A COMPACT NOTATION FOR PECULIAR PROPERTIES CHARACTERIZING INTEGER TETRATION

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ABSTRACT. We introduce a compact notation to express the congruence speed (in radix-10) of an integer tetration base a, along with the cycle of the rightmost non-stable digits of ${}^{b}a$ for unit increments of b. The aforementioned discrete function provides a useful tool for efficiently computing the exact number of frozen digits that characterize the right tail of each nontrivial integer tetration. We also provide an improved upper bound for the minimum hyperexponent $\bar{b}(a)$ that ensures the constancy of the congruence speed of a for all heights $b \geq \bar{b}(a)$. Moreover, we prove that the minimum between the constant congruence speeds of any two integers greater than 1, whose product is not divisible by 10, is always less than or equal to the constant congruence speed of their product. Lastly, we give examples of infinitely many perfect powers whose degree matches their congruence speed at every height above 2, emphasizing the peculiar recurrence relations of hyper-4.

1. INTRODUCTION

For clarity, we denote $\mathbb{N}_0$ as the set of nonnegative integers (including zero) and $\mathbb{N}$ as the set of strictly positive integers (i.e., $\{1, 2, 3, \ldots\}$).

Let $a - 1, b \in \mathbb{N}$ (assuming radix-10), and consider the tetration

$$ {}^{b}a := \begin{cases} a & \text{if } b = 1 \\ a^{\left({}^{(b-1)}a\right)} & \text{if } b \geq 2 \end{cases} $$

which, for each positive integer n, is known to eventually become periodic modulo 10^n (see [8]). This result generally holds for sufficiently large values of the hyperexponent b. For example, for $a = 27057$ and $n = 25$, we must reach height 6 since ${}^{1}27057 \equiv (00000000000000000000)\mathbf{27057} \pmod{10}^{25}$, ${}^{2}27057 \equiv 477797198828352\mathbf{14952}27057 \pmod{10}^{25}$, ${}^{3}27057 \equiv (0)545271142\mathbf{05036}1495227057 \pmod{10}^{25}$, ${}^{4}27057 \equiv 63448\mathbf{63520}050361495227057 \pmod{10}^{25}$, ${}^{5}27057 \equiv 6\mathbf{4499}63520050361495227057 \pmod{10}^{25}$, and finally we get ${}^{6}27057 \pmod{10}^{25} = {}^{7}27057 \pmod{10}^{25} = 1449963520050361495227057$. In detail, we observe that the number of "new" rightmost frozen digits of ${}^{b}27057$ is equal to 5 for $b \leq 4$ and decreases to 4 for all $b \geq 5$.

In the 2011 book "La strana coda della serie $n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}$", the concept of *congruence speed* was introduced to quantify the growth rate of the number of stable digits that appear at the end of ${}^{b}a$ for each unit increase in b. That work also established a connection between the congruence speed and the periodic behavior of the least significant digit of ${}^{b}a$ that is not yet stable at height b. Accordingly, we track the cycle of the rightmost non-stable digit of ${}^{b}a$ at successive heights $b, b+1, b+2, \ldots$ via the *phase shift* array (see Section 3 of [6]).

With this note, we propose a new notation system that efficiently describes the congruence speed of nonnegative integer tetration bases at each height, along with its associated phase shift.

Building upon previous studies in the field, we refine and extend key results, including bounds for the minimum hyperexponent $\bar{b}(a)$ ensuring stability in the congruence speed of a. Our approach relies on modular arithmetic and p-adic analysis to establish rigorous characterizations of congruence speed behaviors, particularly in the context of bases not divisible by 10.

To this end, we include an appendix proving that the constant congruence speed of any integer greater than 1 and not divisible by 10 is greater than or equal to the lowest constant congruence speed among its factors.

Lastly, we give infinitely many perfect powers whose degree equals their congruence speed at every height greater than 2 (in these cases, the congruence speed remains stable for all hyperexponents greater than 1).

This work contributes to the broader study of number theory by offering a structured framework for analyzing the periodic properties of iterated exponentiation and its implications in modular settings.

2. CONGRUENCE SPEED AND PHASE SHIFT OF TETRATION

The present section provides rigorous definitions of the terminology introduced earlier, establishing a formal structure for a more compact and efficient notation to describe the recurrence properties characteristic of the right tail of ${}^{b}a$.

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In [3], the congruence speed of each nonnegative integer a was originally defined as the function

$$\begin{gathered} V : \mathbb{N}_0 \times \mathbb{N} \longrightarrow \mathbb{N}_0 \\ (a, b) \mapsto V(a, b) \end{gathered} ,$$

and the author specified that if the constraint $10 \nmid a$ is imposed along with the sufficient condition $b > a$, the congruence speed no longer depends on b, so we obtain a *constant* congruence speed, the one entry function

$$\begin{gathered} V : \mathbb{N} \setminus \{\text{multiples of } 10\} \longrightarrow \mathbb{N}_0 \\ a \mapsto V(a) \end{gathered}$$

peculiar of hyper-4 and fully described by Equations (3) and (16) of [4].

Since a casual reader could easily be misled by the use of $V(a, b)$ and $V(a)$, and given that describing the congruence speed of a as $V(a, 1)$, $V(a, 2)$, …, $V(a, a)$, and $V(a, a+1) = V(a)$ is not very elegant, let us rewrite the original definitions of congruence speed and constant congruence speed as follows.

Definition 2.1. Let $n \in \mathbb{N}_0$ and assume that $a \in \mathbb{N} \setminus \{1\}$ is not a multiple of 10. Then, given ${}^{b-1}a \equiv {}^{b}a \pmod{10^n} \wedge {}^{b-1}a \not\equiv {}^{b}a \pmod{10^{n+1}}$, for each $b = 1, 2, 3, \ldots$, $v_b(a)$ returns the strictly positive integer such that ${}^{b}a \equiv {}^{b+1}a \pmod{10^{n+v_b(a)}} \wedge {}^{b}a \not\equiv {}^{b+1}a \pmod{10^{n+v_b(a)+1}}$, and we define $v_b(a)$ as the congruence speed of the base a at height b.

Definition 2.2. Let $a \in \mathbb{N} \setminus \{1\}$ not be a multiple of 10. Let $\bar{b}(a) := \min\{b \in \mathbb{N} : v_b(a) = v_{b+k}(a), \forall k \in \mathbb{N}_0\}$. We define as the constant congruence speed of a the nonnegative integer $v_{\bar{b}}(a) := v_{\bar{b}(a)}(a)$.

Assuming $b > 2$, we note that $v_{\bar{b}}(a) = v_{\bar{b}+1}(a) = v_{\bar{b}+2}(a) = \cdots$ and $v_b(a) \geq v_{b+1}(a) \geq v_{b+2}(a) \geq \cdots$ hold for each a greater than 1 and not divisible by 10 (see Equations (17) and (18) of [7]).

