

CRITICISMS AND DISCUSSIONS.

THE "LECTIONES GEOMETRICAE" OF ISAAC BARROW.

In an article which appeared in the February number of *The Open Court* I gave a short summary of the life of this famous mathematician, and endeavored to suggest a reason for the unfair estimate of his worth, especially with regard to his work on the drawing of tangents, formed by contemporary continental mathematicians, and quoted with approval by the writer of the article on "Barrow" in the *Encyclopaedia Britannica*. I suggested that his reading, his training and his disposition all tended to make him a confirmed geometer, with a dislike for, a possible distrust of, and even a certain infacility in, the analytical method of Descartes; that this, together with the accident of his connection with Newton, in whom he recognized a genius peculiarly adapted to analysis, and Barrow's determination to forsake mathematics for divinity, had resulted in his making no attempt to complete the work he had so well begun; and that, therefore, to form a proper conception of his genius, it was necessary to read into his work what might have been got out of it, and not stop short at what was actually published under Barrow's name.

As examples of what can be read into Barrow's work, let us take the following instances, most of them referring to the principles underlying the infinitesimal calculus.

Example 1 (*Lectio VII, 14*).

"If A, B, C, D, E, F are in Arithmetical Progression and A, M, N, O, P, Q are in Geometrical Progression, and the last term F is not less than the last term Q (the number of terms in the two series being equal); then B is greater than M ."

The proof of this is made to depend on a proposition that, if A, B, C, \dots is an arithmetical progression, and A, M, N, \dots is a geometrical progression, such that B is not greater than M ,

then any term in the geometrical progression is greater than the corresponding term in the arithmetical progression. Hence Barrow concludes that if, in the theorem above, B is not greater than M , then F must be less than Q , which is contrary to the hypothesis. He then deduces that, if $F = Q$, then $B > M$, $C > N$, and so on.

Thus Barrow, and no more; now let us see what he might have got out of this if he had so chosen.

If Barrow's final conclusion is expressed differently we have:



Fig. 1.

Suppose that a straight line AB is divided into two parts at C , and the part CB is divided at D, E, F, G in Fig. 1 (i), and at D', E', F', G' in Fig. 1 (ii), so that AC, AD, AE, AF, AG, AB are in arithmetical progression, and $AC, AD', AE', AF', AG', AB$ are in geometrical progression; then $AD > AD', \dots AG > AG'$.

Expressing this algebraically, we see that, if $AC = a$, and $CB = a \cdot x$, and the number of points between C and B is $n-1$, and H is the r th arithmetical and H' is the r th geometrical "mean" point; then the relation $AH > AH'$ becomes

$$a + r \cdot ax/n > a \cdot [\sqrt[n]{\{(a+ax)/a\}}]^r;$$

$$\text{i. e., } 1 + x \cdot r/n > (1+x)^{r/n}; \text{ where } n > r.$$

Also, as CB becomes smaller and smaller the inequality tends to become an equality.

Moreover, if we put $rx/n = y$, and hence $x = ny/r$, then

$$1 + y \cdot n/r < (1+y)^{n/r}; \text{ where } n > r;$$

and the inequality tends to become an equality.

Naturally a man who uses the notation xx for x^2 does not state such a theorem about fractional indices. But none the less he has the approximation to the binomial theorem; that is, all that is necessary for him to obtain the gradient of $x^{n/r}$ or $x^{r/n}$, where $n > r$, although it is concealed in a geometrical form. We may as well say that the ancient geometers did not know the expansion for $\sin(A+B)$, when they used it in the form of Ptolemy's Theorem, as say that Barrow was unaware of the inner meaning of his proposition. Also from the *a fortiori* method of his proof it is evident that he knew that the relative error was less than x/n . It may be objected that

this is insufficient to make the relative error negligible, no matter how small x may be. But these old geometers could use their geometrical facts with far greater skill than many mathematicians of to-day can use their analysis. Barrow does not require to know the magnitude of the error at all; he only requires to know that the inequality in the above example is always in one direction, i. e., the geometric always less than, or always greater than, the corresponding arithmetic mean. The way in which the theorem is used, which indeed is his general method for drawing tangents, is of striking ingenuity. Barrow starts with a very small, so to speak, stock-in-trade; he is able to draw a tangent to a circle, and also to a hyperbola of which the asymptotes are known, and he has the fact that a straight line is everywhere its own tangent. The tool that he most often uses is the hyperbola; and when he cannot immediately find a construction for a tangent to a curve, he draws a hyperbola to touch the curve, and then draws the tangent to the hyperbola. His criterion of tangency is the following:



Fig. 2.

