



# LIX. The mass carried forward by a vortex

W.M. Hicks F.R.S.

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LIX. *The Mass carried forward by a Vortex.**By W. M. HICKS, F.R.S.\**

WHEN a vortex aggregate is moving steadily through an irrotational liquid we can in general distinguish three definite regions of fluid motion, (1) that of the ring or aggregate itself which is in rotational motion and which keeps its identity and constituents throughout however the energy may alter, (2) the portion in irrotational cyclic motion surrounding the first, which also keeps its identity and volume so long as the energy is constant, and which travels uniformly through the liquid like a solid, (3) the irrotational acyclic motion, outside the second region which remains at rest at infinity and no portion of which is ever displaced by more than a small amount. The distinction between the rotational region (1) and the irrotational (2), (3) is fundamental and well known. Less attention than it deserves, however, appears to have been devoted to the discussion of the relationships between the 2nd and 3rd regions. In this regard the following note may be interesting.

When any such aggregate is travelling uniformly through an unlimited fluid with translatory velocity  $U$  relative to the fluid at a great distance, we may in order to study the relative motion suppose the ring brought to rest by imposing everywhere a velocity equal and opposite to  $U$ . In this case the boundaries between the three regions appear as fixed. The first is always a ring surface, even when the aggregate closes up, or the aperture diminishes to zero. The boundary between (2) and (3) may be a ring surface, or it may appear as a singly-connected surface, past which the outer liquid streams. This is what an observer travelling with the ring sees, and we can in this way determine the boundary mass and energy of the portion which goes bodily through the liquid, and the energy of the external part or region (3). Take as an instance a circular unicyclic† vortex ring. When the ratio of the cross section of ring to aperture is very small, it is well known that the velocity at the centre is less than that of translation. Consequently when the system is brought to rest the flow at the centre is in the opposite direction to that of the original translation. In this case the boundary (2), (3) must cut the equatorial plane somewhere

\* Communicated by the Author.

† *I.e.* without bicyclic or gyrostatic quality.

between the centre and the filament, and it will be ring-shaped. As the energy diminishes, the aperture becomes smaller, and the ratio of cross section to aperture larger. The velocity of translation increases, but the velocity at the centre increases at a greater rate, until a state is reached at which the two become equal. In this case the acyclic boundary just loses its ring form and its section has a lemniscate form. As the energy still further increases this boundary cuts across the straight axis of the vortex and the volume of region (2) will ultimately diminish, until the vortex itself closes up into the spherical aggregate, when it entirely disappears. Thereafter it is the actual rotational portion alone which is propagated through the surrounding fluid.

In determining the energies of the three portions it is most convenient first to find their values for the first and second regions when in the stationary state. In this case the stream functions  $\chi$  along the boundaries are constant. If  $E$ ,  $E'$  denote energies of the actual and stationary states respectively

$$E' = \pi\mu(\chi_2 - \chi_1) + 2\pi \iint \omega\chi dx dy,$$

where  $\chi_2$ ,  $\chi_1$  are the constant stream functions along the inner and outer boundaries of a region.

When the vorticities are constant throughout the rotational portions,  $\omega = \text{const.}$  for two-dimensional motion and  $= \lambda x$  for three.

For two dimensions  $\omega \times \text{area of section} = \frac{1}{2}\mu$ ,

$$E' = \frac{1}{2}\mu(\chi_2 - \chi_1) + \frac{\mu}{2A} \iint \chi dx dy.$$

For three dimensions  $\omega = \lambda x$ ,  $\omega dA = \frac{1}{2}d\mu = \lambda x dA = \frac{\lambda}{2\pi} \cdot dV$ ,

$$\therefore \mu = \frac{\lambda}{\pi} \times \text{vol.} = \frac{\lambda m}{\pi},$$

$$E' = \pi\mu(\chi_2 - \chi_1) + \frac{2\pi^2\mu}{m} \iint x\chi dx dy.$$

Passing now to the case where the system is moving with velocity  $U$  through the fluid at rest at infinity, let  $p$ ,  $q$  denote the component velocities in the stationary state, so that  $p + U$ ,  $q$  are the actual ones. Then

$$E = \frac{1}{2} \iint \{ (p + U)^2 + q^2 \} dx dy = E' + \frac{1}{2} \iint U^2 dx dy + U \iint p dx dy,$$

$$\iint p dx dy = \text{momentum in stationary case} = 0.$$

$$\therefore E = E' + \frac{1}{2}(\text{vol. of region}) U^2.$$

But we might have applied the general theory direct to the actual motion. In this case, omitting the rotational region for the moment,  $E = -\pi \int \psi v' ds$ , in which the stream function  $\psi$  is  $\chi + \frac{1}{2} U x^2$ , and  $v'$ , the velocity along the boundary, is  $v + U \frac{dy}{ds}$  where  $v$  is the corresponding velocity in the stationary case. Hence

$$E = -\pi \int (\chi + \frac{1}{2} U x^2) v ds - \pi U \int (\chi + \frac{1}{2} U x^2) dy.$$

