



Transactions of the Royal Society of South Africa

Publication details, including instructions for authors and subscription information:

<http://www.tandfonline.com/loi/ttrs20>

A PROOF BY ELEMENTARY METHODS, WITHOUT COMPLEX QUANTITIES, THAT EVERY ALGEBRAIC FUNCTION (WITH REAL COEFFICIENTS) HAS FACTORS OF THE FORM $(x^2 - px + q)$ (p, q , REAL) AND HENCE, EVERY ALGEBRAIC EQUATION WITH COEFFICIENTS REAL OR IMAGINARY, HAS REAL OR IMAGINARY ROOTS EQUAL IN NUMBER TO THE DEGREE OF THE EQUATION

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Published online: 26 Mar 2010.

To cite this article: W. N. Roseveare M.A. & Lawrence Crawford (1914) A PROOF BY ELEMENTARY METHODS, WITHOUT COMPLEX QUANTITIES, THAT EVERY ALGEBRAIC FUNCTION (WITH REAL COEFFICIENTS) HAS FACTORS OF THE FORM $(x^2 - px + q)$ (p, q , REAL) AND HENCE, EVERY ALGEBRAIC EQUATION WITH COEFFICIENTS REAL OR IMAGINARY, HAS REAL OR IMAGINARY ROOTS EQUAL IN NUMBER TO THE DEGREE OF THE EQUATION, Transactions of the Royal Society of South Africa, 4:1, 215-221, DOI: [10.1080/00359191409519532](https://doi.org/10.1080/00359191409519532)

To link to this article: <http://dx.doi.org/10.1080/00359191409519532>

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A PROOF BY ELEMENTARY METHODS, WITHOUT COMPLEX QUANTITIES, THAT EVERY ALGEBRAIC FUNCTION (WITH REAL COEFFICIENTS) HAS FACTORS OF THE FORM $(x^2 - px + q)$ (p, q , REAL) AND HENCE, EVERY ALGEBRAIC EQUATION WITH COEFFICIENTS REAL OR IMAGINARY, HAS REAL OR IMAGINARY ROOTS EQUAL IN NUMBER TO THE DEGREE OF THE EQUATION.

BY PROFESSOR W. N. ROSEVEARE, M.A.

(Communicated by Professor Lawrence Crawford.)

(Read June 17, 1914.)

Let $|a_0x^n + a_1x^{n-1} + \dots + a_n| \equiv u$, represent any algebraical function of x , of degree n , having its coefficients real: we proceed to consider whether it has a factor of the form $(x^2 - px + q)$, where p, q are real.

§ 1. The remainder when u is divided by $(x^2 - px + q)$ is, as is well known, obtained by putting $x^2 - px + q = 0$, i.e. $x^2 = px - q$, whence it follows that $x^i = p_{i-1}x - q_i$;

$$p_{i-1}, q_i \text{ being determined by } \begin{aligned} x^{i+1} &= p_{i-1}x^2 - q_ix \\ &= p_{i-1}(px - q) - q_ix. \end{aligned}$$

Whence

$$\left. \begin{aligned} p_i &\equiv pp_{i-1} - q_i \text{ and } q_{i+1} \equiv qp_{i-1} \\ \therefore p_i &\equiv pp_{i-1} - qp_{i-2} \end{aligned} \right\} \dots\dots\dots (i)$$

This reduction-formula combined with $p_0 = 1, q_1 = 0$ (i.e. $p_{-1} = 0$), gives all the values of p_i and q_i .

In the above work i may be a negative integer; we now proceed to show that the p 's and q 's for negative integers are closely connected with those for + integers.