Now, let us compactly denote $\mathfrak{V}(a)$ as the congruence speed of the integer tetration base a.

Consequently, if a is greater than 1 and is not a multiple of 10, we can write

$$\mathfrak{V}(a) := (v_1(a), v_2(a), \ldots, v_{\bar{b}-1}(a); v_{\bar{b}}(a))$$

since we indicate $(v_1(a), v_2(a), \ldots, v_{\bar{b}-1}(a), v_{\bar{b}}(a), v_{\bar{b}}(a), v_{\bar{b}}(a), \ldots)$ as $(v_1(a), v_2(a), \ldots, v_{\bar{b}-1}(a); v_{\bar{b}}(a))$.

Then, trivially, $\mathfrak{V}(0) = (; 0)$ and $\mathfrak{V}(1) = (1; 0)$.

On the other hand, we know that the congruence speed of every tetration base divisible by 10 never becomes stable, so we can proceed as follows.

Let $\nu_p(\ldots)$ be the p-adic valuation of the argument. We have that the congruence speed of a is given by

$$\mathfrak{V}(a) := \begin{cases} (v_1(a), v_2(a), \ldots, v_{\bar{b}-1}(a); v_{\bar{b}}(a)) & \text{if} \quad a > 1 \wedge 10 \nmid a \\ (v_1(a), v_2(a), \ldots; +\infty) & \text{if} \quad 10 \mid a \\ (; v_{\bar{b}}(a)) & \text{if} \quad a = 0 \\ (1; v_{\bar{b}}(a)) & \text{if} \quad a = 1 \end{cases} , \tag{2.1}$$

where (see Eq. (3) of [4])

$$v_{\bar{b}}(a) = \begin{cases} 0 & \text{if} \quad a \in \{0, 1\} \\ \min\{\nu_2(a-1), \nu_5(a-1)\} & \text{if} \quad a \equiv 1 \pmod{20} \wedge a \neq 1 \\ \min\{\nu_2(a+1), \nu_5(a-1)\} & \text{if} \quad a \equiv 11 \pmod{20} \\ \nu_5(a^2+1) & \text{if} \quad a \equiv 2, 8 \pmod{10} \\ \min\bigl\{\nu_2(a+1), \nu_5(a^2+1)\bigr\} & \text{if} \quad a \equiv 3, 7 \pmod{20} \\ \min\bigl\{\nu_2(a-1), \nu_5(a^2+1)\bigr\} & \text{if} \quad a \equiv 13, 17 \pmod{20} \\ \nu_5(a+1) & \text{if} \quad a \equiv 4 \pmod{10} \\ \nu_2(a-1) & \text{if} \quad a \equiv 5 \pmod{20} \\ \nu_2(a+1) & \text{if} \quad a \equiv 15 \pmod{20} \\ \nu_5(a-1) & \text{if} \quad a \equiv 6 \pmod{10} \\ \min\{\nu_2(a-1), \nu_5(a+1)\} & \text{if} \quad a \equiv 9 \pmod{20} \\ \min\{\nu_2(a+1), \nu_5(a+1)\} & \text{if} \quad a \equiv 19 \pmod{20} \\ 1 & \text{otherwise} \end{cases} . \tag{2.2}$$

Equation (2.2) follows from the solutions of the fundamental equation $y^5 = y$ in the commutative ring of 10-adic integers (see 2.3 below).

(2.3)

$$y_{1,2,\ldots,14} = \begin{cases} \alpha_1' = 1 = \ldots 0000000000000000000000000000000000000000000000000000000000001 & \text{iff} \quad i = 1 \\ \alpha_1'' = 1 - 2 \cdot \left\{5^{2^n}\right\}_\infty = \ldots 5153153538207781991786760045215487480163574218751 & \text{iff} \quad i = 2 \\ \alpha_2' = \left\{2^{5^n}\right\}_\infty = \ldots 3223304553032451441224165530407839804103263499879186432 & \text{iff} \quad i = 3 \\ \alpha_3' = \left\{5^{2^n}\right\}_\infty - \left\{2^{5^n}\right\}_\infty = \ldots 0390779454884838576212137588152996418333704193 & \text{iff} \quad i = 4 \\ \alpha_3'' = -\left\{5^{2^n}\right\}_\infty - \left\{2^{5^n}\right\}_\infty = \ldots 44317662666830362972182803640476581907922943 & \text{iff} \quad i = 5 \\ \alpha_4' = \left\{5^{2^n}\right\}_\infty - 1 = \ldots 0557423423230896109004106619977392256259918212890624 & \text{iff} \quad i = 6 \\ \alpha_5' = \left\{5^{2^n}\right\}_\infty = \ldots 1000557423423230896109004106619977392256259918212890625 & \text{iff} \quad i = 7 \\ \alpha_5'' = -\left\{5^{2^n}\right\}_\infty = \ldots 99442576576769103890995893380022607743740081787109375 & \text{iff} \quad i = 8 \\ \alpha_6' = 1 - \left\{5^{2^n}\right\}_\infty = \ldots 9442576576769103890995893380022607743740081787109376 & \text{iff} \quad i = 9 \\ \alpha_7' = -\left\{5^{2^n}\right\}_\infty + \left\{2^{5^n}\right\}_\infty = \ldots 09220545115161423787862411847003581666295807 & \text{iff} \quad i = 10 \\ \alpha_7'' = \left\{5^{2^n}\right\}_\infty + \left\{2^{5^n}\right\}_\infty = \ldots 6455682337333169637027817196359523418092077057 & \text{iff} \quad i = 11 \\ \alpha_8' = -\left\{2^{5^n}\right\}_\infty = \ldots 76695446967548558775834469592160195896736500120813568 & \text{iff} \quad i = 12 \\ \alpha_9' = 2 \cdot \left\{5^{2^n}\right\}_\infty - 1 = \ldots 4846846461792218008213239954784512519836425781249 & \text{iff} \quad i = 13 \\ \alpha_9'' = -1 = \ldots 99999999999999999999999999999999999999999999999999999999999 & \text{iff} \quad i = 14 \end{cases}.$$

Now, as we pick each element of the set $\{\alpha_1', \alpha_1'', \alpha_2', \alpha_3', \alpha_3'', \alpha_4', \alpha_5', \alpha_5'', \alpha_6', \alpha_7', \alpha_7'', \alpha_8', \alpha_9', \alpha_9''\}$ and multiply it by each element of the same set (including the original element itself), we obtain Table 1.

Table 1. Multiplicative recurrences of the solutions of the 10-adic equation $y^5 = y$.