Here  $\int \chi dy = \chi \int dy = 0$  along a closed curve.

$$\int x^2 dy \text{ along the two boundaries} = -\frac{1}{\pi} (\text{vol. of region}).$$

$$\therefore E = E' + \frac{1}{2} (\text{vol. of region}) U^2 - \frac{1}{2} \pi U \int x^2 v ds.$$

Comparing with the previous result it follows that  $\int x^2 v ds = 0$ , or that  $\int x^2 v ds$  round one boundary is equal to that round the other.\* Clearly this theorem is general, and states\* that in a stationary condition  $\int x^2 v ds$  is the same along any two stream lines, provided no rotational region intervenes between them.

For the core, the term  $2\pi^2 \mu m^{-1} \iint x \psi dx dy$  becomes

$$\frac{2\pi^2 \mu}{m} \iint x \chi dx dy + \frac{\pi^2 \mu U}{m} \int x^3 dx dy.$$

Now  $2\pi \int x^3 dx dy = \int x^2 \cdot 2\pi x dx dy$  is the moment of inertia of the whole ring round the translation axis  $= mk^2$ , where  $k$  is the corresponding radius of gyration, and is directly expressible in terms of the shape of the section. Thus

$$\begin{aligned} E_1 &= E_1' + \frac{1}{2} m U^2 + \frac{\pi}{2} \mu U k^2 - \frac{1}{2} \pi U \int x^2 v ds \\ &= E_1' + \frac{1}{2} m U^2. \end{aligned}$$

\* The following direct proof of this, I owe to the kindness of Mr. G. H. Livens. Let  $p, q$  denote the component velocities at a point on a stream line. Then integrating along two boundaries

$$\begin{aligned} \int x^2 v ds &= \int x^2 (p dx + q dy) = \iint \left\{ \frac{d}{dx} (x^2 q) - \frac{d}{dy} (x^2 p) \right\} dx dy \\ &= 2 \iint x q dx dy \\ &= \frac{1}{\pi} \times \text{quantity carried forward in the stationary state} \\ &= 0. \end{aligned}$$

Therefore  $\int x^2 v ds$  taken round the stationary core boundary  $= \mu k^2$ .

The corresponding theorem in two dimensions is

$$\int x v ds = \mu \bar{x},$$

where  $\bar{x}$  is the distance of the centre of gravity of the section from the translation axis.

For region (3), the energy in the translation state is

$$\begin{aligned} E_3 &= E_3 + E_2 - E_2 \\ &= \pi \mu \chi_2 - \frac{1}{2} (\text{vol. core}) U^2 + \frac{1}{2} \pi U \int x^2 v ds - E_2, \end{aligned}$$

the integral being taken over the core section

$$= \pi \mu \chi_1 + \frac{1}{2} (\text{total vol. carried forward}) U^2 + \frac{\pi}{2} \mu U k^2.$$

When the portion carried forward is singly-connected,  $\chi_1 = 0$ ,

$$E_3 = \frac{\pi}{2} \mu U k^2 - \frac{1}{2} (\text{translated mass}) U^2.$$

This can be verified at once in the case of the spherical vortex in which  $U = \mu/5c$ ,  $k^2 = \frac{2}{5}c^2$ . This gives  $E_3 = \frac{1}{4}mU^2$  or energy of translation of half mass of fluid displaced by the sphere, and is correct.

The corresponding theorem for two dimensions is on one side of axis of  $x$

$$E_3 = \frac{1}{2} \mu U \bar{x} - \frac{1}{2} (\text{translated mass}) U^2,$$

or for the whole motion

$$E_3 = \mu U \bar{x} - \frac{1}{2} (\text{translated mass}) U^2,$$

$$E_2 = \mu \chi_2 + \frac{1}{2} (\text{mass of region 2}) U^2.$$

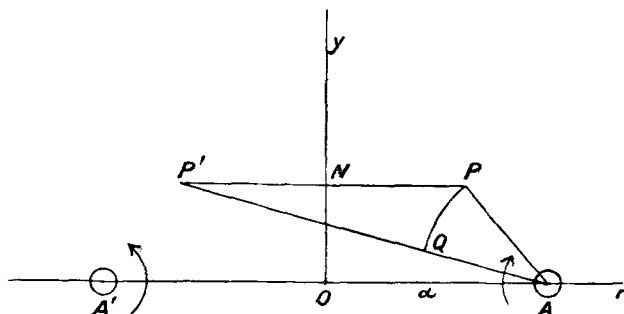
Of this general theory the present note discusses in more detail the two special cases of (a) two parallel straight vortices, (b) a single ring vortex of uniform vorticity.