Thus if

$$x^{-i} = p_{-i-1}x - qp_{-i-2}$$

and

$$x^i = p_{i-1}x - qp_{i-2},$$

we have

$$(p_{-i-1}x - qp_{-i-2})(p_{i-1}x - qp_{i-2}) \equiv 1.$$

$$\therefore p_{-i-1}p_{i-1}(px - q) - qx(p_{-i-1}p_{i-2} + p_{-i-2}p_{i-1}) + q^2p_{-i-2}p_{i-2} \equiv 1.$$

$$\therefore p \cdot p_{-i-1} \cdot p_{i-1} = q(p_{-i-1}p_{i-2} + p_{-i-2}p_{i-1}) \dots \dots \dots (1)$$

and

$$qp_{-i-1}p_{i-1} = q^2p_{-i-2}p_{i-2} - 1 \dots \dots \dots (2)$$

Since

$$pp_{i-1} - qp_{i-2} \equiv p_i, \text{ (1) reduces to } p_{-i-1}p_i = qp_{-i-2}p_{i-1};$$

$$\therefore \frac{p_{-i-1}}{p_{i-1}} = \frac{qp_{-i-2}}{p_i} = \text{by (2), } \frac{1}{q(p_i p_{i-2} - p_{i-1}^2)}.$$

And

$$\begin{aligned} p_i p_{i-2} - p_{i-1}^2 &= (pp_{i-1} - qp_{i-2})p_{i-2} - p_{i-1}(pp_{i-2} - qp_{i-3}) \\ &= q(p_{i-1}p_{i-3} - p_{i-2}^2) = \dots = q^{i-1}(p_i p_{-1} - p_0^2) = -q^{i-1}. \end{aligned}$$

Hence

$$p_{-i-1} = -\frac{p_{i-1}}{q^i}.$$

§ 2. Now the remainder of the division is

$$\begin{aligned} &a_0(p_{n-1}x - qp_{n-2}) + a_1(p_{n-2}x - qp_{n-3}) \dots + a_{n-1}x + a_n \\ &\equiv x(a_0p_{n-1} + a_1p_{n-2} + \dots + a_{n-1}p_0) - q(a_0p_{n-2} + a_1p_{n-3} + \dots + a_{n-2}p_0 + a_{n-1}p_{-1}). \end{aligned}$$

Therefore the conditions for no remainder are

$$\text{and } \left. \begin{aligned} a_0p_{n-1} + a_1p_{n-2} + \dots + a_{n-1}p_0 &= 0 \\ a_0p_{n-2} + a_1p_{n-3} + \dots + a_{n-2}p_0 + a_{n-1}p_{-1} &= 0 \end{aligned} \right\}$$

Employing the relation $p_i = pp_{i-1} - qp_{i-2}$, we can obtain from these two conditions any number of equations of the form

$$a_0p_{n-v} + a_1p_{n-v-1} \dots + a_{n-1}p_{-v} + a_n p_{-v-1} = 0.$$

The notation will be easier to follow if we continue this investigation for the special case of $n=8$: we shall write 0, 1, 2, etc. for a_0, a_1, a_2 , etc.

Using $p_{-1} = \frac{-p_{i-2}}{q^{i-1}}$ we get for $v=2, 3, \dots$ the equations

$$0p_6 + 1p_5 + 2p_4 + 3p_3 + 4p_2 + 5p_1 + 6p_0 - 8\frac{p_0}{q} = 0.$$

$$0p_5 + 1p_4 + 2p_3 + 3p_2 + 4p_1 + 5p_0 - 7\frac{p_0}{q} - 8\frac{p_1}{q^2} = 0.$$

.....

$$0p_2 + 1p_1 + 2p_0 - 4\frac{p_0}{q} - 5\frac{p_1}{q^2} - 6\frac{p_2}{q^3} - 7\frac{p_3}{q^4} - 8\frac{p_4}{q^5} = 0.$$

.....

$$0p_0 - 2\frac{p_0}{q} - 3\frac{p_1}{q^2} - 4\frac{p_2}{q^3} - 5\frac{p_3}{q^4} - 6\frac{p_4}{q^5} - 7\frac{p_5}{q^6} - 8\frac{p_6}{q^6} = 0.$$

These equations are seen to involve coefficients of p_i

- (1) of the form $0 - \frac{8}{q^4}, 1 - \frac{7}{q^3}, \dots$ which we shall write 08, 17, etc., the index of q being half the difference of the figures, and
- (2) of the form $-\frac{8}{q^5}, -\frac{7}{q^6}, \dots$ which we shall write $\cdot 8_5, \cdot 7_6, \dots$ the dot indicating zero here and in vacant places in the following determinants.

