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Iterative Processes

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## ITERATIVE PROCESSES.\*

By H. TODD, B.A.

I. If  $(1 + \sqrt{2})^n = p + q\sqrt{2}$ , where  $p$  and  $q$  are positive integers, then  $(1 - \sqrt{2})^n = p - q\sqrt{2}$ , and by multiplication,

$$p^2 - 2q^2 = (1 + \sqrt{2})^n (1 - \sqrt{2})^n = (-1)^n.$$

$$\therefore p - q\sqrt{2} = (-1)^n / (p + q\sqrt{2}) = (-1)^n / (2q\sqrt{2}) \text{ very nearly,}$$

since  $|p - q\sqrt{2}|$  is small.

Hence  $p/q - \sqrt{2} = (-1)^n / (2q^2\sqrt{2})$ , so that  $p/q$  furnishes a good approximation to  $\sqrt{2}$ , in fact the successive results for  $n=1, 2, \dots$  are  $1/1, 3/2, 7/5, 17/12, 41/29, 99/70, \dots$ , of which the last is in error by less than 1 part in 13,800. This is a well-known result and suggests similar possibilities for  $\sqrt[n]{n}$ .

Let  $a$  and  $b$  be any real numbers not involving  $\sqrt[n]{n}$ .

Then

$$(a + b\sqrt[n]{n})^r = p + q\sqrt[n]{n},$$

and

$$(a - b\sqrt[n]{n})^r = p - q\sqrt[n]{n}.$$

Solving these equations for  $p$  and  $q$ , we find that

$$\begin{aligned} p/(q\sqrt[n]{n}) &= [(a + b\sqrt[n]{n})^r + (a - b\sqrt[n]{n})^r] / [(a + b\sqrt[n]{n})^r - (a - b\sqrt[n]{n})^r] \\ &= [1 + (a - b\sqrt[n]{n})^r / (a + b\sqrt[n]{n})^r] / [1 - (a - b\sqrt[n]{n})^r / (a + b\sqrt[n]{n})^r]. \end{aligned}$$

Now the absolute value of  $(a - b\sqrt[n]{n}) / (a + b\sqrt[n]{n})$  is less than unity; therefore  $\lim_{r \rightarrow \infty} p/(q\sqrt[n]{n}) = 1$ , i.e.  $p/q$  tends to the limit  $\sqrt[n]{n}$ .

Hence to find  $\sqrt[n]{n}$  it is only necessary to raise  $(a + b\sqrt[n]{n})$  to any power and take the ratio of the coefficients of 1 and  $\sqrt[n]{n}$ , and it may be noticed here that, (i)  $a$  and  $b$  should be chosen initially to make  $|a - b\sqrt[n]{n}|$  small in order that rapid convergence may ensue, and (ii) if errors are made in the calculation they will ultimately disappear, for an error would merely mean a change in  $a$  and  $b$  from which the approximation was started.

II. For cube roots a similar process holds good; that is, if  $x = \sqrt[3]{n}$  and  $a, b, c$  are any real positive numbers not involving  $\sqrt[3]{n}$ ; then, if

$$(a + bx + cx^2)^r = p + qx + sx^2,$$

we shall find that  $p/q$  and  $q/s$  both tend to  $\sqrt[3]{n}$  as  $r \rightarrow \infty$ .

To prove this statement, let  $\omega$  be a complex cube root of unity

$$(\omega = \frac{1}{2}(-1 + \sqrt{-3}));$$

then  $\omega^2$  is another.

Also put

$$\lambda_1 = a + bx + cx^2,$$

$$\lambda_2 = a + b\omega x + c\omega^2 x^2,$$

$$\lambda_3 = a + b\omega^2 x + c\omega x^2.$$

Then

$$\lambda_1^r = p + qx + sx^2,$$

$$\lambda_2^r = p + q\omega x + s\omega^2 x^2,$$

$$\lambda_3^r = p + q\omega^2 x + s\omega x^2.$$

Solving these equations for  $p, q, s$ , and recalling that  $1 + \omega + \omega^2 = 0$ , we find that

$$\begin{aligned} p/(qx) &= [\lambda_1^r + \lambda_2^r + \lambda_3^r] / [\lambda_1^r + \omega^2 \lambda_2^r + \omega \lambda_3^r] \\ &= \left[ 1 + \left( \frac{\lambda_2}{\lambda_1} \right)^r + \left( \frac{\lambda_3}{\lambda_1} \right)^r \right] / \left[ 1 + \omega^2 \left( \frac{\lambda_2}{\lambda_1} \right)^r + \omega \left( \frac{\lambda_3}{\lambda_1} \right)^r \right]. \end{aligned}$$

\* A paper read to the Bristol branch of the Mathematical Association.