#### (a) *Two parallel straight vortices.*

Take first the case where they are so far apart that their sections can be regarded as circles of radius  $c$ , with centres at a distance  $2a$ . If  $\omega$  denote the rotational constant and  $\mu$  the circulation,  $\mu = 2\pi c \cdot \omega c = 2\pi c^2 \omega$ . The velocity at any point due to one vortex, at a distance  $r$  from it, is  $\omega c^2/r = \mu/(2\pi r)$ . In the figure A, A' are the centres of the vortices,

$P, P'$  any symmetric points on opposite sides of the axis  $Oy$ . Then the stream function  $\chi$  at  $P$  = flow through  $PN$  due to

Fig. 1.



$A, A'$  = flow through  $PP'$  due to  $A$  alone = flow across  $P'Q$  where  $AQ = AP$ . Hence

$$\chi = \frac{\mu}{2\pi} \int_{AQ}^{AP} \frac{dr}{r} = \frac{\mu}{2\pi} \log \frac{AP'}{AP}.$$

Also velocity of translation is velocity at  $A'$  due to  $A$ , or

$$U = \frac{\mu}{4\pi a}.$$

Hence when brought to rest the stream function is

$$\chi = -Ux + \frac{\mu}{4\pi} \log \frac{(x+a)^2 + y^2}{(x-a)^2 + y^2},$$

or 
$$\frac{\pi}{4\mu} \chi = -\frac{x}{a} + \log \frac{(x+a)^2 + y^2}{(x-a)^2 + y^2}.$$

The velocity at the centre relative to the surrounding liquid is  $\mu/\pi a$  or  $4U$ . Consequently in the case of parallel straight vortices the cyclic boundary (2, 3) is singly-connected. The stream function along it has the same value as for the axis and is zero. The equation to the boundary is therefore, taking the unit of length =  $a$ ,

$$\log \frac{(x+1)^2 + y^2}{(x-1)^2 + y^2} = x$$

or 
$$y^2 = 2x \coth \frac{x}{2} - x^2 - 1.$$

As this equation is independent of  $c$ , the shape of the translated mass remains unaltered, and its area simply varies as  $a^2$ , so long as the filaments are not so close that the shape of the cross sections deviate considerably from circles. So long as they can be treated as such, the shape remains the same however the energy alters. It may therefore be drawn once for all. For this purpose the equation may be written in the form suitable for logarithmic calculation with tables of hyperbolic functions\*.

$$y = \sqrt{\{(\coth x/4 - x)(x - \tanh x/4)\}}.$$

This cuts the plane of the filaments at a distance given by

$$\coth x/4 = x,$$

$$\text{or} \quad \frac{x}{2} = \log_e \frac{x+1}{x-1},$$

$$\text{whence} \quad x = 2.087288 = \alpha a.$$

It also cuts the axis of  $y$  at  $y = \sqrt{3} = \beta a$ .

The calculation is easily carried out by the aid of the Smithsonian tables. The curve when drawn is seen to be very close to a true ellipse. Indeed, if an ellipse be graphically constructed with the same axes it is difficult to distinguish any difference. For  $x=1$ , near which we should expect a maximum deviation, the ordinates of the two are 1.5203 for the ellipse and 1.5257 for the boundary, corresponding to a distance between the two curves of .0035 $a$ . For practical purposes we may therefore take the area to be  $\pi \times 2.0872 \times \sqrt{3}a^2 = 3.61\pi a^2$ , or that of a circle of radius 1.901 $a$ .

The energies are

$$E_1' = \frac{\mu^2}{8\pi},$$

$$E_1 = \frac{\mu^2}{8\pi} + 2 \times \frac{1}{2} \pi c^2 U^2 = \frac{\mu^2}{8\pi} \left( 1 + \frac{c^2}{2a^2} \right),$$

$$E_2 = 2 \times \frac{1}{2} \mu \chi_2 + \frac{1}{2} (\text{mass of region 2}) U^2$$

$$= \frac{\mu^2}{4\pi} \left\{ 2 \log \frac{2a}{c} - 1 - \frac{c^2}{4a^2} \right\} + \frac{1}{2} (\pi \alpha \beta a^2 - 2\pi c^2) \frac{\mu^2}{16\pi^2 a^2}.$$

\* *E. g.* "Hyperbolic Functions." Smithsonian Mathematical Tables.