From the seven equations above $p_6, p_5, \dots p_0$ can be eliminated, giving the determinant equation

$$\begin{vmatrix} 0, & 1, & 2, & 3, & 4, & 5, & 68 \\ \cdot, & 0, & 1, & 2, & 3, & 48, & 57 \\ \cdot, & \cdot, & 0, & 1, & 28, & 37, & 46 \\ \cdot, & \cdot, & \cdot, & 08, & 17, & 26, & 35 \\ \cdot, & \cdot, & \cdot 8_5, & \cdot 7_4, & 06, & 15, & 24 \\ \cdot, & \cdot 8_6, & \cdot 7_5, & \cdot 6_4, & \cdot 5_3, & 04, & 13 \\ \cdot 8_7, & \cdot 7_6, & \cdot 6_5, & \cdot 5_4, & \cdot 4_3, & \cdot 3_2, & 02 \end{vmatrix} = 0.*$$

Moreover, the middle five equations (omitting the first and last) are linear equations for $p_5, p_4, \dots, p_1, p_0$, from which we get

$$\frac{p_1}{p_0} \text{ (i.e. } p) \equiv \begin{vmatrix} 0, & 1, & 2, & 3, & 57 \\ \cdot, & 0, & 1, & 28, & 46 \\ \cdot, & \cdot, & 08, & 17, & 35 \\ \cdot, & \cdot 8_5, & \cdot 7_4, & 06, & 24 \\ \cdot 8_6, & \cdot 7_5, & \cdot 6_4, & \cdot 5_3, & 13 \end{vmatrix} \div \begin{vmatrix} 0, & 1, & 2, & 3, & 48 \\ \cdot, & 0, & 1, & 28, & 37 \\ \cdot, & \cdot, & 08, & 17, & 26 \\ \cdot, & \cdot 8_5, & \cdot 7_4, & 06, & 15 \\ \cdot 8_6, & \cdot 7_5, & \cdot 6_4, & \cdot 5_3, & 04 \end{vmatrix}.$$

Calling the first determinant D_7 and the second pair D'_5 and D_5 we see that if $D_7=0$ gives us a real value of q , $p = \frac{D'_5}{D_5}$ gives us a *unique* corresponding value of p .

§ 3. We proceed to prove that $D_7=0$ has real roots in q (in the result we find that there are always 4 + roots).

- (i) The term independent of q is 07 in the leading diagonal.

* This eliminant can also be obtained by assuming the quotient of the original division to be $b_0x^{n-2} + b_1x^{n-3} + \dots + b_{n-3}x + b_{n-2}$, and determining the b 's; or, as the eliminant of $f(x)$ and $f\left(\frac{1}{x}\right)$. See Dr. Muir, in *Proc. R. S. Ed.*, xxi., p. 360.

- (ii) The highest power of $\frac{1}{q}$ is obtained from the 8's in the other diagonal: its index is $(7+6+\dots+1) = \frac{8 \cdot 7}{2}$ and its sign is $(-)^{6+5+\dots+1}(-)^7 = (-)^{7^2}$, and generally, if the degree of the original function is n , the sign is $(-)^{\overline{n-2}+\overline{n-1}+\dots+1}(-)^{n-1} = (-)^{\frac{n(n-1)}{2}}$.

We shall assume n to be even and a_0 and a_n to be + : for, if n is odd, or a_0, a_n have opposite signs, $a_0x^n + a_1x^{n-1} + \dots + a_n = 0$ is known, on account of its *continuity*, to become 0 for some real value (a) or values of x : thus there are factors $(x-a)$: removing these, we have in every case only to consider the divisibility by $(x^2 - px + q)$ of a function of even degree, with a_0 and a_{n+} .

Thus considering the determinant equation to be one in q' (for $\frac{1}{q}$), it begins and ends, $a_n^{n-1}(-q')^{\frac{n(n-1)}{2}} \dots + a_0^{(n-1)} = 0$, so that the equation certainly has a real root unless n is "even-even."

