NOTE ON THE EVALUATION OF A CERTAIN INTEGRAL CONTAINING BESSEL'S FUNCTIONS

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A PARTICULAR case of the following integral occurs in Prof. H. Lamb's paper "On the Theory of Waves propagated vertically in the Atmosphere" (p. 136 *supra*):—

$$\int_0^\infty J_n(ax)\, J_n(bx)\, J_m(cx)\, x^{1-m}\, dx.$$

Writing
$$W = \int_0^\infty J_n(ax)\, J_n(bx)\, J_m(cx)\, x^{1-m}\, dx$$
and using the relation
$$J_m(cx) = \frac{1}{2\pi\iota}\, c^m x^m \int_{\gamma-\infty\iota}^{\gamma+\infty\iota} e^{\frac{1}{2}t - c^2x^2/2t}\, t^{-m-1}\, dt,$$

where γ is a real positive quantity and $m > -1$, it follows that

$$W = \frac{1}{2\pi\iota}\, c^m \int_0^\infty \int_{\gamma-\infty\iota}^{\gamma+\infty\iota} e^{\frac{1}{2}t - c^2x^2/2t} J_n(ax)\, J_n(bx)\, x t^{-m-1}\, dx\, dt,$$

whence, $n > -1$,
$$W = \frac{1}{2\pi\iota}\, c^{m-2} \int_{\gamma-\infty\iota}^{\gamma+\infty\iota} e^{\frac{1}{2}t - (a^2+b^2)t/2c^2}\, I_n(abt/c^2)\, t^{-m}\, dt,$$

or
$$W = \frac{1}{2\pi\iota}\, a^{m-1} b^{m-1} c^{-m} \int_{\gamma-\infty\iota}^{\gamma+\infty\iota} e^{(c^2-a^2-b^2)t/2c^2}\, I_n(t)\, t^{-m}\, dt.$$

When $c^2 - a^2 - b^2 > 2ab$, writing $c^2 - a^2 - b^2 = 2ab\mu$,

$$\frac{1}{2\pi\iota} \int_{\gamma-\infty\iota}^{\gamma+\infty\iota} e^{\mu t} I_n(t)\, t^{-m}\, dt = \sqrt{\frac{2}{\pi}}\, \frac{\sin(m-n)\pi}{\pi}\, e^{(m-\frac{1}{2})\pi\iota} (\mu^2-1)^{\frac{1}{2}(2m-1)}\, Q_{n-\frac{1}{2}}^{\frac{1}{2}-m}(\mu)\, *;$$

therefore, when $c^2 > (a+b)^2$, $m > -1$, $n > -1$,

$$W = a^{m-1} b^{m-1} c^{-m} \sqrt{\frac{2}{\pi}}\, \frac{\sin(m-n)\pi}{\pi}\, e^{(m-\frac{1}{2})\pi\iota} (\mu^2-1)^{\frac{1}{2}(2m-1)}\, Q_{n-\frac{1}{2}}^{\frac{1}{2}-m}(\mu),$$

where
$$\mu = (c^2 - a^2 - b^2)/2ab.$$

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* Macdonald, *Proc. London Math. Soc.*, Vol. XXXV., p. 436.

It remains to evaluate W when $(a+b)^2 > c^2$, for this purpose using the relation

$$I_n(t) = \frac{1}{\sqrt{(2\pi)}}\, t^l \int_{-1}^{1} e^{t\nu}\, (1-\nu^2)^{\frac{1}{2}(2l-1)}\, P_{n-\frac{1}{2}}^{\frac{1}{2}-l}(\nu)\, d\nu,\text{*}$$

where $n-l$ is an integer greater than -1.