A straight line and a curve, or two curves, will touch one another if one curve lies totally outside or inside the other line. That is, the curves ABA, CBC, touch one another, if $OA < OC$, whether O is supposed to be some fixed point, or the straight lines CAO are all drawn parallel to some straight line fixed in position. This criterion is important, as it will be referred to later.

In the next example chosen he does not however use any of the above three tools; for, finding that the curves formed from the arithmetical and geometrical means of the same order are such that he can draw a tangent at any point of the former in a very simple manner, he uses this as his auxiliary curve to find the tangent at any point of the latter.

Example 2 (*Lectio IX*, 1).

“Let the straight lines AB, VD be parallel to one another; and let a straight line DB, given in position, cut them; also let the lines

EBE, FBF pass through B and be so related that, if any straight line PG is drawn parallel to DB, then PF is always an arithmetical mean of the same given order between PG and PE; also let BS touch the curve EBE. It is required to find the tangent at B to the curve FBF."

The construction given is:

Make $DS:DT = FG:EG$; and join BT. Then BT is the required tangent (see Fig. 3).

The proof is as follows:

$FG:EG = DS:DT = LG:KG$; hence, since $KG < EG$, $\therefore LG < FG$. Therefore BT is the tangent.*

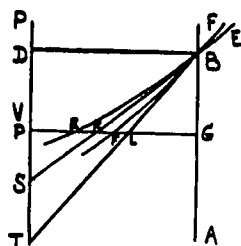


Fig. 3.

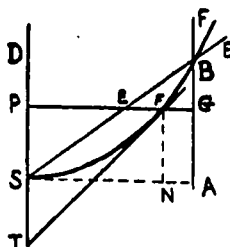


Fig. 4.

Barrow then makes use of the theorem on arithmetical and geometrical means, given as our first example, to show that the same construction holds good if PF is a geometrical mean of the same order between PG and PE, by proving that the curve formed from the geometrical means touches the curve formed from the arithmetical means at B. Lastly, he shows, by the use of an analogous curve, that a similar construction can be used for drawing the tangent at any point F on the curve FBF, provided that the tangent at the corresponding point E on the curve EBE is known (see Fig. 4). He then adds the remarkable note:

"It is to be noted that if EBE is supposed to be a straight line, the line FBF is one of the parabolas or paraboliform curves. Wherefore, what is generally known about these curves (deduced by calculation, and verified by a sort of induction, yet not anywhere proved geometrically, as far as I am aware) flows from an im-*

* This undoubtedly refers to the work of Wallis.

* Note, in passing, that this is equivalent to saying that the gradient of $f[x.r/n + a.(n-r)/n]$ is r/n times the gradient of $f(x)$ at the point where $x = a$.

mensely more fruitful source, and covers innumerable curves of other kinds."†

Now if, in Fig. 4, which shows Barrow's method of drawing the tangent at any point F of the paraboliform FBF, we take SA and SD as the axes of coordinates, and suppose that PF is the r th mean, out of n means,‡ between PG and PE, so that $PT:PS = n:r$, and $SA = a$, $PE = b$, $SP = mb$, where m is the gradient of EBE; then for the curve FBF, we have

$y = FN = SP = mb$; and $x = SN = PF = a \cdot (b/a)^{r/n} = b^{r/n} \cdot a^{(n-r)/n}$; and the equation to the curve FBF is

$$(y/m)^{r/n} = x/a^{(n-r)/n} \text{ or } y = K x^{n/r};$$

whilst the gradient of the tangent at F is

$$PT/PF = (n/r) \cdot (PS/PF) = (n/r) \cdot (y/x) = (n/r) \cdot K x^{n/r-1}.$$

Thus the gradient is found for any curve of the form $y = K x^{p/q}$, where $p > q$; and, by interchanging the axes, for any curve of the form $y = K x^{p/q}$, where $p < q$.