$\cdot$	$\boldsymbol{\alpha_1'}$	$\boldsymbol{\alpha_1''}$	$\boldsymbol{\alpha_2'}$	$\boldsymbol{\alpha_3'}$	$\boldsymbol{\alpha_3''}$	$\boldsymbol{\alpha_4'}$	$\boldsymbol{\alpha_5'}$	$\boldsymbol{\alpha_5''}$	$\boldsymbol{\alpha_6'}$	$\boldsymbol{\alpha_7'}$	$\boldsymbol{\alpha_7''}$	$\boldsymbol{\alpha_8'}$	$\boldsymbol{\alpha_9'}$	$\boldsymbol{\alpha_9''}$
$\boldsymbol{\alpha_1'}$	α_1'	α_1''	α_2'	α_3'	α_3''	α_4'	α_5'	α_5''	α_6'	α_7'	α_7''	α_8'	α_9'	α_9''
$\boldsymbol{\alpha_1''}$	α_1''	α_1'	α_2'	α_3''	α_3'	α_4'	α_5''	α_5'	α_6'	α_7''	α_7'	α_8'	α_9''	α_9'
$\boldsymbol{\alpha_2'}$	α_2'	α_2'	α_4'	α_6'	α_6'	α_8'	0	0	α_2'	α_4'	α_4'	α_6'	α_8'	α_8'
$\boldsymbol{\alpha_3'}$	α_3'	α_3''	α_6'	α_9'	α_9''	α_2'	α_5'	α_5''	α_8'	α_1''	α_1'	α_4'	α_7''	α_7'
$\boldsymbol{\alpha_3''}$	α_3''	α_3'	α_6'	α_9''	α_9'	α_2'	α_5''	α_5'	α_8'	α_1'	α_1''	α_4'	α_7'	α_7''
$\boldsymbol{\alpha_4'}$	α_4'	α_4'	α_8'	α_2'	α_2'	α_6'	0	0	α_4'	α_8'	α_8'	α_2'	α_6'	α_6'
$\boldsymbol{\alpha_5'}$	α_5'	α_5''	0	α_5'	α_5''	0	α_5'	α_5''	0	α_5''	α_5'	0	α_5'	α_5''
$\boldsymbol{\alpha_5''}$	α_5''	α_5'	0	α_5''	α_5'	0	α_5''	α_5'	0	α_5'	α_5''	0	α_5''	α_5'
$\boldsymbol{\alpha_6'}$	α_6'	α_6'	α_2'	α_8'	α_8'	α_4'	0	0	α_6'	α_2'	α_2'	α_8'	α_4'	α_4'
$\boldsymbol{\alpha_7'}$	α_7'	α_7''	α_4'	α_1''	α_1'	α_8'	α_5''	α_5'	α_2'	α_9'	α_9''	α_6'	α_3''	α_3'
$\boldsymbol{\alpha_7''}$	α_7''	α_7'	α_4'	α_1'	α_1''	α_8'	α_5'	α_5''	α_2'	α_9''	α_9'	α_6'	α_3'	α_3''
$\boldsymbol{\alpha_8'}$	α_8'	α_8'	α_6'	α_4'	α_4'	α_2'	0	0	α_8'	α_6'	α_6'	α_4'	α_2'	α_2'
$\boldsymbol{\alpha_9'}$	α_9'	α_9''	α_8'	α_7''	α_7'	α_6'	α_5'	α_5''	α_4'	α_3''	α_3'	α_2'	α_1'	α_1''
$\boldsymbol{\alpha_9''}$	α_9''	α_9'	α_8'	α_7'	α_7''	α_6'	α_5''	α_5'	α_4'	α_3'	α_3''	α_2'	α_1''	α_1'

Table 2 (also referenced in the Appendix) synthesizes the behavior of the last digit/two digits of the congruence classes modulo 10/modulo 20 (respectively) enumerated in 2.2. In detail, we note that all the even entries in Table 2 (i.e., 2, 4, 6, and 8) are assumed modulo 10, while the remaining entries (i.e., 1, 11, 3, 13, 7, 17, 5, 15, 9, and 19) are considered modulo 20.

Table 2. Transformation through the product of all pairs of congruence classes modulo 10 or 20 considered by 2.2.

·	**1**	**11**	**2, 8**	**3, 7**	**13, 17**	**4**	**5**	**15**	**6**	**9**	**19**
1	1	11	2, 8	3, 7	13, 17	4	5	15	6	9	19
11	11	1	2, 8	13, 17	3, 7	4	15	5	6	19	9
2, 8	2, 8	2, 8	4, 6	4, 6	4, 6	2, 8	0	0	2, 8	2, 8	2, 8
3, 7	3, 7	13, 17	4, 6	9, 1	19, 11	2, 8	15	5	2, 8	3, 7	13, 17
13, 17	13, 17	3,7	4, 6	19, 11	9,1	2, 8	5	15	2, 8	13, 17	3, 7
4	4	4	2, 8	2, 8	2, 8	6	0	0	4	6	6
5	5	15	0	15	5	0	5	15	0	5	15
15	15	5	0	5	15	0	15	5	0	15	5
6	6	6	2, 8	2, 8	2, 8	4	0	0	6	4	4
9	9	19	2, 8	3, 7	13, 17	6	5	15	4	1	11
19	19	9	2, 8	13, 17	3, 7	6	15	5	4	11	1

As a general result, we note that, assuming $q, \frac{a}{q} \in \mathbb{N} \setminus \{1\} : a \not\equiv 0 \pmod{10}$,

$$v_{\bar{b}}\left(q \cdot \frac{a}{q}\right) \geq \min\left\{v_{\bar{b}}(q), v_{\bar{b}}\left(\frac{a}{q}\right)\right\} \tag{2.4}$$

always holds (see the Appendix for the proof).

Thus, for each tetration base a as above, $q \mid a$ implies $v_{\bar{b}}(a) \geq \min\left\{v_{\bar{b}}(q), v_{\bar{b}}\left(\frac{a}{q}\right)\right\}$.

In this regard, we note that if $a := 999\ldots999$ (repunit 9's), $q := 111\ldots111$ (repunit 1's), and $k \in \mathbb{N}$ are such that $10^{k-1} < q < a < 10^k$, then the difference $v_{\bar{b}}(a) - \min\left\{v_{\bar{b}}(q), v_{\bar{b}}\left(\frac{a}{q}\right)\right\}$ is always equal to $k-1$ (since $v_{\bar{b}}(999\ldots999) = v_1(10^k - 1) = k$ while $v_{\bar{b}}(9) = v_1(10^1 - 1) = 1 = v_2(111\ldots111) = v_{\bar{b}}(111\ldots111)$).

In the end, the constant congruence speed of every integer greater than 1 and not divisible by 10 is necessarily greater than or equal to the minimum of the constant congruence speeds of its factors.