Now  $\alpha\beta = 2.0872 \sqrt{3} = 3.6151$ ,

$$E_2 = \frac{\mu^2}{2\pi} \left\{ \log \frac{2a}{c} - .2740 - \frac{c^2}{4a^2} \right\},$$

$$E_3 = \mu U a - \frac{1}{2} \times 3.6151 U^2$$

$$= \frac{1}{2} \times 4.3849 \pi a^2 U^2$$

$$= \frac{1}{2} \times 1.2130 \times \text{translated mass} \cdot U^2$$

$$= .1339 \frac{\mu^2}{\pi}.$$

The last result can be verified from the fact, that since the form of the translated mass differs only inappreciably from that of an elliptic cylinder, the energy of the external motion is  $\alpha a/\beta a$  of the liquid displaced by the cylinder. Now  $\alpha/\beta = 1.205$ . The difference from 1.2130 is probably due more to the velocity changes due to the slight change in form, than to the difference in area of the two cross sections.

The curious fact emerges, that so long as the filaments are not so close as to appreciably affect the shape of their cross sections, the energy of the external fluid is constant and is independent of the velocity of propagation. As the velocity increases, the quantity carried forward diminishes so that this result follows. This fact might have been foreseen from a consideration of dimensions, remembering that with the proviso above, there is only one length  $a$  at disposal to define the system.

The most interesting portion of the motion, when the two filaments close in to be almost in contact, is unfortunately not at present capable of being treated. It is easy to show, however, that if the shape of the core-section referred to its centre of gravity be given by  $r = a(1 + b_2 \xi^2 \cos 2\theta + \dots)$  where  $\xi = c/a$ , the change in  $U$  depends on  $\xi^4$ . The above values are therefore approximately good even when a considerable amount of deformation is present.

### (b) *The circular ring.*

When with diminishing energy the aperture just closes up, the aggregate, as is well known, takes the form of a sphere.



If  $c$  denote its radius

$$\text{mass} = m = \frac{4}{3} \pi c^3,$$

$$U = \frac{\mu}{5c},$$

$$E_1 = \frac{2}{35} \pi \mu^2 c,$$

$$E_2 = 0,$$

$$E_3 = \frac{1}{4} m U^2 = \frac{\pi \mu^2 c}{75} = \frac{7}{30} E_1.$$

Thus the external energy is less than one quarter the internal.

The only other state in which an approximate solution is attainable with the present state of theory is the ring state when the ratio  $b/a$  is not large, where  $b$  is the radius of the cross section, and  $a$  that of the circular axis. Here

$$m = 2\pi a \times \pi b^2 = 2\pi^2 a b^2 = \frac{4}{3} \pi c^3$$

$$\text{or} \quad a b^2 = \frac{2}{3\pi} c^3.$$

For this case

$$U = \frac{\mu}{4\pi a} \left\{ \log \frac{8a}{b} - \frac{1}{4} \right\} = \frac{\mu}{\pi} V \quad (\text{say}),$$

$$\chi = \frac{\mu}{\pi} \sqrt{(ax)} \left\{ \frac{1+k'^2}{2k} F - \frac{1}{k} E \right\},$$

$$\text{where} \quad k^2 = \frac{4ax}{(x+a)^2 + y^2}, \quad k'^2 = \frac{(x-a)^2 + y^2}{(x+a)^2 + y^2},$$

and  $\chi$  denotes  $\frac{1}{2\pi} \times$  flow through disk parallel to the plane of the ring whose rim passes through the point  $x, y$ . When brought to rest, therefore, the stream function in the stationary condition is given by

$$\frac{\pi \chi}{\mu} = \sqrt{(ax)} \left\{ \frac{1+k'^2}{2k} F - \frac{1}{k} E \right\} - \frac{1}{2} V x^2,$$

$$\text{where now} \quad V = \frac{1}{4a} \left( L - \frac{1}{4} \right), \quad L \text{ denoting } \log_e \frac{8a}{b}.$$

In general the velocity at the centre of the ring is less than that of translation. In this case region (2) is also

ring shaped. To find the condition that it may be singly-connected it must be possible to find a point on the axis where the flow is zero. Write

$$\frac{1+k^{1/2}}{2k} F - \frac{1}{k} E \equiv X.$$

Near the axis, that is  $k$  very small, it is easy to show that

$$X = \frac{\pi}{32} k^3.$$

At a point of bifurcation  $d\chi/dx=0$ ,

$$\therefore \frac{1}{2} \sqrt{\frac{a}{x}} X + \sqrt{(ax)} \frac{dX}{dk} \cdot \frac{dk}{dx} - Vx = 0;$$

$$\text{also } \frac{dk}{dx} = \frac{1}{2} \frac{k}{x}, \quad x = \frac{k^2}{4a} (a^2 + y^2) \quad \text{near the axis,}$$

whence it easily follows that

$$a^2 + y^2 = a^2 \left( \frac{2\pi}{L - \frac{1}{4}} \right)^{2/3}.$$

Hence

$$L - \frac{1}{4} < 2\pi,$$

$$L < 6.53318,$$

$$\frac{b}{a} > .01163.$$

For this, and considerably larger, values of  $b/a$ , the expressions for  $\chi$ ,  $U$  will still hold with great approximation, and we shall be able to obtain some information as to the singly connected states. For  $b/a = .0116$ ,  $y=0$ , and the boundary has a node at the centre. The two configurations require separate treatment.