$$W = \frac{1}{2\pi\iota}\, a^{m-1}\, b^{m-1}\, c^{-m}$$

$$\times \frac{1}{\sqrt{(2\pi)}} \int_{\gamma-\infty\iota}^{\gamma+\infty\iota} \int_{-1}^{1} e^{(c^2-a^2-b^2+2ab\nu)t/2ab}\, (1-\nu^2)^{\frac{1}{2}(2l-1)}\, P_{n-\frac{1}{2}}^{\frac{1}{2}-l}(\nu)\, t^{l-m}\, d\nu\, dt,$$

where l is chosen less than m. Now

$$\frac{1}{2\pi\iota} \int_{\gamma-\infty\iota}^{\gamma+\infty\iota} e^{\kappa t}\, t^{l-m}\, dt \quad [m > l]$$

vanishes when κ is negative, and has the value $\kappa^{m-l-1}/\Pi(m-l-1)$ when κ is positive; therefore, when $(a-b)^2 > c^2$, $W = 0$ and, when

$$(a+b)^2 > c^2 > (a-b)^2,$$

$$W = a^{m-1}\, b^{m-1}\, c^{-m}\, \frac{1}{\sqrt{(2\pi)}}\, \frac{1}{\Pi(m-l-1)} \int_{\mu}^{1} (\nu-\mu)^{m-l-1}\, (1-\nu^2)^{\frac{1}{2}(2l-1)}\, P_{n-\frac{1}{2}}^{\frac{1}{2}-l}(\nu)\, d\nu,$$

where $\mu = (a^2+b^2-c^2)/2ab$, and, using the relation

$$P_{n-\frac{1}{2}}^{\frac{1}{2}-l}(\nu) = \frac{1}{\Pi(l-\frac{1}{2})} \left(\frac{1+\nu}{1-\nu}\right)^{\frac{1}{2}(1-2l)} F[-n+\tfrac{1}{2},\, n+\tfrac{1}{2},\, l+\tfrac{1}{2},\, \tfrac{1}{2}(1-\nu)],$$

$$W = a^{m-1}\, b^{m-1}\, c^{-m}\, \frac{1}{\sqrt{(2\pi)}}\, \frac{1}{\Pi(m-l-1)\, \Pi(l-\frac{1}{2})}$$

$$\times \int_{\mu}^{1} (\nu-\mu)^{m-l-1}\, (1-\nu)^{l-\frac{1}{2}}\, F[-n+\tfrac{1}{2},\, n+\tfrac{1}{2},\, l+\tfrac{1}{2},\, \tfrac{1}{2}(1-\nu)]\, d\nu,$$

that is,

$$W = a^{m-1}\, b^{m-1}\, c^{-m}\, \frac{1}{\sqrt{(2\pi)}}\, \frac{1}{\Pi(m-l-1)\, \Pi(-n-\frac{1}{2})\, \Pi(n-\frac{1}{2})}$$

$$\times \sum_{0}^{\infty} \frac{\Pi(r-n-\frac{1}{2})\, \Pi(r+n-\frac{1}{2})}{\Pi(l+r-\frac{1}{2})\, \Pi(r)}\, 2^{-r} \int_{\mu}^{1} (\nu-\mu)^{m-l-1}\, (1-\nu)^{l+r-\frac{1}{2}}\, d\nu.$$

Making the substitution $\nu-\mu = (1-\mu)\zeta$,

$$\int_{\mu}^{1} (\nu-\mu)^{m-l-1}\, (1-\nu)^{l+r-\frac{1}{2}}\, d\nu = (1-\mu)^{m+r-\frac{1}{2}} \int_{0}^{1} \zeta^{m-l-1}\, (1-\zeta)^{r+l-\frac{1}{2}}\, d\zeta,$$

that is, the value of this integral is

$$(1-\mu)^{m+r-\frac{1}{2}}\, \frac{\Pi(m-l-1)\, \Pi(l+r-\frac{1}{2})}{\Pi(m+r-\frac{1}{2})};$$

* *Proc. London Math. Soc.*, Vol. xxxv., p. 431.