Remark 1. Despite the asymmetrical nature of relation 2.4, which defines only a (weak) link between the constant congruence speed of a given integer (greater than 1 and not a multiple of 10) and its factorization, Section 3 of [4] shows that the only prime numbers greater than 5 with a unit constant congruence speed are necessarily congruent to 2, 3, 4, 6, 8, 9, 11, 12, 13, 14, 16, 17, 19, 21, 22, or 23 modulo 5^2. Furthermore, Theorem 3 of [3] establishes the existence of infinitely many prime numbers characterized by each positive value of constant congruence speed, so the constraint $v_{\bar{b}}(a) = 1$ is neither a sufficient nor a necessary condition for the primality of a.

As a result, if we combine the popular primality criterion that every prime greater than 3 is congruent to 1 or 5 modulo 6 with the additional requirement that the constant congruence speed must equal 1, the observed increase in the frequency of primes (within a range of natural numbers) is merely a trivial consequence of the fact that we are also excluding all multiples of 5 from the residual list of candidate primes.

We do not explore this aspect further, as the aim of the present remark is to outline a brief application tip of the constant congruence speed formula introduced in [3], beyond its use in recreational mathematics contexts – such as proving that the exact number of stable digits of the well-known Graham's number, g_{64}, is precisely $\operatorname{slog}_3(g_{64}) - 1$, where $\operatorname{slog}_3(g_{64})$ denotes the base-3 super-logarithm of Graham's number itself [6].

As stated in Definition 3.2 of [6], "(. . .) we call phase shift of a at height b the congruence class modulo 10 of the difference between the rightmost non-stable digit of ${}^{b}a$".

For brevity purposes, similarly to Definition we set $s_{\bar{b}}(a) := s_{\bar{b}(a)}(a)$ and then we can compactly indicate the phase shift each integer tetration base $a \in \mathbb{N}_0$ as $\boldsymbol{\mathfrak{o}}(a)$, where $\boldsymbol{\mathfrak{o}}(a)$ is defined as follows:

$$\boldsymbol{\mathfrak{o}}(a) := \begin{cases} (s_1(a), s_2(a), \ldots, s_{\bar{b}-1}(a); [s_{\bar{b}}(a), s_{\bar{b}+1}(a), s_{\bar{b}+2}(a), s_{\bar{b}+3}(a)]) & \text{if} \quad s_{\bar{b}}(a) \neq s_{\bar{b}+2}(a) \\ (s_1(a), s_2(a), \ldots, s_{\bar{b}-1}(a); [s_{\bar{b}}(a), s_{\bar{b}+1}(a)]) & \text{if} \quad (s_{\bar{b}}(a) = s_{\bar{b}+2}(a) \land s_{\bar{b}}(a) \neq s_{\bar{b}+1}(a)) \\ (s_1(a), s_2(a), \ldots, s_{\bar{b}-1}(a); [s_{\bar{b}}(a)]) & \text{if} \quad s_{\bar{b}}(a) = s_{\bar{b}+1}(a) \\ (s_1(a), s_2(a); [s_{\bar{b}}(a)]) & \text{if} \quad 10 \mid a \\ (; [9, 1]) & \text{if} \quad a = 0 \\ (; [0]) & \text{if} \quad a = 1 \end{cases} . \tag{2.5}$$

In particular, we call *asymptotic* phase shift the array $[s_{\bar{b}}(a), s_{\bar{b}+1}(a), s_{\bar{b}+2}(a), s_{\bar{b}+3}(a)]$ (or $[s_{\bar{b}}(a), s_{\bar{b}+1}(a)]$, or $[s_{\bar{b}}(a)]$) of $\boldsymbol{\mathfrak{o}}(a)$ (see Section 3 of [6]).

In 2.5, the statement "$10 \mid a \Rightarrow \text{ס}(a) = (s_1(a), s_2(a); [s_{\bar{b}}(a)])$", arises from the trivial consideration that $\bar{b}(a) \leq 3$ holds for all $a \in \{\text{multiples of } 10\}$ (see the comments of the OEIS sequence A377124 [2]).

Assuming that $a \in \mathbb{N} \setminus \{1\}$ is such that $10 \nmid a$, from [4], we have that $\bar{b}(a) - 2 \leq \tilde{\nu}(a)$, where

$$\tilde{\nu}(a) \coloneqq \begin{cases} \nu_5(a-1) & \text{if} \quad a \equiv 11 \pmod{20} \\ \nu_5(a^2+1) & \text{if} \quad a \equiv 3, 7 \pmod{10} \\ \nu_5(a+1) & \text{if} \quad a \equiv 9 \pmod{20} \\ 2 & \text{if} \quad a = 5 \\ 1 & \text{if} \quad (a \equiv 5 \pmod{10} \wedge a \neq 5) \\ 1 & \text{if} \quad a \equiv 2, 6, 16, 18, 19 \pmod{20} \\ 0 & \text{if} \quad a \equiv 1, 4, 8, 12, 14 \pmod{20} \end{cases} . \tag{2.6}$$

Accordingly, let $\tilde{b}(a) \coloneqq \tilde{\nu}(a) + 2$ and $\bar{b}(a) \leq \tilde{b}(a)$ follows.

Remark 2. In (2.6), we distinguish between the case $a = 5$ (characterized by $\bar{b}(a) = 4$) and $a : (a \equiv 5 \pmod{10} \wedge a \neq 5)$ (where $\bar{b}(a) = 3$ holds). Then, we observe that 5 is the only integer greater than 1 whose third tetration is congruent to its second tetration modulo 1 plus the number of the digits of its second tetration (i.e., the only solution in $\mathbb{N} \setminus \{1\}$ of ${}^3a \equiv {}^2a \pmod{10^{\lfloor \log_{10}({}^2a) \rfloor + 2}}$ is 5 – see [6], page 3). Thus, for each a congruent to 5 modulo 10, a sufficient but not necessary condition that ensures the perfect match between the congruence speed and the number of new stable digits of ${}^ba$ is given by $b > 2$, so we can finally write $\text{ע}(5) = (1, 4, 3; 2)$ (instead of $(1, 3, 4; 2)$) and conclude that ${}^ba$ exhibits exactly $1 + 4 + 3 + (b-3) \cdot 2$ stable digits at heights $3, 4, 5, \ldots$ (since $v_1(5) + v_2(5) + v_3(5) = 1 + 4 + 3 = 1 + 3 + 4$, $\bar{b}(5) = 4$, and $v_{\bar{b}}(5) = 2$).