*Singly-connected configuration.*—Here  $b/a > .0116$ . The equation to the boundary is given by  $\chi=0$ . It cuts the axis at a point given by

$$y/a = \sqrt{\left\{ \left( \frac{2\pi}{L - \frac{1}{4}} \right)^{2/3} - 1 \right\}}.$$

With given  $b/a$ ,  $U$  is calculated. The equation to the boundary can then be written

$$X = \frac{1}{2} V a \cdot \left( \frac{x}{a} \right)^{3/2}$$

$$\begin{aligned} \text{or } \frac{x}{a} &= \left( \frac{2X}{aV} \right)^{2/3} = \left( \frac{8}{L - \frac{1}{4}} \right)^{2/3} X^{2/3} \\ &= A X^{2/3} \quad (\text{say}). \end{aligned}$$

Now  $X$  is a function of  $k$  alone. The curve is then traced by giving a series of values to  $k$ , and calculating  $x$  from this equation. The corresponding ordinate is found from the circle  $k=\text{const.}$  or  $(x+a)^2 + y^2 = \frac{4ax}{k^2}$ . If a complete set of these circles (corresponding to bipolar co-ordinates) be drawn the points corresponding to given  $x$  can be found graphically, although the method is not susceptible of great exactness when the points are in the neighbourhood of the equatorial plane. If the boundary be drawn for a given value of  $A$ , that for any other suitable value can be drawn at once in the following way. Suppose the new value of  $A$  is  $f \cdot A$ . If the boundary for  $A$  cuts one of the circles in  $P$ , draw  $PN$  perpendicular to the axis and on it take  $P'N = f \cdot PN$ . The perpendicular from  $P'$  to the axis of  $x$  (or equatorial plane) will cut the same circle at  $Q$ , which is a corresponding point on the new boundary. It will therefore be sufficient to draw the curve accurately for a particular value of  $A$  only. The curve I in fig. 2 is drawn for  $A=2$ .  $A=1$  corresponds to  $b/a = .002$  and is too small to give a singly-connected space.  $A=2$  corresponds to  $b/a = .368$ , which is far too large for our approximate formulæ to hold. But this is immaterial for a standard curve from which to draw others for which the approximation holds. For the limiting case  $A=1.1741$  or  $f=.587$ . In general

$$\begin{aligned} \frac{8}{L - \frac{1}{4}} &= A^{3/2} = (2f)^{3/2}, \\ L &= \frac{1}{4} + \left(\frac{2}{f}\right)^{3/2}, \\ b/a &= 8e^{-L}. \end{aligned}$$

These give  $b/a$  when  $f$  is given, or  $f$  when  $b/a$  is given. The table\* at the end gives the values of  $X$ ,  $\log X$ ,  $X^{2.3}$  for a series of values of  $k$  used in the calculations of this paper.

Curves II, III, V, give the singly-connected boundaries for the three cases of  $b/a = .1, .05, .0116$ —the last being the limiting case. They were drawn graphically from I by the method indicated above.

The areas and volumes found graphically from the curves are :

	Area.	Vol.
II. ....	$1.220 a^2$	$2.18 a^3$
III. ....	$.478 a^2$	$1.685 a^3$
V. ....	$.268 a^2$	$1.362 a^3$ .

\* The values of  $F$ ,  $E$  are taken from the values given in Bertrand's *Calcul Intégral*.

In connexion with the construction circles  $k = \text{const.}$ , they cut the axis of  $x$  at points

$$x = a \frac{1-k'}{1+k'} \quad , \quad a \frac{1+k'}{1-k'} ,$$

so that

$$\text{rad} = \frac{2k'}{k^2} , \quad \text{distance of centre from axis} = \frac{1+k'^2}{k^2} = \frac{2}{k^2} - 1.$$