Note. The axes are not necessarily rectangular in Barrow's figure; though of course in the consideration of the gradient they are taken as rectangular.

In the face of the note quoted in italics above, I submit that it is idle to contend that Barrow was not aware of the significance of his theorem; but as before, he was not prepared to use the index notation, let alone fractional indices. For this reason, most probably, he also leaves the point that, if EBE is a hyperbola, so that $PS \cdot PE$ is a constant, m say, then $y = m/b$, and the equation of the curve FBF is of the form $y = K x^{-p/q}$.§

Thus Barrow proves geometrically and rigidly, without any difficulty about the convergence of the binomial theorem, that in general, if $y = K x^n$, then $dy/dx = n \cdot y/x$. He could have drawn the tangent, or found its gradient, by the method which he either thought little of, or affected to despise—*ex calculo* (observe the half-sneering comparison between the methods of calculation adopted by Wallis (?) and a geometrical proof, in the parenthesis in Barrow's

† In other words, the gradient of $f(x^{r/n} \cdot a^{(n-r)/n})$ is r/n times the gradient of $f(x)$, at the point where $x = a$.

‡ It should be observed that Barrow defines previously such a curve as the locus of F as "having an exponent r/n ."

§ He does this in a considerably harder way in *Lectio IX*, 10; from this general theorem the case when EBE is a straight line is deduced in exactly the same way as for the paraboliforms, and yields the hyperboliforms $y = K x^{-p/q}$.

note, as quoted above). Thus Barrow is in possession of a method for differentiating any explicit algebraic function of x ; for he has another theorem connecting the tangents to two allied curves, the ordinate of one being proportional to a power of that of the other. For instance, he could have differentiated such a function as

$$(x+a)^{2/3} + (x^2-a^2)^{3/4}.$$

Of course Barrow does not consider such a case as this; at least, he has not got a theorem to draw a tangent to a curve, whose ordinates are the sum of the ordinates of two other curves, of which the tangents at every point are known.* Such a construction is easy; but the point I make is that Barrow was in a position to do any differentiation of this kind, by calculation, if he had had a mind to.

Further, by combining this method with the "differential triangle" method (the well-known " a and e " method—the prototype of the " h and k " method of the ordinary beginner's text-book of to-day), he could have differentiated implicit functions also, again by calculation. As examples of the "differential triangle" method Barrow takes the *Folium of Descartes* and the *Quadratrix* amongst others. A third example is of even more interest. Barrow finds the subtangent of a curve, which turns out to have an equation $y = \tan x$; moreover, he leaves it in such a form (namely, $t:m = rr:rr+mm$), that it is only necessary to put $r=1$ and $m=y$, in order to obtain

$$dy/dx = m/t = 1 + y^2 = 1 + \tan^2 x = \sec^2 x.$$

In addition, the pair of figures that he gives could equally well have been used to find the subtangent for $y = \sin x$, in a form that immediately yields $dy/dx = m/t = \cos x$; but he winds up by saying, "These would seem to be sufficient to explain this method."

It is of course well known that Barrow was the first to perceive that differentiation and integration were inverse operations. This is proved in a very simple manner by means of a theorem and its converse.

In Fig. 5, ZGEG is a curve such that the ordinates to an axis VD continually increase (or decrease) from left to right. VIFI is

* This ability to deal with irrational algebraic functions, and that too without the binomial theorem, constitutes perhaps Barrow's greatest advance on the work of his predecessors on the infinitesimal calculus; although it by no means constitutes his only claim to great genius.

the space ADLK shall always be equal to the rectangle contained by R and DB.

For if $DE = R$, and the rectangle BDHI is completed, and MN is taken to be an indefinitely small arc of the curve AB, and MEX, NOS are drawn parallel to AD; then we have

$$NO : MO = TF : FM = R : FZ ;$$

$$\therefore NO.FZ = MO.R, \text{ or } FG.FZ = ES.EX.$$

Hence since the sum of such rectangles as $FG.FZ$ differs only in the slightest degree from the space ADLK, and the rectangles $ES.EX$ from the rectangle DHIB, the proposition follows quite obviously.