Now, if we are not able to determine the exact value of $\bar{b}(a)$ for some given integer tetration base a above 1, we can still write the congruence speed of a as

$$\tilde{\text{ע}}(a) \coloneqq \begin{cases} (v_1(a), v_2(a), \ldots, v_{\tilde{b}-1}(a); v_{\tilde{b}}(a)) & \text{if} \quad 10 \nmid a \\ (v_1(a), v_2(a), \ldots; +\infty) & \text{if} \quad 10 \mid a \end{cases} . \tag{2.7}$$

Similarly, we can express the phase shift of every $a \in \mathbb{N} \setminus \{1\}$ in the form

$$\tilde{\text{ס}}(a) \coloneqq \begin{cases} (s_1(a), s_2(a), \ldots, s_{\tilde{b}-1}(a); [[s_{\tilde{b}}(a), s_{\tilde{b}+1}(a), s_{\tilde{b}+2}(a), s_{\tilde{b}+3}(a)]]) & \text{if} \;\; s_{\tilde{b}}(a) \neq s_{\tilde{b}+2}(a) \\ (s_1(a), s_2(a), \ldots, s_{\tilde{b}-1}(a); [[s_{\tilde{b}}(a), s_{\tilde{b}+1}(a)]]) & \text{if} \;\; (s_{\tilde{b}}(a) = s_{\tilde{b}+2}(a) \wedge s_{\tilde{b}}(a) \neq s_{\tilde{b}+1}(a)) \\ (s_1(a), s_2(a), \ldots, s_{\tilde{b}-1}(a); [s_{\tilde{b}}(a)]) & \text{if} \;\; s_{\tilde{b}}(a) = s_{\tilde{b}+1}(a) \\ (s_1(a), s_2(a); [s_3(a)]) & \text{if} \;\; 10 \mid a \end{cases} , \tag{2.8}$$

where $[[s_{\tilde{b}}(a), s_{\tilde{b}+1}(a), s_{\tilde{b}+2}(a), s_{\tilde{b}+3}(a)]]$ is always one of the four circular permutations of the asymptotic phase shift of a (i.e., $[s_{\bar{b}}(a), s_{\bar{b}+1}(a), s_{\bar{b}+2}(a), s_{\bar{b}+3}(a)]$ is necessarily equal to $[[s_{\tilde{b}}(a), s_{\tilde{b}+1}(a), s_{\tilde{b}+2}(a), s_{\tilde{b}+3}(a)]]$, or $[[s_{\tilde{b}+1}(a), s_{\tilde{b}+2}(a), s_{\tilde{b}+3}(a), s_{\tilde{b}}(a)]]$, or $[[s_{\tilde{b}+2}(a), s_{\tilde{b}+3}(a), s_{\tilde{b}}(a), s_{\tilde{b}+1}(a)]]$, or $[[s_{\tilde{b}+3}(a), s_{\tilde{b}}(a), s_{\tilde{b}+1}(a), s_{\tilde{b}+2}(a)]]$, and similarly $[s_{\bar{b}}(a), s_{\bar{b}+1}(a)] = [[s_{\tilde{b}}(a), s_{\tilde{b}+1}(a)]]$ or $[s_{\bar{b}}(a), s_{\bar{b}+1}(a)] = [[s_{\tilde{b}+1}(a), s_{\tilde{b}}(a)]]$), while it is trivial to point out that $s_{\tilde{b}}(a) = s_{\tilde{b}+1}$ implies $[s_{\tilde{b}}(a)] = [s_{\bar{b}}(a)]$ (so choosing to write $[[s_{\tilde{b}}(a)]]$ instead of $[s_{\tilde{b}}(a)]$ or even $[s_{\bar{b}}(a)]$ would be pointless).

In this case, we call *modular* phase shift the array $[[s_{\tilde{b}}(a), s_{\tilde{b}+1}(a), s_{\tilde{b}+2}(a), s_{\tilde{b}+3}(a)]]$ (or $[[s_{\tilde{b}}(a), s_{\tilde{b}+1}(a)]]$, or $[s_{\tilde{b}}(a)]$) of $\tilde{\text{ס}}(a)$.

Lastly, for each $a \coloneqq j \cdot 10^c$ such that $j, c \in \mathbb{N} \setminus \{1\}$ (which implies $10 \mid a$ by construction), the congruence speed of such tetration bases can be compactly rewritten (for each $b \in \mathbb{N}$) as

$$v_b(j \cdot 10^c) = \begin{cases} 0 & \text{iff} \;\; j = 0 \\ c \cdot \left({}^{b-1}(j \cdot 10^c) - {}^{b-2}(j \cdot 10^c)\right) & \text{otherwise} \end{cases} ,$$

by [5], and thus, for positive integers c and j, we can finally state that

$$\text{ע}(a) \coloneqq \begin{cases} (v_1(a), v_2(a), \ldots, v_{\bar{b}-1}(a); v_{\bar{b}}(a)) & \text{if} \quad a > 1 \wedge 10 \nmid a \\ \\ (c, c \cdot \left({}^{0}(j \cdot 10^c) - {}^{-1}(j \cdot 10^c)\right), c \cdot \left({}^{1}(j \cdot 10^c) - {}^{0}(j \cdot 10^c)\right), c \cdot \left({}^{2}(j \cdot 10^c) - {}^{1}(j \cdot 10^c)\right), \ldots; +\infty) \\ & \text{if} \quad a = j \cdot 10^c \;\; (j, c \in \mathbb{N}) \\ \\ (; v_{\bar{b}}(a)) & \text{if} \quad a = 0 \\ \\ (1; v_{\bar{b}}(a)) & \text{if} \quad a = 1 \end{cases} . \tag{2.9}$$

Let g be a composite integer. We define the g-adic valuation of an integer d as the function $\gamma_g : \mathbb{Z} \to \mathbb{N} \cup \{+\infty\}$ such that

$$\gamma_g(d) := \begin{cases} \max\{q \in \mathbb{N}_0 : g^q \mid d\} & \text{if} \quad d \neq 0 \\ +\infty & \text{if} \quad d = 0 \end{cases}.$$

Trivially, $\gamma_g(d)$ corresponds to the largest exponent q such that g^q divides d.

For each $b > 2$, we have $\left({}^{b}a \equiv {}^{b+1}a \pmod{10^{\sum_{k=1}^{b} v_k(a)}}\right) \wedge \left({}^{b}a \not\equiv {}^{b+1}a \pmod{10^{\left(\sum_{k=1}^{b} v_k(a)\right)+1}}\right)$.

Thus, if $b > \bar{b}(a)$ and $1 < a : a \not\equiv 0 \pmod{10}$ are given,

$$\left({}^{b}a \equiv {}^{b+1}a \pmod{10^{v_1(a)+v_2(a)+\cdots+v_{\bar{b}(a)-1}+\left(b(a)-\left(\bar{b}(a)-1\right)\right)\cdot v_{\bar{b}}(a)}}\right) \wedge$$
$$\left({}^{b}a \not\equiv {}^{b+1}a \pmod{10^{v_1(a)+v_2(a)+\cdots+v_{\bar{b}(a)-1}+\left(b(a)-\left(\bar{b}(a)-1\right)\right)\cdot v_{\bar{b}}(a)+1}}\right)$$

follows by construction.