therefore

$$W = a^{m-1} b^{m-1} c^{-m}$$
$$\times \frac{1}{\sqrt{(2\pi)}} \frac{1}{\Pi(n-\frac{1}{2})\,\Pi(-n-\frac{1}{2})} \sum_0^\infty \frac{\Pi(r+n-\frac{1}{2})\,\Pi(r-n-\frac{1}{2})}{\Pi(r+m-\frac{1}{2})\,\Pi(r)} 2^{-r}(1-\mu)^{m+r-\frac{1}{2}}$$

or $$W = a^{m-1} b^{m-1} c^{-m} \frac{1}{\sqrt{(2\pi)}} \frac{(1-\mu)^{m-\frac{1}{2}}}{\Pi(m-\frac{1}{2})} F[-n+\tfrac{1}{2}, n+\tfrac{1}{2}, m+\tfrac{1}{2}, \tfrac{1}{2}(1-\mu)],$$

that is, $$W = \frac{1}{\sqrt{(2\pi)}} a^{m-1} b^{m-1} c^{-m} (1-\mu^2)^{\frac{1}{4}(2m-1)} P^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\mu).$$

Collecting the results,

$$W = 0 \quad [(a-b)^2 > c^2],$$

$$W = \frac{1}{\sqrt{(2\pi)}} a^{m-1} b^{m-1} c^{-m} (1-\mu^2)^{\frac{1}{4}(2m-1)} P^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\mu),$$

where $$\mu = (a^2+b^2-c^2)/2ab \quad [(a+b)^2 > c^2 > (a-b)^2],$$

$$W = a^{m-1} b^{m-1} c^{-m} \sqrt{\frac{2}{\pi}} \frac{\sin(m-n)\pi}{\pi} e^{(m-\frac{1}{2})\pi\iota} (\mu^2-1)^{\frac{1}{4}(2m-1)} Q^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\mu),$$

where $$\mu = (c^2-a^2-b^2)/2ab \quad [c^2 > (a+b)^2],$$

and $$n > -1, \qquad m > -\tfrac{1}{2}.$$

When $n-m$ is an integer greater than -1 these results become

$$W = 0 \quad [(a-b)^2 > c^2],$$

$$W = \frac{1}{\sqrt{(2\pi)}} a^{m-1} b^{m-1} c^{-m} (1-\mu^2)^{\frac{1}{4}(2m-1)} P^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\mu) \quad [(a+b)^2 > c^2 > (a-b)^2],$$

$$W = 0 \quad [c^2 > (a+b)^2],$$

where $\mu = (a^2+b^2-c^2)/2ab$, and in this case $P^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\mu)$ can be expressed in finite terms. The result in the particular case when $n = m$ is, remembering that

$$P^{\frac{1}{2}-n}_{n-\frac{1}{2}}(\mu) = \frac{2^{\frac{1}{2}-n}}{\Pi(n-\frac{1}{2})} (1-\mu^2)^{\frac{1}{4}(2n-1)},$$

$$W = 0 \quad [(a-b)^2 > c^2],$$

$$W = \frac{a^{-m} b^{-m} c^{-m}}{2^{3m-1}\sqrt{\pi}\,\Pi(m-\frac{1}{2})} \{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)\}^{m-\frac{1}{2}}$$
$$[(a+b)^2 > c^2 > (a-b)^2],$$

$$W = 0 \quad [c^2 > (a+b)^2];$$

this particular result has been given by Sonine.*

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* *Math. Ann.*, Bd. XVI.

When $m = \frac{1}{2}$, remembering that

$$J_{\frac{1}{2}}(cx) = \left(\frac{2}{\pi cx}\right)^{\frac{1}{2}} \sin cx$$

and writing
$$V = \int_0^\infty J_n(ax)\, J_n(bx) \sin cx\, dx,$$

the above results become

$$V = 0 \quad [(a-b)^2 > c^2],$$

$$V = \tfrac{1}{2} a^{-\frac{1}{2}} b^{-\frac{1}{2}} P_{n-\frac{1}{2}}(\mu) \qquad [(a+b)^2 > c^2 > (a-b)^2],$$