These proofs compare favorably with the usual analytical proofs; and they show that Barrow not only appreciated the fact that differentiation and integration are inverse operations, but also recognized the necessity of proving the fact both directly and conversely. As I have mentioned, this is fairly well known; but what does not seem to have been remarked is that Barrow ever made any use of the theorems. However in the appendix to *Lectio XI*, where he develops the work of Huygens on the measurement of the circle, Barrow quotes formulas for the area and the position of the center of gravity of any paraboliform; but he states "*of which the proofs follow without much difficulty in various ways from what has already been shown,*" and leaves the rest to the reader. As a matter of fact, the proofs do follow quite easily, as is shown below; moreover Barrow could have found the radius of gyration of a paraboliform, or other power summations, practically amounting to $\int y^n dx$, by means of theorems previously given.

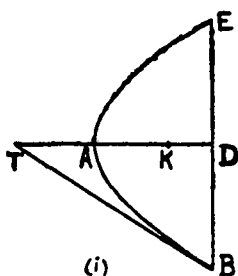


Fig. 7 (i)

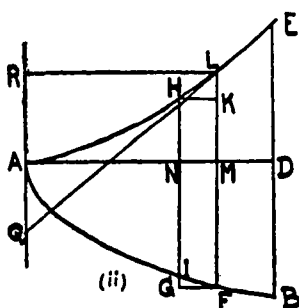


Fig. 7 (ii)

"If BAE is a paraboliform curve whose axis is AD and base or

ordinate BDE, BT a tangent to it, and K the center of gravity; then, if its exponent is n/m , we have

Area of BAE = $m/(m+n)$ of AD.BE; $TD = m/n$ of AD;
and $KD = m/(n+2m)$ of AD." [See Fig. 7 (i).]

Suppose, in Fig. 7 (ii), that AHLE is a paraboliform whose exponent is $r/s=1/a$, say; let H be a near point to L on the curve, so that HLK is Barrow's "differential triangle"; then $LK/HK = \text{gradient} = QR/RL = a$. $AR/RL = a$. LM/AM ; and conversely.

Let AIFB be another curve, such that $FM/R = LK/HK = a$. LM/AM always, then, as has been shown, area AFBD = R.DE always.

But in this case we have

$$\begin{aligned} IG:FM &= LM/AM - HN/AN:LM/AM, \\ &= AM.LK - LM.HK:LM.AN, \\ &= (a-1).LM.HK:LM.AN; \\ \therefore FG/GI &= 1/(a-1) \text{ of } AM/FM. \end{aligned}$$

Hence AIFB is a paraboliform, vertex A, axis AD, and exponent equal to $a-1$. Conversely, if AIFB is a paraboliform whose exponent is $n/m (=a-1)$; then the integral curve AHLE is a paraboliform whose exponent is $1/a$ or $m/(n+m)$; and since $DB/R = a$. DE/AD , the area AIFBD = R.DE = $m/(n+m)$ of AD.DB.

Similarly, area ALED = AD.DE - $(n+m)/(n+2m)$ of AD.DE
= $m/(n+2m)$ of AD.DE;

$$\therefore R.a. \text{ area ALED} : \text{AD. area AFBD} = n+m : n+2m.$$

Now since $FM/R = a$. LM/AM , $\therefore FM.AM.MN = R.a.LM.HK$; hence, summing, we have $AK.\text{area AFBD} = R.a.\text{area ALED}$;

$$AK:AD = n+m:n+2m, \text{ or } KD = n/(n+2m) \text{ of AD.}$$

In a similar manner the radius of gyration could have been found from the sum of $FM.MN.AM^2 = R.a.LM.HK.AM$; and so on for higher powers of AM.

There are many other ingenious propositions, although these are perhaps not of such general interest as those that have already been given. But they all go to show how far above the ordinary

the genius of Barrow was, especially when we remember how short was Barrow's professional connection with mathematics, and the relatively large and varied amount of matter that came from him in this time.

For instance he proves that, if $ZD + AD$ is constant, then $ZD^m \cdot AD^{m-2n}$ is a maximum, when $ZD:AD = m:m-2n$.