Under the constraints above, the integer tetration ${}^{b}a$ exhibits exactly $\sum_{k=1}^{\bar{b}(a)-1} v_k(a) + \left(b - \left(\bar{b}(a) - 1\right)\right) \cdot v_{\bar{b}}(a)$ stable digits, and then the bound $(b-2) \cdot v_{\bar{b}}(a) \leq v_1(a) + v_2(a) + \cdots + v_b(a) \leq (b+1) \cdot v_{\bar{b}}(a)$ holds for each pair (a, b) as above (e.g., ${}^{400}2$ has exactly $0 + 0 + (400 - 2) \cdot 1$ frozen digits).

For example, let $c, k, t \in \mathbb{N}$ be such that $t \geq 2 + \gamma_{10}(c)$. Then, for any given c, the congruence speed of $a := \left(10^{k+t} + 10^{t-\gamma_{10}(c)} + 1\right)^c$ is

$$\text{ע}(a) = (2 \cdot t; t), \tag{2.10}$$

which does not depend on k, so (2.10) proves the existence of infinitely many c-th perfect powers whose constant congruence speed is also equal to c (for this purpose, it is sufficient to assume $t = c$ and $\text{ע}\left(\left(10^{k+c} + 10^{c-\gamma_{10}(c)} + 1\right)^c\right) = (2 \cdot c; c)$ follows – see [7]).

We provide an improved version of the above statement in Remark 3.

Remark 3. Let $c, k, m+1, t \in \mathbb{N}$ be such that $t \geq 2 + \gamma_{10}(c)$ (i.e., t greater than 1 plus the number of trailing 0's at the end of c, if any), $k \geq t$, and assume $m \equiv 0, 1 \pmod{3}$. Then, $\left(3 \cdot m \cdot 10^{k+t} + 10^{k+t} + 10^{t-\gamma_{10}(c)} + 1\right)^c$ is always a c-th perfect power (i.e., exactly a c-th perfect power by the *digit sum divisible by* 3 *and not by* 3^2 argument) characterized by a congruence speed of $(2 \cdot t; t)$.

On this purpose, we note that, for each positive integer c, also $\text{ע}\left(\left(10^{c-\gamma_{10}(c)} + 1\right)^c\right) = (2 \cdot c; c)$, and then $\left(10^{c-\gamma_{10}(c)} + 1\right)^c$ is always a perfect power of degree exactly c with the same constant congruence speed (by Mihăilescu's theorem [1] – since 10^n is a perfect power not equal to 2^3).

3. CONCLUSION

The notation we have introduced for the congruence speed and phase shift of tetration may facilitate more sophisticated investigations of these properties in the future.

We expect that this specialized nomenclature has not only supported our proof in the Appendix, which shows how the relationship between the constant congruence speed and the corresponding multiplicative group ensures that

$$v_{\bar{b}}(a) \geq \min\left\{v_{\bar{b}}(q), v_{\bar{b}}\left(\frac{a}{q}\right)\right\}$$

holds for each pair $\left(q, \frac{a}{q}\right)$ of integers greater than 1 and not divisible by 10, but may also inspire further research aimed at providing a comprehensive description of the set M of all pairs of positive integers whose minimum constant congruence speed equals that of their product. Such a classification would allow us to determine in advance whether $v_{\bar{b}}\left(q \cdot \frac{a}{q}\right) = \min\left\{v_{\bar{b}}(q), v_{\bar{b}}\left(\frac{a}{q}\right)\right\}$ is satisfied for the given pair $\left(q, \frac{a}{q}\right)$ (at present, for example, we can confirm that $(624, 22943) \in M$ only after directly computing $v_{\bar{b}}(14316432) = 4$ and verifying that $\min\{v_{\bar{b}}(624), v_{\bar{b}}(22943)\} = \min\{4, 5\} = 4$ also holds).

Our study of congruence speed and phase shift has been entirely focused on radix-10; however, extending these results to different numeral systems will surely be a key objective for future research in the field.

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Appendix

This Appendix is devoted to proving that the constant congruence speed of each integer greater than 1 and not a multiple of 10 is greater than or equal to the minimum of the constant congruence speeds of every subset of its factors whose product equals the given tetration base (so that $v_{\bar{b}}\left(q \cdot \frac{a}{q}\right) \geq \min\left\{v_{\bar{b}}(q), v_{\bar{b}}\left(\frac{a}{q}\right)\right\}$ holds for all $q, \frac{a}{q} \in \mathbb{N} \setminus \{\{\text{multiples of } 10\} \cup \{1\}\}$).

For this purpose, given $m := \frac{a}{q}$, we consider 2.2 to show that 2.4 holds for each pair of positive integers (q, m) whose product $q \cdot m$ is greater than 1 and not a multiple of 10. Since, in 2.4, q and m (i.e., $\frac{a}{q}$) can independently span the entire set $\mathbb{N} \setminus \{\{\text{multiples of } 10\} \cup \{1\}\}$. This result implies the general property that the constant congruence speed of every integer greater than 1 (whose last digit is not 0) cannot be lower than the lowest constant congruence speed among its factors (as we can ideally iterate the same process as many times as needed). Consequently, we will proceed case by case to fulfill all the lines of 2.2.

Since all the cases mentioned above flow in the 7 types of multiplicative classes listed in Table 3, we write the symbol (r) next to some class products to make clear that, in the corresponding case of the present proof, the variables q and m should be swapped with each other.

Table 3. The seven fundamental multiplicative classes enumerated in 2.2.

$\cdot$	**1**	**11**	**2, 8**	**3, 7**	**13, 17**	**4**	**5**	**15**	**6**	**9**	**19**
1	I	I; II	III	III; II	III; I	II	I	II	I	II; I	II
11	I; II	I; IV	III	III; IV	III; II(r)	II	II(r)	IV	I	II; II(r)	II; IV
2, 8	III	III	V; VI	V; VI	V; VI	VII	/	/	III(r)	VII	VII
3, 7	III; I	III; IV	V; VI	IV; V; VI	V; II(r); VI	VII	II(r)	IV	III(r)	VII; II(r)	IV; VII
13, 17	III; I	III; II(r)	V; VI	V; II(r); VI	V; I; VI	VII	I	II	III(r)	VII; I	VII; II
4	II	II	VII	VII	VII	IV	/	/	II(r)	IV	IV
5	I	II(r)	/	II(r)	I	/	I	II	/	I	II
15	II	IV	/	IV	II	/	II	IV	/	II(r)	IV
6	I	I	III(r)	III(r)	III(r)	II(r)	/	/	I	II	II
9	II; I	II; II(r)	VII	VII; II(r)	VII; I	IV	I	II(r)	II	IV; I	IV; II
19	II	II; IV	VII	IV; VII	VII; II	IV	II	IV	II	IV; II	IV

From here on, we will assume that q and m are two integers greater than 1 and not divisible by 10, so their constant congruence speeds also belong to $\mathbb{N}$. The same happens for the constant congruence speed of their product (as $(q \cdot m) \in \mathbb{N} \setminus \{\{\text{multiples of } 10\} \cup \{1\}\}$ implies that $v_{\bar{b}}(q \cdot m) \in \mathbb{N}$).