§ 4. Continuing the work for $n=8$, we notice that D_5 is a determinant concentric with D_7 .

Let (48), (37), (26), (15), (04) in brackets denote the minors of the corresponding elements of the last column of D_5 , and treat D_7 thus:—

Multiply the 1st row by (48), and the 3rd by $(48)q$; the 2nd and 4th by (37), $(37)q$; 3rd and 5th by (26), $(26)q$; 4th and 6th by (15), $(15)q$; the 5th and 7th by (04), $(04)q$; and add columns for a new last row.

The scheme is represented below, the multipliers being written against the rows, and D_5 being screened.

$$D_7 \equiv \begin{vmatrix} 0, & 1, & 2, & 3, & 4, & 5, & 68 \\ \cdot, & \boxed{0, & 1, & 2, & 3, & 48}, & 57 \\ \cdot, & \cdot, & 0, & 1, & 28, & 37, & 46 \\ \cdot, & \cdot, & \cdot, & 08, & 17, & 26, & 35 \\ \cdot, & \cdot, & \cdot 8_5, & \cdot 7_4, & 06, & 15, & 24 \\ \cdot, & \cdot 8_6, & \cdot 7_5, & \cdot 6_4, & \cdot 5_3, & 04, & 13 \\ \cdot 8_7, & \cdot 7_6, & \cdot 6_5, & \cdot 5_4, & \cdot 4_3, & \cdot 3_2, & 02 \end{vmatrix} \begin{matrix} \dots\dots\dots(48) \\ \dots\dots\dots(37) \\ \dots\dots\dots(26) + (48)q \\ \dots\dots\dots(15) + (37)q \\ \dots\dots\dots(04) + (26)q \\ \dots\dots\dots(15)q \\ \dots\dots\dots(04)q. \end{matrix}$$

The elements of the new last row are determinants consisting of the first four columns of D_5 with varying fifth columns as shown below.

$$\begin{vmatrix} 0, & 1, & 2, & 3 & 0 & 1 & 2+0q & 3+1q & 48+2q & 57+3q & 48q \\ \cdot, & 0, & 1, & 28 & \cdot & 0 & 1 & 28+0q & 37+1q & 46+28q & 37q \\ \cdot, & \cdot, & 08, & 17 & \cdot & \cdot & 08 & 17 & 26+08q & 35+17q & 26q \\ \cdot, & \cdot 8_5, & \cdot 7_4, & 06 & \cdot & \cdot 8_5 & \cdot 7_4 & 06+\cdot 8_5q & 15+\cdot 7_4q & 24+06q & 15q \\ \cdot 8_6, & \cdot 7_5, & \cdot 6_4, & \cdot 5_3 & \cdot 8_6 & \cdot 7_5 & \cdot 6_4+\cdot 8_6q & \cdot 5_3+\cdot 7_5q & 04+\cdot 6_4q & 13+\cdot 5_3q & 04q \end{vmatrix}$$

Thus, noticing that the original last row was multiplied by $(04)q$, which is seen to be $0 \cdot D_3q$, where D_3 is the concentric determinant of order 3, we have

[illegible]

$$qD_7 \cdot D_3 = q \cdot D_5 \cdot D_5 - D'_5 \cdot D'_5 + D_5 \cdot D''_5 \text{ (say).}$$

Now treat D_5 in the same way, using minors of D_3 ; we get

$$q_{D_5 \cdot 0 \cdot D_x} \equiv \begin{vmatrix} 0, & 1, & 2, & 3, & 48 \\ ., & 0, & 1, & 28, & 37 \\ ., & ., & 08, & 17, & 26 \\ ., & \cdot 8_5, & \cdot 7_4, & 06, & 15 \\ ., & ., & D_3, & D'_3, & q_{D_3 + D'_3} \end{vmatrix} \quad \text{where } D''_3 = \begin{vmatrix} 0, & 1, & 46 \\ ., & 08, & 35 \\ \cdot 8_5, & \cdot 7_4, & 24 \end{vmatrix}$$

$$\therefore qD_5 D_1 \equiv (qD_3 + D_3'')D_3 - D_3'^2 + D_3 \cdot D_3^* \text{ say.}$$

Now

$$D_I \equiv 08 \equiv a_0 - \frac{a_8}{q^4},$$

and when 08 is zero

$$D_3 = \begin{vmatrix} 0 & 1 & 28 \\ \cdot & \cdot & 17 \\ \cdot & \frac{17}{q} & 06 \end{vmatrix} = -\frac{0 \cdot 17^2}{q}.$$

Therefore when $08=0$, D_3 has the opposite sign to q .*

* A similar treatment reduces dialytic and circulant determinants to two or three terms of like form to those in the text.