where
$$\mu = (a^2+b^2-c^2)/2ab,$$

$$V = \frac{\cos n\pi}{\pi}\, a^{-\frac{1}{2}} b^{-\frac{1}{2}} Q_{n-\frac{1}{2}}(\mu) \quad [c^2 > (a+b)^2],$$

where
$$\mu = (c^2-a^2-b^2)/2ab.$$

When n is half a positive odd integer these results become

$$V = 0 \quad [(a+b)^2 > c^2],$$

$$V = \tfrac{1}{2} a^{-\frac{1}{2}} b^{-\frac{1}{2}} P_{n-\frac{1}{2}}(\mu) \quad [(a+b)^2 > c^2 > (a-b)^2],$$

$$V = 0 \quad [c^2 > (a+b)^2].$$

The various known series for the spherical harmonics enable the values of W or V to be calculated to any degree of accuracy; the values in the neighbourhood of the discontinuities will be given below.

When $c^2-(a-b)^2$ is small and tends to zero, it follows from the series for $P_{n-\frac{1}{2}}^{\frac{1}{2}-m}(\mu)$ in powers of $(1-\mu)$ that W tends to

$$\frac{1}{\sqrt{(2\pi)}}\, a^{m-1} b^{m-1} c^{-m} \frac{(1-\mu)^{m-\frac{1}{2}}}{\Pi(m-\frac{1}{2})},$$

that is, to zero when $m > \frac{1}{2}$; also, that V tends to the value $\frac{1}{2}a^{-\frac{1}{2}}b^{-\frac{1}{2}}$ when $c^2-(a-b)^2$ tends to zero. When $(a+b)^2-c^2$ is small μ is nearly equal to -1, and it is convenient to replace $P_{n-\frac{1}{2}}^{\frac{1}{2}-m}(\mu)$ by harmonics of $-\mu$. If $m-\frac{1}{2}$ is not an integer,

$$P_{n-\frac{1}{2}}^{\frac{1}{2}-m}(\mu) = \frac{\cos n\pi}{\cos m\pi} P_{n-\frac{1}{2}}^{\frac{1}{2}-m}(-\mu) + \frac{\sin(n-m)\pi}{\cos m\pi} \frac{\Pi(n-m)}{\Pi(n+m-1)} P_{n-\frac{1}{2}}^{m-\frac{1}{2}}(-\mu),$$

and therefore, when $n-m$ is not an integer, W tends to the value

$$\frac{1}{\sqrt{\pi}}\, 2^{m-1} a^{m-1} b^{m-1} c^{-m} \frac{\sin(n-m)\pi}{\cos m\pi} \frac{\Pi(n-m)}{\Pi(n+m-1)\,\Pi(\frac{1}{2}-m)},$$

that is, to the value

$$-\frac{1}{\pi\sqrt{\pi}}\, 2^{m-1} a^{m-1} b^{m-1} c^{-m} \sin(n-m)\pi\, \frac{\Pi(n-m)\,\Pi(m-\frac{3}{2})}{\Pi(n+m-1)}.$$

If $m-\frac{1}{2}$ is an integer,

$$P^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\mu) = \cos(n-m)\pi\, P^{\frac{1}{2}-m}_{n-\frac{1}{2}}(-\mu) - \frac{2}{\pi}\sin(n-m)\pi\, Q^{\frac{1}{2}-m}_{n-\frac{1}{2}}(-\mu),$$

and, making use of the appropriate expression for $Q^{\frac{1}{2}-m}_{n-\frac{1}{2}}(-\mu)$,* it follows that, when $n-m$ is not an integer, W tends to the value

$$-\frac{1}{\pi\sqrt{\pi}}\, 2^{m-1} a^{m-1} b^{m-1} c^{-m} \sin(n-m)\pi\, \frac{\Pi(n-m)\,\Pi(m-\frac{3}{2})}{\Pi(n+m-1)},$$

when $m > \frac{1}{2}$.