The proof of this theorem is generally ascribed to Cardinal Ricci, who published it in 1666. Remembering that these lectures were given in 1664-5-6, there is at least a doubt whether Barrow had not anticipated him. Even if he did not, Ricci's proof is made to depend on a lemma that if a magnitude is divided into r equal parts, their continued product is greater than that obtained by dividing it into r parts in any other manner. Barrow deduces it as an easy and immediate consequence of his theorem on a tangent to a paraboliform already quoted; so that Barrow's proof is inde-

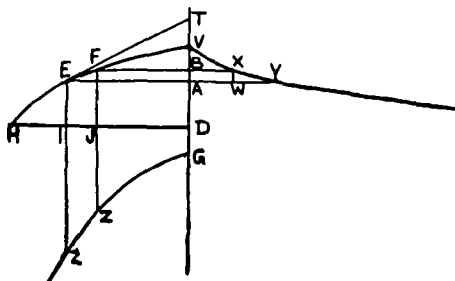


Fig. 8.

pendent of Ricci. Barrow also shows that $ZD^m \cdot AD^{2n-m}$, where $2n > m$, is a minimum under similar circumstances.

Again, he shows, by means of his beloved paraboliforms, that if AB is the arc of a circle whose center is C , and BD is drawn perpendicular to the radius AC , then the arc AB lies between

$(3CA \cdot DB)/(2CA + CD)$ and $(2CA \cdot DB + CD \cdot DB)/(CA + 2CD)$; hence, taking the arc to subtend 30 degrees and the radius of the circle to be 113, he finds that the limits of the semi-circumference are 355+ and 355-; thus verifying in a rigid manner the ratio $355/113$ or $31\frac{9}{113}$, which was found by Metius in the 16th century, by an unjustifiable but fairly obvious manipulation of the two limits $31\frac{5}{106}$ and $31\frac{7}{120}$. In the course of proving the preliminary lemmas for the geometrical limits given above, Barrow in effect integrates the function $a \cdot \cos^{-1} x/a$.

Another striking instance of Barrow's (shall I call it con-

tributary laziness?) is the omission of the proof of the theorem of Lecture XI, § 27.

"Let VEH be any curve, whose axis is VD and base DH, and let any straight line ET touch it; draw EA parallel to HD. Also let GZZ be another curve such that, when any straight line EZ is drawn from E parallel to VD cutting the base HD in I and the curve GZZ in Z, and a straight line of given length R is taken; then at all times $DA^2 : R^2 = DT : IZ$.

"Then $DA : AE = R^2 : \text{space DGZI}$."

The omitted proof would have run as follows:

Let VXY be a curve such that, if EA produced meets it in Y, then $EA : AD = AY : R$. Divide the arc EV into an infinite number of parts at F, M, etc. and draw FBX, MCX, etc. parallel to HD, meeting VD in B, C, etc. and the curve VXY in the points X; also draw FJZ, MKZ, etc. meeting HD in J, K, etc. and the curve GZZ in the points Z.

Then $AY \cdot AD \cdot BD = R \cdot EA \cdot BD = R \cdot (EA \cdot AD + EA \cdot AB)$,

and $BX \cdot AD \cdot BD = R \cdot FB \cdot AD = R \cdot (EA \cdot AD - IJ \cdot AD)$;

hence, if XW, drawn parallel to VD, cuts AY in W, we have

$WY \cdot AD^2 = WY \cdot AD \cdot BD = R \cdot (EA \cdot AB + IJ \cdot AD)$.

But, as in previous theorems, $EA : AT = IJ : AB$, $AB \cdot AE = AT \cdot IJ$;

$WY \cdot AD^2 = R \cdot (AT \cdot IJ + IJ \cdot AD) = R \cdot DT \cdot IJ$.

Now $DA^2 : R^2 = DT : IZ = DT \cdot IJ : IZ \cdot IJ$;

$R^2 : IZ \cdot IJ = AD^2 : DT \cdot IJ = R : WY$.

Hence, since the sum of the rectangles $IZ \cdot IJ$ only differs in the least degree from the space DGZI, and the sum of the lengths WY is AY; it follows immediately that

$R^2 : \text{space DGZI} = R : AY = DA : AE$.