But

$$\begin{aligned} |\lambda_2| &= |a + b\omega x + c\omega^2 x^2| \\ &< |a| + |b\omega x| + |c\omega^2 x^2|, \\ \text{i.e.} \quad &< a + bx + cx^2 \quad (\text{since } |\omega| = |\omega^2| = 1), \\ \text{i.e.} \quad &< \lambda_1. \end{aligned}$$

Similarly  $|\lambda_3| < \lambda_1$ , so that both  $(\lambda_2/\lambda_1)^r$  and  $(\lambda_3/\lambda_1)^r$  tend to zero as  $r \rightarrow \infty$ .

Hence  $p/(qx) \rightarrow 1$ , or  $p/q \rightarrow x$ , and similarly it may be shown that  $q/s \rightarrow x$ .

Once again it is evident that if  $a, b, c$  are suitably chosen, the convergence can be made very rapid; for example, if  $x = \sqrt[3]{5}$ , it will be found that  $a=41$ ,  $b=24$ , and  $c=14$  gives for the second power approximations valid to four places. Also it is evident that any errors committed in the work will ultimately disappear for the same reason as before.

III. For the  $n^{\text{th}}$  roots we shall merely state the theorem as follows, the proof being an obvious extension of that for the previous case.

If  $x = \sqrt[n]{r}$ , and

$$(a + bx + cx^2 + \dots + kx^{n-1})^p = A + Bx + Cx^2 + \dots + Kx^{n-1},$$

where  $a, b, c, \dots, k$  are real positive quantities not involving  $\sqrt[n]{r}$ ; then in the limit, as  $p \rightarrow \infty$ , the ratios  $A/B, B/C, C/D, \dots$  all tend to  $\sqrt[n]{r}$ .

As an example take  $x = \sqrt[4]{2}$ , and  $a=b=c=d=1$ .

$$\text{Now} \quad (1 + x + x^2 + x^3)^4 = 195 + 164x + 138x^2 + 116x^3,$$

and  $195/164 = 1.18902\dots$ , the correct result being  $1.1893\dots$  (For the actual multiplication a method of detached coefficients shortens the work, remembering that  $x^5 = 2x$ , etc.)

IV. These results in the solution of the equation  $x^n = r$  suggest that similar results will hold for the roots of any algebraic equation, and with certain restrictions this proves to be the case. Let  $x^2 + px - q = 0$  be any quadratic whose roots are real and different, being  $x_1$  and  $x_2$ , with  $|x_1| > |x_2|$ .

Then  $x^n$  will reduce to a linear function of  $x$  in virtue of  $x^2 = -px + q$ .

$$\text{Let} \quad x^n = a_n x + b_n.$$

$$\text{Then} \quad x_1^n = a_n x_1 + b_n,$$

$$\text{and} \quad x_2^n = a_n x_2 + b_n.$$

So, by solving these equations for  $a_n$  and  $b_n$ ,

$$\begin{aligned} a_n/b_n &= (x_1^n - x_2^n)/(x_1 x_2^n - x_2 x_1^n) \\ &= [1 - (x_2/x_1)^n]/[-x_2 + x_1(x_2/x_1)^n]. \end{aligned}$$

But  $|x_1| > |x_2|$ , so that  $(x_2/x_1)^n \rightarrow 0$  as  $n \rightarrow \infty$ .

$$\text{Hence} \quad a_n/b_n \rightarrow -1/x_2 = x_1/q.$$

So this gives a method of finding the roots of a quadratic when they are real.

$$\text{Example.} \quad x^2 = x + 1, \quad \text{here, } x_1 = \frac{1}{2}(\sqrt{5} + 1) = 1.618\dots,$$

$$x^3 = 2x + 1, \quad q = 1,$$

$$x^6 = 8x + 5, \quad 144/89 = 1.6179\dots$$

$$x^{12} = 144x + 89;$$

It is easily proved that the magnitude of the error committed in taking  $a_n/b_n$  for  $x_1/q$  is of the order of  $(x_2/x_1)^{n-1} \cdot x_1/q$ , which furnishes a means of knowing how far to proceed in order to get an approximation to any desired degree of accuracy.