If $m = \frac{1}{2}$, and $n-\frac{1}{2}$ is not an integer, V tends to the value

$$-\frac{1}{2\pi}\cos n\pi\, a^{-\frac{1}{2}} b^{-\frac{1}{2}} \log\{(a+b)^2-c^2\},$$

when $(a+b)^2-c^2$ tends to zero. If $n-m$ is an integer and $m > \frac{1}{2}$, W tends to zero, when $(a+b)^2-c^2$ tends to zero. If $m = \frac{1}{2}$ and $n-\frac{1}{2}$ is an integer, V tends to the value $\frac{1}{2}\sin n\pi\, a^{-\frac{1}{2}} b^{-\frac{1}{2}}$, when $(a+b)^2-c^2$ tends to zero.

When $c^2 > (a+b)^2$ the expression for W can be replaced by

$$W = \frac{1}{\pi} a^{m-1} b^{m-1} c^{-m} 2^{1-m} \sin(m-n)\pi \times \frac{1}{\Pi(m-1)} \int_\eta^\infty e^{-n\zeta}(2\cosh\zeta - 2\cosh\eta)^{m-1}\, d\zeta,$$

where $$2ab\cosh\eta = c^2-a^2-b^2,$$

and m is positive.

When η tends to zero, this expression, writing $e^{-\zeta} = x$, tends to

$$\frac{1}{\pi} a^{m-1} b^{m-1} c^{-m} 2^{1-m} \sin(n-m)\pi\, \frac{1}{\Pi(m-1)} \int_0^1 x^{n-m}(1-x)^{2m-2}\, dx,$$

provided $m > \frac{1}{2}$. Hence, if $m > \frac{1}{2}$, W tends to the value

$$\frac{1}{\pi\sqrt{\pi}}\, 2^{m-1} a^{m-1} b^{m-1} c^{-m} \sin(m-n)\pi\, \frac{\Pi(n-m)\,\Pi(m-\frac{3}{2})}{\Pi(n+m-1)},$$

when $c^2-(a+b)^2$ tends to zero. When $m = \frac{1}{2}$,

$$V = \frac{1}{\pi}\cos n\pi\, a^{-\frac{1}{2}} b^{-\frac{1}{2}} \int_\eta^\infty e^{-n\zeta}(2\cosh\zeta - 2\cosh\eta)^{-\frac{1}{2}}\, d\zeta,$$

and the limit to which

$$\int_\eta^\infty e^{-n\zeta}(2\cosh\zeta - 2\cosh\eta)^{-\frac{1}{2}}\, d\zeta$$

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* Macdonald, *Proc London Math. Soc.*, Vol. XXXI., p. 276.

tends when η tends to zero is $-\log\eta$; therefore V tends to

$$-\frac{1}{2\pi}a^{-\frac{1}{2}}b^{-\frac{1}{2}}\cos n\pi\log\{c^2-(a+b)^2\},$$

when $c^2-(a+b)^2$ tends to zero.

[*Added January* 19*th*, 1909.]

The following integral is closely related to the integral investigated above, but differs from it inasmuch as it is continuous for all real values of a, b, c. Writing

$$U=\int_0^\infty J_n(ax)\,J_n(bx)\,K_m(cx)\,x^{m+1}\,dx,$$

and making use of the relation

$$K_m(cx)=\tfrac{1}{2}\int_0^\infty e^{-\frac{1}{2}cx(s+s^{-1})}\,s^{m-1}\,ds,$$

which is equivalent to

$$K_m(cx)=\tfrac{1}{2}(cx)^{-m}\int_0^\infty e^{-\frac{1}{2}s-\frac{1}{2}c^2x^2s^{-1}}\,s^{m-1}\,ds,$$

it follows that

$$U=\tfrac{1}{2}c^{-m}\int_0^\infty\int_0^\infty e^{-\frac{1}{2}s-\frac{1}{2}c^2x^2s^{-1}}\,J_n(ax)\,J_n(bx)\,s^{m-1}\,x\,dx\,ds;$$

hence, changing the order of integration and using the relation

$$\int_0^\infty e^{-\frac{1}{2}c^2x^2s^{-1}}\,J_n(ax)\,J_n(bx)\,x\,dx=\frac{s}{c^2}\,e^{-(a^2+b^2)s/2c^2}\,I_n\left(\frac{abs}{c^2}\right),$$

$$U=\tfrac{1}{2}c^{-m-2}\int_0^\infty e^{-\frac{1}{2}s-\frac{1}{2}(a^2+b^2)sc^{-2}}\,I_n\left(\frac{abs}{c^2}\right)s^m\,ds,$$

when $n>-\frac{1}{2}$.

