ALGEBRAIC AND TRANSCENDENTAL NUMBERS.

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1. In a former paper [1]) it was shown that the trigonometric functions and the hyperbolic functions represent transcendental numbers whenever the argument is an algebraic number other than zero. It was likewise shown that the arguments of the trigonometric functions and of the hyperbolic functions are transcendental numbers whenever the functions are algebraic numbers. Thus, in the equation $y = \sin x$, $y = \tanh x$, etc., x and y cannot both be algebraic numbers excepting when $x = 0$.

In the above theorems either the functions or the argument is an algebraic number. Should a transcendental value be assigned to either the function or to the argument, no information would be given in regard to the nature of the number represented by the other.

The question naturally arises, is it not possible to find a class of numbers, say T, so that the nature of the function becomes known when the argument assumes a value of the class T, or vice versa. The inverse trigonometric functions, used as argument, represents such a class of transcendental numbers. In the present paper other classes of transcendental numbers are determined which possess the desired property.

2. ***Notation.*** — Algebraic numbers are represented by a, a_1, a_2; transcendental numbers are represented by t, t_1, t_2; rational numbers are represented by r; positive integers by n; and the square root of (-1) by i.

3. ***Preliminary theorem.*** — *If e^{ix} represents an algebraic number, each trigonometric function of x represents an algebraic number; if e^{ix} represents a transcendental number, the trigonometric functions of x are also transcendental.*

Demonstration. — We have $e^{ix} = \cos x + i \sin x$. Let $e^{ix} = y$. Then the equation $\cos x + i \sin x = y$ gives $\cos x = \dfrac{y^2 + 1}{2y}$.

If y is an algebraic number, $\cos x$ is an algebraic number, and it follows from

[1]) G. N. Bauer and H. L. Slobin, *Some Transcendental Curves and Numbers* [Rendiconti del Circolo Matematico di Palermo, t. XXXVI (2° semestre 1913), pp. 327-332].

the relations between the trigonometric functions that all the trigonometric functions of x are algebraic numbers.

If y is a transcendental number, $\cos x$ is also a transcendental number, for the assumption that it is algebraic, say a, leads to the equation $y^2 - 2ay + 1 = 0$, which is impossible if y is a transcendental number.

COROLLARY I. — If a trigonometric function of x represents an algebraic number, e^{ix} represents an algebraic number; if a transcendental number, e^{ix} also represents a transcendental number.

COROLLARY II. — If the argument of a trigonometric function is of the form $i \log a$, the function represents an algebraic number; if of the form $i \log t$, the function represents a transcendental number.

COROLLARY III. — If a trigonometric function represents an algebraic number, its argument is of the form $i \log a$; if it represents a transcendental number its argument is of the form $i \log t$.

4. THEOREM. — *If the argument of a trigonometric function is of the form $r\pi$, the function represents an algebraic number.*

Demonstration. — We may write

$$\cos r\pi + i \sin r\pi = e^{ir\pi} = (e^{i\pi})^r = (-1)^r.$$

But every algebraic number raised to a *rational* power represents analgebraic number. Hence $(-1)^r$ is an algebraic number. Hence the theorem follows from the theorem of Art. 3.

COROLLARY I. — The function $e^{ir\pi}$ represents an algebraic number.

COROLLARY II. — If the argument of a trigonometric function is of the form $i \log a + r\pi$, the function represents an algebraic number; if of the form $i \log t + r\pi$, the function represents a transcendental number.

COROLLARY III. — The converse of Corollary 2 is true.

Application. — To find the algebraic equation whose roots are

$$\frac{1}{3}\cos\frac{2\pi}{7}, \qquad \frac{1}{3}\cos\frac{4\pi}{7}, \qquad \frac{1}{3}\cos\frac{8\pi}{7}.$$

From

$$\cos\frac{2\pi}{7} + i \sin\frac{2\pi}{7} = (e^{i\pi})^{\frac{2}{7}} = (-1)^{\frac{2}{7}},$$

we find

$$\frac{1}{3}\cos\frac{2\pi}{7} = \frac{(-1)^{\frac{4}{7}} + 1}{6(-1)^{\frac{2}{7}}} \equiv a.$$

Likewise

$$\frac{1}{3}\cos\frac{4\pi}{7} = \frac{-(-1)^{\frac{1}{7}} + 1}{6(-1)^{\frac{4}{7}}} \equiv a_1$$

and

$$\frac{1}{3}\cos\frac{8\pi}{7} = \frac{-(-1)^{\frac{2}{7}} - 1}{6(-1)^{\frac{1}{7}}} \equiv a_2 .$$

The equation

$$(x - a)(x - a_1)(x - a_2) = 0$$

reduces to

$$216x^3 + 36x^2 - 12x - 1 = 0.$$

The symbol $(-1)^{\frac{1}{7}} \equiv k$ is defined by the equation $k^7 = -1$, with the added condition that no smaller power of k is equal to -1.