*Toroidal configuration.*—Passing now to the consideration of the case where the second region is ring-shaped, the boundary is given by the loop of that stream line which cuts itself in the equatorial plane, at a point where the velocity is zero. The value of the stream function for this is clearly negative, since in the aperture between the centre and this point the velocity is everywhere negative. As before the stream function is given by

$$\frac{\pi\chi}{\mu} = \sqrt{(ax)}X - \frac{1}{2}Vx^2.$$

Under given conditions, *i. e.*  $V$  given, we have to find the value of  $x$  which makes  $d\chi/dx = 0$  when  $y = 0$ . Substituting this value of  $x$  and  $y = 0$  in  $\chi$  then gives the constant value (say  $\chi_1$ ) of the bounding stream line, and the equation of the latter is  $\chi = \chi_1$ . The finding of the root of  $d\chi/dx$  can be carried out by ordinary approximation when numerical examples are required. Our present purpose, however, is not so much to get the result for a given state (value of  $b/a$ ) as to follow the changes as the states vary. For this purpose it is best to choose positions for sets of nodes, and calculate the values of  $U$  required to satisfy  $d\chi/dx = 0$ . The corresponding value of  $b/a$  is then found from  $U$ . For instance choose  $k$ , where  $x/a = (1-k')/(1+k')$  since now  $x < a$ . Then

$$\frac{1}{2} \sqrt{\frac{a}{x}} X + \sqrt{(ax)} \frac{dX}{dk} \cdot \frac{dk}{dx} - Vx = 0,$$

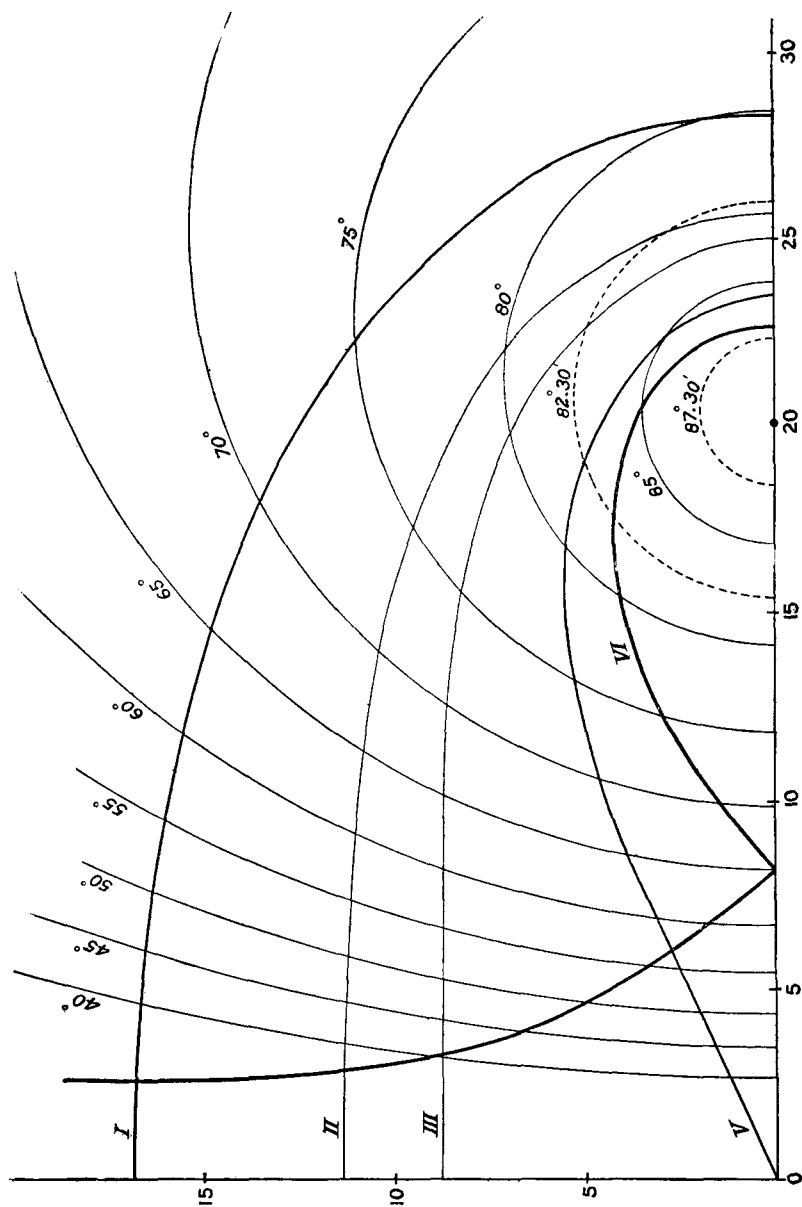
$$\frac{dk}{dx} = \frac{k k'}{2x} \quad \text{when } y = 0.$$

Thus 
$$X + k k' \frac{dX}{dk} - 2aV \left(\frac{x}{a}\right)^{3/2} = 0.$$

It follows after an easy reduction that

$$aV = \frac{1}{4} \frac{1+k'}{k'} (k'F + E).$$

Fig. 2.



