdm} \right)^2 \dots \right\} \times \\ \times \left\{ (x^2 + m^2)^{\frac{1}{2}} - \frac{1}{2} m \log \frac{(x^2 + m^2)^{\frac{1}{2}} + m}{(x^2 + m^2)^{\frac{1}{2}} - m} \right\}, \quad (16)$$

and the coefficients are given by (13, 14).

These are the formulæ given in the British Association Report.

So far as tabulation will ordinarily be required, it will be sufficient in general, even for only moderate values of  $x$  or  $m$  ( $x > 10$  or  $m > 10$ ) to take the first terms of  $T$  and  $t$  only, if a three-figure accuracy is required. The order of accuracy possessed by the formulæ is similar to that of the ordinary semiconvergent expression for  $J_0(x)$  where  $x$  is real.

The first approximations may be written

$$\left. \begin{aligned} I_m(x) &= (2\pi x \cosh \beta)^{-\frac{1}{2}} e^{x(\cosh \beta - \beta \sinh \beta)} \\ K_m(x) &= \pi (2\pi x \cosh \beta)^{-\frac{1}{2}} e^{-x(\cosh \beta - \beta \sinh \beta)} \end{aligned} \right\} \quad (17)$$

where  $\beta$  is defined by  $m = x \sinh \beta$ .

A useful substitution in the final formulæ has been suggested to me by Prof. Alfred Lodge. If an angle  $\theta$  be chosen such that

$$x = m \tan \theta,$$

then

$$t = m(\sec \theta + \log_e \tan \frac{1}{2}\theta), \quad \dots \quad (18)$$

and this logarithm has already been exhaustively tabulated. Thus the tabulation of the Bessel functions may be performed very rapidly, and this applies also when the higher approximations are used.

CVI. *On Non-Newtonian Mechanical Systems, and Planck's Theory of Radiation.* By J. H. JEANS, M.A., F.R.S.\*

1. **P**LANCK'S treatment of the radiation problem, introducing as it does the conception of an indivisible atom of energy, and consequent discontinuity of motion, has led to the consideration of types of physical processes which were until recently unthought of, and are to many still unthinkable. The theory put forward by Planck would probably become acceptable to many if it could be stated physically in terms of continuous motion, or mathematically in terms of differential equations. Larmor† has recently made an extremely interesting suggestion as to how it might perhaps be possible to do this, but has not so far carried out the analysis necessary to determine whether his suggestion leads to a solution of the difficulty or not.

The question discussed in the present paper includes that

\* Communicated by the Author.

† Bakerian Lecture, 1909, Proc. Roy. Soc. A. vol. lxxxiii. and Phil. Mag. xx. p. 350.