The equation $\cos r\pi + i\sin r\pi = (-1)^r$ gives us a direct method of finding the algebraic equation whose roots are given as trigonometric functions of the argument $r\pi$. The resulting equations generally have coefficients that are not rational numbers, but the theorem given by BACHMANN [2]) that if x is a root of an equation whose coefficients are algebraic numbers, x is an algebraic number, relieves us of the necessity of showing that our forms can always be reduced to equations with rational coefficients.

5. THEOREM. — *If the argument of a trigonometric function is of the form* $(a + r\pi)$, *the function represents a transcendental number, excepting when* $a = 0$.

Demonstration. — We have $e^{i(a+r\pi)} = e^{ia}e^{ir\pi}$. By LINDEMANN's theorem [3]) e^{ia} represents a transcendental number, excepting when $a = 0$; and by Cor. I, Art. 4, $e^{ir\pi}$ represents an algebraic number. Hence $e^{i(a+r\pi)}$ is a transcendental number. The theorem then follows from Art. 3.

COROLLARY. — If the argument of a trigonometric function is of the form $i\log a + a_1 + r\pi$, the function represents a transcendental number.

The converse of the corollary is not true. If the function represents certain transcendental numbers the argument will be of the form $i\log t + a + r\pi$.

6. THEOREM. — *If a trigonometric function represents a number of the form* $a_1\pi + a_2$, *the argument is of the form* $i\log t\pi$.

Demonstration. — Let us write $a_1\pi + a_2 = \cos i\log x$. By Cor. III of Art. 3, x must be a transcendental number since $a_1\pi + a_2$ is a transcendental number. Then let $x = t_1$. The transcendental numbers are either of the form $a\pi$ or $t\pi$.

Now t_1 cannot be equal to $a\pi$ as may be shown as follows:

From

$$\cos i\log t_1 + i\sin i\log t_1 = e^{i(i\log t_1)} = \frac{1}{t_1},$$

we find

$$\cos i\log t_1 = \frac{t_1^2 + 1}{2t_1}.$$

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[2]) P. BACHMANN, *Vorlesungen über die Natur der Irrationalzahlen* (Leipzig, Teubner, 1892), p. 20.

[3]) F. LINDEMANN, *Ueber die Zahl π* [Mathematische Annalen, Bd. XX (1882), pp. 213-225].

Then by hypothesis

$$\frac{t_1^2 + 1}{2t_1} = a_1\pi + a_2.$$

Now if t_1 can be of the form $a\pi$, let us substitute $t_1 = a\pi$ in the last equation. This leads to $(a^2 - 2aa_1)\pi^2 + 2aa_2\pi + 1 = 0$, an algebraic equation satisfied by π, which is impossible. Hence t_1, cannot be of the form $a\pi$, and must therefore be of the form $t\pi$.

Corollary. — If a trigonometric function represents a number of the form $a_1\pi$, the argument is of the form $i \log t\pi$.

This follows directly by letting $a_2 = 0$ in the above demonstration.

7. The relations existing between the trigonometric functions and the hyperbolic functions show that if the trigonometric functions of x represent algebraic or transcendental numbers, the hyperbolic functions of ix represent algebraic or transcendental numbers respectively.

8. Summary, *for argument given.*

Argument	*Trigonometric Function*
$i \log a$	algebraic
$i \log t$	transcendental
$r\pi$	algebraic
$i \log a + r\pi$	algebraic
$i \log t + r\pi$	transcendental
$a + r\pi, \quad a \neq 0$	transcendental
$i \log a_1 + a_2$	transcendental

Summary, *for functions given.*

Trigonometric Function	*Argument*
a	$i \log a_1$
t	$i \log t_1$
a	$i \log a_1 + r\pi$
t	$i \log t_1 + r\pi$
$a_1\pi + a_2$	$i \log t\pi$
$a\pi$	$i \log t\pi$

Similar results are true for the exponential function and the hyperbolic functions, as pointed out in Arts. 3 and 7.

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