The important points about this theorem are

1. that Barrow says "Perhaps at some time or other the following theorem, deduced from what has gone before, will be of service; *it has been so to me repeatedly*";

2. that, if DT and DH are taken as the coordinate axes, and it is taken into account that the tangent ET makes an obtuse angle

with the x -axis, then $DT = x - y dx/dy$; also $IJ = dy$, and WY is $d(y/x)$. Hence the analytical equivalent of the equality

$$WY \cdot AD^2 = R \cdot DT \cdot IJ \text{ is } Rx^2 \cdot d(y/x) = R \cdot (x - y dx/dy) dy;$$

or

$$d(y/x) = (x dy - y dx)/x^2.$$

Thus Barrow had the geometrical equivalent of the *differentiation of a quotient*, and found it of service repeatedly.

I will make one more quotation. As an example of a method of construction given for drawing, in general, curves such as the one given below, we have the following:

"Let AEG be a curve whose axis is RAD, such that, when through any point E taken in it a straight line EDM is drawn perpendicular to AD, and AE is joined, then AE is always a mean proportional between a given length AR and AP, of the order whose exponent is n/m. It is required to find the curve AMB of which the tangent at M is parallel to AE."

"I note, about the curve AM, that $n:m = AE : \text{arc } AM$."

"If $n/m = 1/2$ (or AE is the simple geometrical mean between AR and AP), then, AEG being a circle, AMB is the primary cycloid. Hence the measurement of the latter comes out of a general rule."

Thus Barrow obtains the fact that the arc AM of a cycloid is twice the corresponding chord of the circle. Most of the theorems on the cycloid are due to Pascal; but in the *Encyclopaedia Britannica* the rectification of the cycloid is ascribed to Wren. If the reference there given to the *Phil. Trans.* of 1673 is correct, it follows that Wren was anticipated by Barrow. It is well known that previously only one curve, the semi-cubical parabola, had been rectified.

Lastly it may be noted that many of Barrow's theorems in *Lectio XI*, when translated into analytical form, are nothing more or less than theorems on the change of the independent variable in integration. Thus he shows that

$$\int y dx = \int y / (dy/dx) dy, \quad \int r^2 d\theta = \int r^2 (d\theta/dr) dr.$$

Many other points might be made, but, in Barrow's words,

Haec sufficere videntur.

The two points now remaining to be considered are:

1. Why, if Barrow's genius and knowledge were so great, did he not complete the work he had so ably begun, and be hailed universally as the real originator of the calculus?

2. What influence did his predecessors have on Barrow, and what influence did Barrow and Newton have upon one another?

On the question as to the sources from which Barrow derived his ideas, there is some difficulty in deciding; and the narrowness of my reading makes me diffident in writing anything that might be considered dogmatic on this point; so that the following remarks are put forward more or less in the fashion of suggestions.

The general opinion would seem to be that Barrow was a mere improver on Fermat. But if we are to believe in Barrow's honesty the source of his ideas could not have been the work of Fermat. For Barrow religiously gives references to the ancient and contemporary mathematicians whose work he quotes. These include Cartesius, Hugenus, Galilaeus, Gregorius a St. Vincentio, Gregorius Aberd. (James Gregory of Aberdeen; in connection with this name, Barrow makes the noteworthy statement that he does not care to put his "*sickle into another man's harvest*"—the reference being to Gregory's work on evolutes and involutes), Euclides, Aristoteles, Apollonius and many others; but no mention is made of Fermat, nor does he use Fermat's method of determining the tangent by a maximum or minimum ordinate. On the other hand he may have deliberately omitted reference to Fermat, because his criterion of tangency of lines and curves was so similar to this method, that he might have provoked by the reference accusations of plagiarism. There is a distinct admiration shown for the work of Galileo, and the idea of time as the independent variable obsesses the first few lectures, an idea which he evidently obtained in the first place from Galileo, as did Newton also. But, like Newton, he simply intends this as a criterion by means of which he can be sure that one of his variables shall increase uniformly. Also, we learn from the preface that these preliminary chapters, in which he discusses time, were an afterthought; Barrow says "falling in with his (*Librarius*—the publisher, query Collins) wishes, I will not say unwillingly, I added the first five lectures."