From V, the corresponding value of  $b/a$  is found, and

$$\begin{aligned}\frac{\pi\chi_1}{\mu a} &= \frac{1-k'}{k} \left\{ X_1 - \frac{1}{2}aV \left( \frac{1-k'}{k} \right)^3 \right\} \\ &= \tan \frac{\theta}{2} \left\{ X_1 - \frac{1}{2}aV \tan^3 \frac{\theta}{2} \right\}, \quad \text{where } k = \sin \theta.\end{aligned}$$

These give  $\chi_1$ , V,  $b/a$  for a configuration in which the boundary of the translated fluid cuts the equatorial plane at a distance  $x = a(1-k')/(1+k') = a \tan^2 \frac{\theta}{2}$  from the centre.

The boundary is drawn as before by taking a series of values of  $k$ , greater than that chosen for the node, solving for  $x$  from the equation  $\chi = \chi_1$ , and taking the corresponding point on the  $k$  circle. The equation however now is the quartic

$$\left(\frac{x}{a}\right)^2 - \frac{2}{aV} X \cdot \left(\frac{x}{a}\right)^{\frac{3}{2}} + \frac{2}{aV} \chi_1 = 0,$$

$$\text{say} \quad z^4 - pz - q = 0$$

where  $z^2 = x/a$  and  $q$  is essentially positive since  $\chi_1$  is negative. The left-hand member may be written

$$(z^2 + az + b)(z^2 - az + b') = 0,$$

where

$$b + b' - a^2 = 0,$$

$$ab' - ab = -p,$$

$$bb' = -q.$$

Hence

$$2b' = a^2 - \frac{p}{a},$$

$$2b = a^2 + \frac{p}{a}.$$

Since  $a^2 - 4b$  is negative, the first factor gives imaginary roots and the real roots are

$$z = \frac{a}{2} \pm \frac{1}{2} \sqrt{\left(\frac{2p}{a} - a^2\right)},$$

$$\frac{x}{a} = z^2 = \frac{p}{2a} \pm a \sqrt{\left(\frac{p}{2a} - \frac{a^2}{4}\right)},$$

in which the + sign has to be taken since the smaller root will be found to give  $y$  imaginary.

The equation to find  $a$  is

$$a^6 + 4qa^2 = p^2.$$

To solve this put  $a^2 = 4\sqrt{\left(\frac{q}{3}\right)}\xi$ . Then

$$4\xi^3 + 3\xi = \frac{3\sqrt{3}}{16} \frac{p^2}{q^{3/2}} = A \text{ say.}$$

If then  $\sinh u = A$ ,

$$\xi = \sinh \frac{1}{3}u,$$

$$a^2 = \sqrt{(16q/3)} \sinh \frac{1}{3}u,$$

and the solution is obtained at once by the use of the Smithsonian tables\*.

In order to get some idea of the general shape of the boundaries the case with the node at  $k = \sin 65^\circ$  is taken. The curve as calculated is the loop portion of curve VI in fig. 2. Here  $k = \sin 65^\circ$ ;  $k' = \cos 65^\circ$ ;  $X = .2173$ ;  $\log F = .36338$ ;  $\log E = .06589$ , whence  $aU = 1.8005\mu/\pi$ ;  $b/a = .00464$ ;  $\pi\chi_1/\mu = -.00985$ . Also the equatorial axial radius expressed in terms of the standard sphere of same volume is  $a = 21.56c$ ,

$$U = .133 U.$$

The equation in  $z$  is

$$z^4 - 1.1108X \cdot z - .01094 = 0,$$

with

$$\log p = .04564 + \log X,$$

$$\log A = 2.54388 + 2 \log X,$$

$$\log 4\sqrt{(q/3)} = 1.38315.$$

It will be sufficient to illustrate the method of calculation by a single example—say where the boundary cuts the

\* If these are not at hand, the following will enable ordinary tables to be used. The eqn.  $\sinh u = A$  regarded as an equation in  $e^u$  gives

$$e^u = \sqrt{(A^2 + 1)} + A$$

$$e^{-u} = \sqrt{(A^2 + 1)} - A.$$

$$\text{If then } \cot a = A, \quad e^u = \cot \frac{a}{2}, \quad e^{-u} = \tan \frac{a}{2},$$

$$\sinh \frac{1}{3}u = \frac{1}{2} \left\{ \left( \cot \frac{a}{2} \right)^{\frac{1}{3}} - \left( \tan \frac{a}{2} \right)^{\frac{1}{3}} \right\}.$$

When  $A$  is so large as to require values outside the Smithsonian tables, the trigonometrical tables are also inapplicable since  $a$  is only a few minutes of arc. In this case the approximation

$$\sinh \frac{1}{3}u = \frac{1}{2} \left\{ (2A)^{\frac{1}{3}} - (2A)^{-\frac{1}{3}} \right\}$$

is sufficient.