CLASS I PRODUCTS

Let $q \equiv m \equiv 1 \pmod{20}$. We consider the number $q \cdot m$, which is also congruent to 1 modulo 20, aiming to evaluate its constant congruence speed (we are interested in comparing $v_{\bar{b}}(q \cdot m)$ with $v_{\bar{b}}(q)$ and $v_{\bar{b}}(m)$).

With reference to 2.2, we can first suppose that $\min\{\nu_5(q \cdot m - 1), \nu_2(q \cdot m - 1)\} = \nu_5(q \cdot m - 1)$, $\min\{\nu_2(q - 1), \nu_5(q - 1)\} = \nu_5(q - 1)$, and $\min\{\nu_2(m - 1), \nu_5(m - 1)\} = \nu_5(m - 1)$. If so, $\nu_5(q \cdot m - 1)$ can be rewritten as $\nu_5(q - 1 + (m - 1) \cdot q)$ and then, by the properties of p-adics, we get

$$\nu_5(q - 1 + (m - 1) \cdot q) \geq \min\{\nu_5(q - 1), \nu_5((m - 1) \cdot q)\}.$$

Using the properties of p-adics on the product,

$$\nu_5((m - 1) \cdot q) \geq \nu_5(m - 1)$$

easily follows (since $\nu_5((m - 1) \cdot q) = \nu_5(m - 1) + \nu_5(q)$).

Thus, the 5-adic valuation of $(q \cdot m - 1)$ is greater than or equal to the minimum between $\nu_5(q - 1)$ and $\nu_5(m - 1)$.

On the other hand, if $\min\{\nu_2(q-1), \nu_5(q-1)\} = \nu_2(q-1)$ and $\min\{\nu_2(m-1), \nu_5(m-1)\} = \nu_2(m-1)$, the proof would remain unchanged, provided that we explicitly state the inequalities $\nu_5(q-1) \geq \nu_2(q-1)$ and $\nu_5(m-1) \geq \nu_2(m-1)$.

Conversely, if $\min\{\nu_5(q \cdot m-1), \nu_2(q \cdot m-1)\} = \nu_2(q \cdot m-1)$, we can assume that $\min\{\nu_2(q-1), \nu_5(q-1)\} = \nu_2(q-1)$ and also that $\min\{\nu_2(m-1), \nu_5(m-1)\} = \nu_2(m-1)$.

In the same way, we observe that $\nu_2(q \cdot m-1)$ can be rewritten as $\nu_2(q-1+(m-1) \cdot q)$ and then, applying the well-known p-adic properties, we find that the 2-adic valuation of $(q \cdot m-1)$ is greater than or equal to the minimum between $\nu_2(q-1)$ and $\nu_2(m-1)$.

CLASS II PRODUCTS

Here we consider the case where $q \equiv 1 \pmod{20}$ and $m \equiv 11 \pmod{20}$ (see 2.2), so the product $q \cdot m$ belongs to the congruence class 11 modulo 20.

Then, we assume that $\min\{\nu_2(q \cdot m+1), \nu_5(q \cdot m-1)\} = \nu_2(q \cdot m+1)$, $\min\{\nu_2(q-1), \nu_5(q-1)\} = \nu_2(q-1)$, and $\min\{\nu_2(m+1), \nu_5(m-1)\} = \nu_2(m+1)$.

At this point, we note how $\nu_2(q \cdot m+1)$ can be rewritten as $\nu_2(m+1+m \cdot (q-1))$ so that, from the properties of p-adic valuations, we get

$$\nu_2(m+1+m \cdot (q-1)) \geq \min\{\nu_2(m+1), \nu_2((q-1) \cdot m)\}.$$

Hence, from $\nu_2((q-1) \cdot m) = \nu_2(q-1) + \nu_2(m)$, it follows that $\nu_2((q-1) \cdot m) \geq \nu_2(q-1)$.

Thus, the 2-adic valuation of $(q \cdot m+1)$ is greater than or equal to the minimum between $\nu_2(m+1)$ and $\nu_2(q-1)$.

On the contrary, if we had supposed that $\min\{\nu_2(q-1), \nu_5(q-1)\} = \nu_5(q-1)$ and $\min\{\nu_2(m+1), \nu_5(m-1)\} = \nu_5(m-1)$, the proof of this case would not have changed substantially, provided that we had explicitly stated that $\nu_2(q-1) \geq \nu_5(q-1)$ and $\nu_2(m-1) \geq \nu_5(m-1)$.

CLASS III PRODUCTS

Let $q \equiv 1 \pmod{20}$ and $m \equiv 2, 8 \pmod{10}$. We evaluate $q \cdot m$ knowing that it is congruent to 2 or 8 modulo 10.

With reference to 2.2, we consider $\nu_5((q \cdot m)^2+1)$ and assume that $\min\{\nu_2(q-1), \nu_5(q-1)\} = \nu_5(q-1)$. The other value to be taken into account is $\nu_5(m^2+1)$, which is associated with the congruence classes 2 and 8 modulo 10.

We note that $\nu_5((q \cdot m)^2+1)$ can be rewritten as $\nu_5((q^2-1) \cdot (m^2-1)+q^2+m^2)$, which is greater than or equal to

$$\min\left\{\nu_5((q^2-1) \cdot (m^2-1)), \nu_5(q^2+m^2)\right\}.$$

Since $\nu_5((q^2-1) \cdot (m^2-1)) = \nu_5((q-1) \cdot (q+1) \cdot (m-1) \cdot (m+1))$, by the properties of the p-adic valuations,

$$\nu_5((q-1) \cdot (q+1) \cdot (m-1) \cdot (m+1)) \geq \nu_5(q-1)$$

easily follows.

At the same time, from $\nu_5(q^2+m^2) = \nu_5(m^2+1+q^2-1)$, we have that $\nu_5(m^2+1+q^2-1) \geq \min\left\{\nu_5(m^2+1), \nu_5(q^2-1)\right\}$ and, since $\nu_5(q^2-1) = \nu_5((q-1) \cdot (q+1)) = \nu_5(q-1) + \nu_5(q+1)$, it follows that $\nu_5(q^2-1) \geq \nu_5(q-1)$.

Thus, the 5-adic valuation of $((q \cdot m)^2+1)$ is greater than or equal to the minimum between $\nu_5(q-1)$ and $\nu_5(m^2+1)$.

In contrast, if $\min\{\nu_2(q-1), \nu_5(q-1)\} = \nu_2(q-1)$ had been the case, the proof would be analogous by noting that $\nu_5(q-1) \geq \nu_2(q-1)$.