The mental picture that I form of Barrow is that of the teacher, who has to deliver lessons on a subject, reading up everything he can lay his hands on, and then pugnaciously deciding that, although most of it is very good stuff, yet he can and will "go one better." In the course of his work he happens on the paraboliforms, perceives their usefulness, and is immediately led on to the great discovery of the "differential triangle" method. I think if any one

compares the figures used, (i) for the proof of tangency in the case of the paraboliforms, and (ii) for the infinitesimal method, he will no longer inquire for the source from which Barrow got his ideas.

Personally I have not the slightest doubt that it was a flash of inspiration suggested by the former figure (indeed it was this resemblance which caused me to put into analytical form the theorem chosen as example 2 above, and led me on to the translation of the whole work); it was Barrow's luck to have first of all had occasion to draw that figure, and secondly to have had the genius to have noticed its significance and to be able to follow up the clue thus afforded. As further corroborative evidence that Barrow's ideas were in great part his own creations we have the facts that he was alone in considering a curve as a collection of indefinitely short straight lines, and that, as he states in one place, he could not see any difference between indefinitely narrow rectangles and straight lines as the constituent parts of an area.

The answer to the question as to why Barrow did not com-

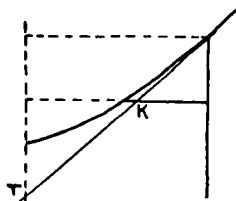


Fig. 9(i).

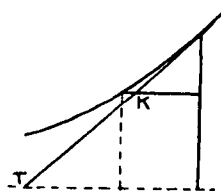


Fig. 9(ii).

plete the work he had begun is, I think, inseparably bound up with his connection with Newton; and I can imagine that Barrow's interest, as a confirmed geometer, would have been first really aroused by Newton's poor show in his scholarship paper on Euclid, for which Barrow was the examiner. This was in April, 1664, the year of the delivery of Barrow's first lectures as Lucasian professor, and, according to Newton's own words, just about the time that he (Newton) discovered his method of infinite series, led thereto by his reading of the work of Wallis and Descartes. Newton doubtless attended these lectures of Barrow, and the probability is that he would have shown to Barrow his work on infinite series (this seems to have been the custom of the time, for it is on record that Newton five years later, in 1669, communicated to Collins, through Barrow, a compendium of his method of fluxions). Bar-

row would be struck with the incongruity of a man of Newton's ability not appreciating Euclid; at the same time the one great mind would be drawn to the other, and the connection thus begun would inevitably have developed. Here we must consider that Barrow was professor of Greek from 1660 to 1662, then professor of geometry at Gresham College from 1662 to 1664, and Lucasian professor from 1664 to 1669; that Newton was in residence as a member of Trinity College from 1661 until he was forced from Cambridge by the plague in the summer of 1665; that, from manuscript notes in Newton's handwriting, it was probably during this enforced absence from Cambridge (and Barrow) that he began to develop his method of fluxions. From these dates I argue that Barrow most probably developed his geometrical work from researches begun for the necessities of lectures at Gresham College in the years 1662-3-4, and further elaborated them in the years 1664-5-6; that Newton would have not only heard these lectures before he had to leave Cambridge, but also would have had the manuscript to read, as a loan to a pupil from a master who had begun to take a strong interest in him; and that thus Newton would have got the germ of the idea from Barrow, but that the accident of the forced disconnection at this time made Newton follow the idea up in the manner and style which was essentially his own.

The similarity of the two methods of Barrow and Newton is far too close to admit of them being anything else but the outcome of one single idea. For the fluxional method the procedure is as follows:

1. Substitute $x + \dot{x}o$ for x and $y + \dot{y}o$ for y in the given equation connecting the fluents x and y .
2. Subtract the original equation and divide through by o .
3. Regard o as an evanescent quantity, and neglect o and its powers.

Barrow's rules are, altered in order for the sake of the correspondence:

2. After the equation has been formed (Newton's rule 1) reject all terms consisting of letters denoting constant or determined quantities or terms which do not contain a or e (which are equivalent to Newton's $\dot{y}o$ and $\dot{x}o$ respectively); for these terms brought over to one side of the equation will always be equal to zero (Newton's rule 2, first part).