Writing $\quad a^2+b^2+c^2=2ab\mu,\quad abs=c^2t,$

this becomes $\quad U=\tfrac{1}{2}c^m(ab)^{-m-1}\displaystyle\int_0^\infty e^{-\mu t}\,I_n(t)\,t^m\,dt,$

that is, $\quad U=\tfrac{1}{2}c^m(ab)^{-m-1}\displaystyle\sum_0^\infty\frac{1}{2^{n+2k}\,\Pi(n+k)\,\Pi(k)}\int_0^\infty e^{-\mu t}\,t^{n+m+2k}\,dt;$

hence $\quad U=\tfrac{1}{2}c^m(ab)^{-m-1}\displaystyle\sum_0^\infty\frac{\Pi(n+m+2k)}{2^{n+2k}\,\Pi(n+k)\,\Pi(k)}\,\frac{1}{\mu^{n+m+2k+1}},$

provided that $n+m>-1$, that is,

$$U=\tfrac{1}{2}c^m(ab)^{-m-1}\sum_0^\infty\frac{2^{n+m+2k}\,\Pi\left(\dfrac{n+m}{2}+k\right)\Pi\left(\dfrac{n+m-1}{2}+k\right)}{2^{n+2k}\,\Pi(n+k)\,\Pi(k)\,\Pi(-\frac{1}{2})\,\mu^{n+m+2k+1}};$$

and therefore

$$U = 2^{m-1}\pi^{-\frac{1}{2}}c^m(ab)^{-m-1}\frac{\Pi\left(\frac{n+m}{2}\right)\Pi\left(\frac{n+m-1}{2}\right)}{\Pi(n)}\,\frac{1}{\mu^{n+m+1}}$$

$$\times F\left(\frac{n+m+2}{2},\ \frac{n+m+1}{2},\ n+1,\ \frac{1}{\mu^2}\right).$$

Now $$Q_{n-\frac{1}{2}}^{m+\frac{1}{2}}(\mu) = e^{(m+\frac{1}{2})\pi\iota}\frac{\Pi(n+m)\,\Pi(-\frac{1}{2})}{2^{n+\frac{1}{2}}\,\Pi(n)}\,\frac{(\mu^2-1)^{\frac{1}{4}(2m+1)}}{\mu^{n+m+1}}$$

$$\times F\left(\frac{n+m+2}{2},\ \frac{n+m+1}{2},\ n+1,\ \frac{1}{\mu^2}\right);$$

therefore

$$U = 2^{n+m-\frac{1}{2}}\pi^{-\frac{1}{2}}c^m(ab)^{-m-1}e^{-(m+\frac{1}{2})\pi\iota}\frac{\Pi\left(\frac{n+m}{2}\right)\Pi\left(\frac{n+m-1}{2}\right)}{\Pi(n+m)\,\Pi(-\frac{1}{2})}$$

$$\times(\mu^2-1)^{-\frac{1}{4}(2m+1)}\,Q_{n-\frac{1}{2}}^{m+\frac{1}{2}}(\mu);$$

whence $$U = 2^{-\frac{1}{2}}\pi^{-\frac{1}{2}}e^{-(m+\frac{1}{2})\pi\iota}c^m(ab)^{-m-1}(\mu^2-1)^{-\frac{1}{4}(2m+1)}\,Q_{n-\frac{1}{2}}^{m+\frac{1}{2}}(\mu);$$