CLASS IV PRODUCTS

Let $q \equiv m \equiv 11 \pmod{20}$. We consider the product $q \cdot m$, which belongs to the congruence class 1 modulo 20.

Following 2.2, we suppose that $\min\{\nu_2(q \cdot m-1), \nu_5(q \cdot m-1)\} = \nu_2(q \cdot m-1)$, $\min\{\nu_2(q+1), \nu_5(q-1)\} = \nu_2(q+1)$, and $\min\{\nu_2(m+1), \nu_5(m-1)\} = \nu_2(m+1)$.

From $\nu_2((q+1) \cdot m - m - 1) = \nu_2((q+1) \cdot m - (m+1))$, we get $\nu_2((q+1) \cdot m - (m+1)) \geq \min\{\nu_2((q+1) \cdot m), \nu_2(m-1)\}$.

Thus, $\nu_2((q+1) \cdot m) = \nu_2(q+1) + \nu_2(m)$ so that $\nu_2((q+1) \cdot m) \geq \nu_2(q+1)$.

As a result, the 2-adic valuation of $(q \cdot m-1)$ is greater than or equal to the minimum between $\nu_2(q+1)$ and $\nu_2(m-1)$.

Lastly, if $\min\{\nu_2(q+1), \nu_5(q-1)\} = \nu_5(q-1)$ and $\min\{\nu_2(m+1), \nu_5(m-1)\} = \nu_5(m-1)$ had been the case, the proof remains the same by noting that $\nu_2(q+1) \geq \nu_5(q-1)$ and $\nu_2(m-1) \geq \nu_5(m-1)$.

CLASS V PRODUCTS

Let $q \equiv 2, 8 \pmod{10}$ and $m \equiv 2, 8 \pmod{10}$. Specifically, when $q \equiv 2 \pmod{10}$ and $m \equiv 2 \pmod{10}$, the product $q \cdot m$ belongs to the congruence class 4 modulo 10. In this case, following 2.2, we consider $\nu_5(q \cdot m + 1)$ to show that it cannot fall below the minimum between $\nu_5(q^2+1)$ and $\nu_5(m^2+1)$.

Let $\beta := \min\{\nu_5(q^2+1), \nu_5(m^2+1)\}$. Then, $q^2 + 1 \equiv 0 \pmod{5^\beta}$ implies

$$q^2 \equiv -1 \pmod{5^\beta} \tag{3.1}$$

and, symmetrically, $m^2 + 1 \equiv 0 \pmod{5^\beta}$ implies

$$m^2 \equiv -1 \pmod{5^\beta}. \tag{3.2}$$

By multiplying term by term both sizes of 3.1 and 3.2, we get

$$q^2 \cdot m^2 \equiv 1 \pmod{5^\beta}.$$

Hence,

$$(q \cdot m)^2 \equiv 1 \pmod{5^\beta}.$$

It follows that $q \cdot m \equiv -1 \pmod{5^\beta}$ or $q \cdot m \equiv 1 \pmod{5^\beta}$.

Since here we are considering $q \cdot m \equiv 4 \pmod{10}$, we pick

$$q \cdot m \equiv -1 \pmod{5^\beta} \tag{3.3}$$

so that $q \cdot m + 1 \equiv 0 \pmod{5^\beta}$.

By 3.3, $\nu_5(q \cdot m + 1) \geq \beta$ follows, and so we have proved that

$$\nu_5(q \cdot m + 1) \geq \min\left\{\nu_5(q^2+1), \nu_5(m^2+1)\right\}.$$

CLASS VI PRODUCTS

As $q \equiv 2 \pmod{10}$ and $m \equiv 8 \pmod{10}$ are given, the product $q \cdot m$ belongs to the congruence class 6 modulo 10.

In 2.2, we need to consider $\nu_5(q \cdot m - 1)$ to show that it is always greater than or equal to the minimum between $\nu_5(q^2+1)$ and $\nu_5(m^2+1)$.

Consequently, we can reconstruct the proof of the Class V products by following the same steps, as we choose $q \cdot m \equiv 1 \pmod{5^\beta}$ instead of $q \cdot m \equiv -1 \pmod{5^\beta}$, since here we assume $q \cdot m \equiv 6 \pmod{10}$.

In this way, we easily show that $\nu_5(q \cdot m - 1) \geq \beta$, from which $\nu_5(q \cdot m - 1) \geq \min\{\nu_5(q^2+1), \nu_5(m^2+1)\}$ follows.

CLASS VII PRODUCTS

Lastly, we consider $q \equiv 2, 8 \pmod{10}$ and $m \equiv 4 \pmod{10}$. Then, $q \cdot m \equiv 2, 8 \pmod{10}$ and, from 2.2, we have to evaluate $\nu_5((q \cdot m)^2 + 1)$ by comparing it to the minimum between $\nu_5(q^2+1)$ and $\nu_5(m+1)\}$.

We note that the proof would be almost identical to the one we have provided for the Class III products, but now we need to adopt a different approach, consistent with the new hypotheses. In detail, we focus on the product $\nu_5((q \cdot m)^2 + 1)$ and observe that it can be rewritten as $\nu_5((q^2-1) \cdot (m^2-1) + q^2 + m^2)$, which is greater than or equal to $\min\{\nu_5((q^2-1) \cdot (m^2-1)), \nu_5(q^2+m^2)\}$.

From $\nu_5((q^2-1) \cdot (m^2-1)) = \nu_5((q-1) \cdot (q+1) \cdot (m-1) \cdot (m+1))$, we get $\nu_5((q^2-1) \cdot (m^2-1)) \geq \nu_5(m+1)$ by applying the well-known property of the p-adic product (i.e., $\nu_5((q-1) \cdot (q+1) \cdot (m-1) \cdot (m+1)) = \nu_5(q-1) + \nu_5(q+1) + \nu_5(m-1) + \nu_5(m+1)$).

Similarly, from $\nu_5(q^2+m^2) = \nu_5(m^2 - 1 + q^2 + 1)$, we obtain $\nu_5(q^2+m^2) \geq \min\{\nu_5(q^2+1), \nu_5(m^2-1)\}$, and since

$$\nu_5(m^2-1) = \nu_5((m-1) \cdot (m+1)) = \nu_5(m-1) + \nu_5(m+1),$$

$\nu_5(m^2-1) \geq \nu_5(m+1)$ follows.

Thus, there is no exception to the general rule among all Class I to Class VII products.

Therefore, $v_{\bar{5}}(q \cdot m) \geq \min\{v_{\bar{5}}(q), v_{\bar{5}}(m)\}$ holds for every pair (q, m) of integers greater than 1 and that do not end in 0. This concludes the proof of 2.4 (as $m := \frac{a}{q}$).