Thus we have the result :—

Build up a determinant of order $\overline{n-1}$, with the successive concentric determinants.

$$D_1 \equiv a_0 - \frac{a_n}{q^2}, D_3 \equiv \begin{vmatrix} a_0 & a_1 & a_2 - \frac{a_n}{q^{2-1}} \\ & \cdot & a_0 - \frac{a_n}{q^2} & a_1 - \frac{a_{n-1}}{q^{2-1}} \\ -\frac{a_n}{q^{2+1}} & -\frac{a_{n-1}}{q^2} & a_0 - \frac{a_{n-2}}{q^{2-2}} \end{vmatrix}$$

and so on.

It will be found that of each set of three successive determinants, the extremes have for + values of q opposite signs when the middle one is 0.

Also $D_1 \equiv a_0 - a_n q'^2 = 0$ (n being even) has a + root; and D_3, q' have opposite signs when $D_1 = 0$. The beginning and end of $D_3, D_5, D_7 \dots$ are

$$D_3 \equiv a_0^3 \dots + a_n^3 q'^{\frac{3n}{2}}; \quad D_5 \equiv a_0^5 \dots - a_n^5 q'^{\frac{5n}{2}}; \quad D_7 \equiv a_0^7 \dots + a_n^7 q'^{\frac{7n}{2}}.$$

Hence the following scheme of + roots and signs (0 on a line indicating a group of an odd number of roots).

	$q'=0$	$q'=+\infty$	Degree.
D_1	+	-	$\frac{n}{2}$
D_3	+	+	$\frac{3n}{2}$
D_5	+	-	$\frac{5n}{2}$
D_{n-1}	$\frac{(n-1)n}{2}$

Thus $D_{n-1} = 0$ has at least $\frac{1}{2}(\overline{n-1}+1)$, i.e. $\frac{n}{2}$ real + roots.

$\therefore a_0 x^n + a_1 x^{n-1} + \dots + a_n = 0$ has $\frac{n}{2}$ factors of the form $(x^2 - px + q)$.

Q.E.D.

NOTES.

A. If $(\alpha \pm i\beta)$ is a pair of imaginary roots, $\alpha^2 + \beta^2 =$ one of the $\frac{n}{2}$ real values of q . Thus the above scheme giving regions for these values of q gives some indication of the values of the moduli of imaginary roots.

B. If the given equation has some of its coefficients imaginary

$$(a_0 + ia'_0)x^n + (a_1 + ia'_1)x^{n-1} + \dots = 0 \dots\dots\dots(1)$$

$$\therefore (a_0x^n + a_1x^{n-1} + \dots)^2 + (a'_0x^n + a'_1x^{n-1} + \dots)^2 = 0 \dots\dots\dots(2)$$

an equation of degree $2n$ with real coefficients. We have proved that this has n pairs of factors of the form $x - (a \pm i\beta)$.

Since (2) is the product of (1) and its conjugate $(a_0 - ia'_0)x^n \dots$, (1) or its conjugate must hold when $x = a + i\beta \equiv r(\cos \theta + i \sin \theta)$.

$$\begin{aligned} \text{And if } (a_0 + ia'_0)r^n(\cos n\theta + i \sin n\theta) \dots &= 0, \\ (a_0 - ia'_0)r^n(\cos n\theta - i \sin n\theta) \dots &= 0 \text{ follows.} \end{aligned}$$

Therefore we may say that of the $2n$ roots of (2) those of form $a + i\beta$ are roots of (1) and those of form $a - i\beta$ are roots of the conjugate.