1. In the calculation omit all terms containing a power of a or e ,

or products of these, for these are of no value (Newton's rule 2, second part, and rule 3).

3. Now substitute m , the ordinate, for a , and t , the subtangent, for e . This corresponds to Newton's next step, the obtaining of the ratio $\dot{x}:\dot{y}$, which is exactly the same as Barrow's $e:a$.

The only difference is that Barrow's way is more suitable to his geometrical purpose of finding the "quantity of the subtangent," and Newton's method is peculiarly adapted for analysis, especially in problems on motion. It is particularly to be observed that Barrow, in giving a description of his way, writes throughout in the first person singular. Although at the time of publication of the lectures Barrow had seen the fluxional method, or "a compendium" of it, as it passed through his hands on its way to Collins, yet he left his own method as it stood; probably he used it freely (he applies to it the words *usitatum a nobis*—the word *usitatum* being elsewhere written to denote *familiar* or *well known*; also mark Barrow's use of the more or less usual plural *nobis* in opposition to the first person singular when describing the method) to obtain hints for his tangent propositions, but not thinking much of it as a method compared with a strictly geometrical method, probably because he could not always find a geometrical construction to correspond; yet he admits it into his work "on the advice of a friend" on account of its generality. On the other hand Newton perceives the immense possibilities of the analytical methods introduced by Descartes, and develops the idea on his own lines, possibly owing to the accident of his being removed from the influence of Barrow for a short time.

There is however another possibility. In the preface we read that "as delicate mothers are wont, I committed to the foster care of friends, not unwillingly, my discarded child".... These two friends Barrow mentions by name, "*Isaac Newton*.... (a man of exceptional ability and remarkable skill) has revised the proof, warning me of many matters to be corrected, and adding some things of his own work"*.... "*John Collins* has attended to the publication." It is just possible that Newton showed Barrow the idea of his fluxional method before he had developed it fully, and that Barrow developed it in some small degree as a tool for the purpose mentioned above, and inserted it into his work. At any rate it seems to be fairly plain that Newton was the friend on whose

* Most probably in the *Optics*.

advice the method was inserted. I think however that the more probable alternative, judging from the later work of Newton, is that first given. This would explain the lack of what I have endeavored to make out to be the true appreciation of Barrow's genius. Barrow saw that the correct development of his idea was on purely analytical lines, he recognized his own disability in this direction and the peculiar aptness of Newton's genius for the task; and the growing desire to forsake mathematics for divinity made him only too willing to hand over his discarded child to the foster care of Newton and Collins "to be led out and set forth as might seem good to them," as he says in his preface. Who can tell what might have appeared in a second edition, "revised and enlarged," if Barrow, on his return to Cambridge as Master of Trinity and afterwards Vice-Chancellor, had had the energy to make one; or if Newton had made a treatise of it instead of a book of "Scholastic Lectures," as Barrow warns his readers that it is? But Barrow died two years later, and Newton was far too occupied with other matters.

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[Note.—Since writing the above article, the author has found that the *Lectiones Geometricae* form a perfect calculus. This will be explained in a forthcoming volume of the *Open Court Classics of Science and Philosophy*. —Ed.]

POLYXENA CHRISTIANA.*

A REVIEW OF BOUSSET'S "KYRIOS CHRISTOS."

"But she, though dying, none the less
Great forethought took, in seemly wise to fall."

—Eur., *Hek.*, 568f.

By odds the most imposing and important apologetic of recent years is the deep-learned, deep-felt and deep-thoughted *Kyrios Christos* of Prof. Wilhelm Bousset, well known by his *Religion des Judentums*, his *Offenbarung Johannis*, his *Hauptprobleme der Gnosis*, and as editor with Wilhelm Heitmüller of the *Theologische*

* This review, written in the first half of the year 1914, has been withheld from the press thus far, along with several other such essays, in the hope that after the cessation of hostilities in Europe it might more readily "fit audience find, though few"; but the coming of such a season seems now too likely to be indefinitely delayed.