circle  $k = \sin 75^\circ$ . Here

$$\begin{aligned}
 \log X &= \bar{1} \cdot 61752, \\
 \log p &= \bar{1} \cdot 66316, \\
 \log A &= \bar{1} \cdot 77840 \\
 &= \log \sinh \left( 4 \cdot 78 + \frac{353}{454} \right) = \log \sinh 4 \cdot 7881, \\
 \log \sinh 1 \cdot 5960 &= \quad \cdot 37388 \\
 \text{add} \quad \bar{1} \cdot 38315 \\
 \log a^2 &= \bar{1} \cdot 75703 = \log \cdot 57152 \\
 \log p/a &= \bar{1} \cdot 66316 \\
 &\quad - \bar{1} \cdot 87851 \\
 &\quad \bar{1} \cdot 78465 = \log \cdot 60905 \\
 \frac{p}{2a} - \frac{a^2}{4} &= \quad \cdot 30452 \\
 &\quad - \cdot 14288 \\
 &\quad \cdot 16164 \\
 \log a \sqrt{\left( \frac{p}{2a} - \frac{a^2}{4} \right)} &= \bar{1} \cdot 87851 \\
 &\quad \bar{1} \cdot 60427 \\
 &\quad \bar{1} \cdot 48278 = \log \cdot 30394 \\
 \frac{x}{a} &= \cdot 30452 + \cdot 30394 = \cdot 60846.
 \end{aligned}$$

The point on the boundary is then found either graphically by the point on the circle  $k = \sin 75^\circ$  whose distance from the axis is  $\cdot 608$ , or it may be calculated direct as  $y = \cdot 145a$ .

The area of the loop measured graphically from the curve is  $\cdot 1099a^2$  and the distance of its centre of gravity from the axis  $= \cdot 8112a$ . The volume of the liquid carried forward is therefore  $\cdot 5600a^3 = 1181m$ .

*The energies.*—The respective energies are now determined by substituting the values of  $\chi_2$ ,  $\chi_1$ , and the graphically measured volumes in the expressions already determined. The value of  $\chi_2$  is obtained by inserting the co-ordinates of the point where the boundary of the core cuts the equatorial plane, viz.

$$x = a - b, \quad k' = b/(2a - b),$$

where

$$\frac{\pi}{\mu} \chi_2 = \sqrt{(a^2 - ab)} \left\{ \frac{1 + k'^2}{2k} F - \frac{1}{k} E \right\} - \frac{(a - b)^2}{8a} \left( L - \frac{1}{4} \right),$$

$$F = \log \frac{8a}{b} + \frac{1}{4} k'^2 (L - 1) \dots$$

$$E = 1 + \frac{1}{2} k'^2 (L - \frac{1}{2}) \dots$$



Hence to the lowest order

$$\chi_2 = \frac{\mu a}{8\pi} \left( 3L - \frac{31}{4} \right),$$

also  $(\text{rad. gyr.})^2 = a^2 + \frac{3}{4}b^2 = a^2$  to lowest order.

The numerical values for the four cases considered are given in the following Table.

	II.	III.	V.	VI.
$b/a$ .....	·1	·05	·0116	·0046
$a/c$ .....	2·768	4·395	11·64	21·56
$b/c$ .....	·277	·220	·135	·099
$U/U$ .....	·584	·437	·215	·133
$\pi\chi_1/\mu a$ .....	0	0	0	—·00985
$\pi\chi_2/\mu a$ .....	·6745	·9356	1·4823	1·8291
$\text{vol.}/m$ .....	9·72	30·09	452	1181
$E_3/E_1$ .....	6·139	11·72	39·92	96·21
$E_2/E_1$ .....	12·06	25·86	107·3	231·6

Table for X.  $k = \sin \theta$ .

$\theta$ .	X.	$X^2/3$ .	$\log X$ .
10 .....	·000527	·00652	4·7218
15 .....	·001793	·014759	3·25358
20 .....	·004309	·026479	3·63437
25 .....	·008565	·041861	3·93272
30 .....	·015139	·061197	2·18009
35 .....	·024711	·084838	2·39289
40 .....	·0382	·1134	2·58206
45 .....	·0564	·1471	2·75128
50 .....	·0811	·1873	2·90902
55 .....	·1140	·2351	1·05690
60 .....	·1579	·2921	1·19838
65 .....	·2173	·3614	1·33706
70 .....	·2984	·4465	1·47480
75 .....	·4145	·5559	1·61752
77, 30 .....	·4931	·6241	1·69293
80 .....	·5931	·7059	1·77312
82, 30 .....	·7270	·8085	1·86153
85 .....	·9214	·9469	1·96444
86 .....	1·0302	1·0200	·01288
86, 30 .....	1·0957	1·0628	·03969
87 .....	1·1717	1·1114	·06881
87, 30 .....	1·2619	1·1677	·10102
88 .....	1·3726	1·2351	·13754
89 .....	1·7290	1·4405	·23779
89, 30 .....	2·0641	1·6211	·31473

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