and therefore

$$\int_0^\infty J_n(ax)\,J_n(bx)\,K_m(cx)\,x^{m+1}\,dx$$

$$= 2^{-\frac{1}{2}}\pi^{-\frac{1}{2}}e^{-(m+\frac{1}{2})\pi\iota}c^m(ab)^{-m-1}(\mu^2-1)^{-\frac{1}{4}(2m+1)}\,Q_{n-\frac{1}{2}}^{m+\frac{1}{2}}(\mu),$$

where $$2ab\mu = a^2+b^2+c^2 \quad (n > -\tfrac{1}{2},\ n+m > -1).$$

Remembering that $K_{-m} = K_m$, it follows that

$$\int_0^\infty J_n(ax)\,J_n(bx)\,K_m(cx)\,x^{1-m}\,dx$$

$$= 2^{-\frac{1}{2}}\pi^{-\frac{1}{2}}e^{(m-\frac{1}{2})\pi\iota}c^{-m}(ab)^{m-1}(\mu^2-1)^{\frac{1}{4}(2m-1)}\,Q_{n-\frac{1}{2}}^{\frac{1}{2}-m}(\mu),$$

where $$n > -\tfrac{1}{2}, \quad m < n+1.$$

Substituting $m = -\frac{1}{2}$ in the first result, and writing for $K_{-\frac{1}{2}}(cx)x^{\frac{1}{2}}$, its value $2^{-\frac{1}{2}}\pi^{\frac{1}{2}}c^{-\frac{1}{2}}e^{-cx}$, it becomes

$$\int_0^\infty e^{-cx}J_n(ax)\,J_n(bx)\,dx = \frac{1}{\pi}a^{-\frac{1}{2}}b^{-\frac{1}{2}}Q_{n-\frac{1}{2}}(\mu),$$

a result previously obtained by the writer.*

* *Proc. London Math. Soc.*, Vol. XXVI., pp. 160, 161, 165.

Again, substituting $m = 0$, the first result becomes

$$\int_0^\infty J_n(ax)\,J_n(bx)\,K_0(cx)\,x\,dx = 2^{-\frac{1}{2}}\pi^{-\frac{1}{2}}e^{-\frac{1}{2}\pi\iota}(ab)^{-1}(\mu^2-1)^{-\frac{1}{4}}Q_{n-\frac{1}{2}}^{\frac{1}{2}}(\mu),$$

that is, $$\int_0^\infty J_n(ax)\,J_n(bx)\,K_0(cx)\,x\,dx = \frac{1}{2ab\sinh\psi}\,e^{-n\psi},$$

where $$\cosh\psi = \mu = (a^2+b^2+c^2)/2ab.$$

Substituting in the first result $m = n$, it becomes

$$\int_0^\infty J_n(ax)\,J_n(bx)\,K_n(cx)\,x^{n+1}\,dx$$
$$= 2^{-\frac{1}{2}}\pi^{-\frac{1}{2}}c^n(ab)^{-n-1}e^{-(n+\frac{1}{2})\pi\iota}(\mu^2-1)^{-\frac{1}{4}(2n+1)}Q_{n-\frac{1}{2}}^{n+\frac{1}{2}}(\mu),$$

that is,

$$\int_0^\infty J_n(ax)\,J_n(bx)\,K_n(cx)\,x^{n+1}\,dx$$
$$= 2^{-\frac{1}{2}}\pi^{-\frac{1}{2}}c^n(ab)^{-n-1}(\mu^2-1)^{-\frac{1}{4}(2n+1)}\frac{\Pi(2n)\,\Pi(-\frac{1}{2})}{2^{n+\frac{1}{2}}\Pi(n)}(\mu^2-1)^{-\frac{1}{4}(2n+1)};$$

and therefore

$$\int_0^\infty J_n(ax)\,J_n(bx)\,K_n(cx)\,x^{n+1}\,dx$$
$$= 2^{n-1}\pi^{-\frac{1}{2}}\Pi(n-\tfrac{1}{2})\,c^n(ab)^{-n-1}(\mu^2-1)^{-n-\frac{1}{2}} \qquad (n > -\tfrac{1}{2}).$$