V. The extension to cubics is as follows:

If  $x^3 + px^2 + qx + r = 0$  is an equation whose root  $x$  having the greatest modulus is real, and if  $x^n$  be reduced in virtue of  $x^3 = -px^2 - qx - r$  to the form

$$x^n = \lambda_n + \mu_n x + \nu_n x^2, \dots \dots \dots (1)$$

we shall have in the limit,

$$\lambda_n/(x^2 + px + q) = \mu_n/(x + p) = \nu_n/1.$$

The proof of this statement is similar to the quadratic case, for if  $x_1, x_2$  are the other roots of the original cubic, we get more equations from (i) with  $x_1, x_2$  respectively substituted for  $x$ . These are solved for  $\lambda_n, \mu_n, \nu_n$ , and the limit is found as before.

It is not difficult to prove that the error committed in taking  $\mu_n/\nu_n$  for  $x+p$  is of order  $A(x_1/x)^n + B(x_2/x)^n$ , where  $A$  and  $B$  are independent of  $n$ , and if  $x_1$  and  $x_2$  are complex numbers, the error is of order  $C\sqrt{x^{-3n}}$ , where  $C$  is independent of  $n$ .

The case of chief interest is Cardan's "irreducible case," for then all the roots are real and the conditions are satisfied.

*Example.*  $x^3 = x + 1$ : in this case the other roots  $x_1$  and  $x_2$  are complex, but

$$|x_1| = |x_2| < x;$$

$$x^9 = 3x^2 + 4x + 2;$$

$$x^{18} = 37x^2 + 49x + 28.$$

$49/37 = 1.324\dots$ , and the required result is  $1.3247\dots$ .

VI. All the foregoing results are special cases of a general theorem concerning algebraic equations (and itself a special case of a theorem on linear transformations) that it may be of interest to state.

Let  $x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_n = 0$  be any algebraic equation with real coefficients whose roots are  $x, x_1, x_2, \dots, x_{n-1}$ .

Let  $\varphi(x) = a + bx + cx^2 + \dots + kx^{n-1}$ , where  $a, b, \dots, k$  are real, and

$$[\varphi(x)]^r = a_r + b_rx + c_rx^2 + \dots + j_rx^{n-2} + k_rx^{n-1}.$$

Then if  $x$  is real and  $|\varphi(x)| > |\varphi(x_1)|, |\varphi(x_2)|, \dots, |\varphi(x_{n-1})|$ , we have in the limit as  $r \rightarrow \infty$ ,

$$a_r / (x^{n-1} + p_1x^{n-2} + \dots + p_{n-1}) = b_r / (x^{n-2} + p_1x^{n-3} + \dots + p_{n-2}) = \dots = j_r / (x + p_1) = k_r / 1.$$

H. TODD.

**136.** Of all the pursuits of human ingenuity, that of mathematics demands the intensest application. It is related of one well known in the records of science, that after the exhaustion of some minute astronomical experiments he has been driven to count the drops of rain at the window, or watch the race of two flies along the glass, in order that by an utter repose of thought the intellect might recover its elasticity.—*Conversations at Cambridge, 1836.* [Who was the astronomer?]

**137.** "... Given certain factors, and a sound brain should always evolve the same fixed product with the certainty of Babbage's calculating machine.

"What a satire, by the way, is that machine on the mere mathematician! A Frankenstein-monster, a thing without brains and without heart, too stupid to make a blunder; that turns out results like a corn-sheller, and never grows any wiser or better, though it grind a thousand bushels of them!

"I have an immense respect for a man of talents *plus* 'the mathematics.' But the calculating power alone should seem to be the least human of qualities, and to have the smallest amount of reason in it; since a machine can be made to do the work of three or four calculators, and better than any one of them. Sometimes I have been troubled that I had not a deeper intuitive apprehension of the relations of numbers. But the triumph of the ciphering hand-organ has consoled me. I always fancy I can hear the wheels clicking in a calculator's brain. The power of dealing with numbers is a kind of 'detached lever' arrangement, which may be put into a mighty poor watch. I suppose it is about as common as the power of moving the ears voluntarily, which is a moderately rare endowment."—Oliver Wendell Holmes, *The Autocrat of the Breakfast-Table*, 1858, chap. i. [Per Dr. J. M'Whan.]