

## The Approximate Solution of Various Boundary Problems by Surface Integration combined with Freehand Graphs

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VII. *The Approximate Solution of Various Boundary Problems by Surface Integration\* combined with Freehand Graphs.*

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§ 1. General.

§ 2.  $(\partial^4/\partial x^4 + 2\partial^4/\partial x^2\partial y^2 + \partial^4/\partial y^4)\gamma = 0$  in the body; and on the boundary  $\gamma$  and  $\frac{\partial\gamma}{\partial n}$  given.

§ 2.1. General.

§ 2.2. Example—Stress in a stretched notched plate.

§1. The method of solving boundary problems now to be described, consists in drawing, at a guess, a rough graph of the lines along which the integral is constant, and testing its accuracy by perimeter integration applied (in the approximate form of summation) to circles (or other simple figures) enclosed within the given boundary, in such a way as to find the integral at the centre of each circle, in terms of the integral and its space rates round the circumference; then embodying the corrections so found in a new graph and repeating the process until improvement becomes slow. In doing this one begins with a circle as large as will go into the region, or if the region is long and narrow, with several such circles. This gives a first approximation to the integral in the centre of the region. The graph is then re-drawn, and circles of half the radius of the first are used to test the parts extending from the centre to the boundary. Then, after re-drawing, still smaller circles are used for testing nooks and corners. Next the largest size of circles is used again, and so on.

The above process is for equations such as  $\nabla^2\phi=0$ ,  $\nabla^4\phi=0$ , but when we have on the right-hand side of the equations not zero but a known function of  $\phi$  and of the independents, then a certain volume integral has to be taken as well as the perimeter integrals. This can be calculated with sufficient accuracy by adding up a number of arithmetical values. When, as com-

\* The particular kind of surface integration referred to is that which gives the value, at a single point inside the surface, of the integral of an equation which holds throughout the volume inside the surface. When, by symmetry, it is only necessary to apply it to a line lying in the surface it will be called "perimeter integration," not contour integration as the word "contour" is reserved to mean a line along which a function of position is constant.

monly, these are the values of a function of two functions of position, then the work is made easier by drawing a graph of the contours of one of the functions on tracing paper and laying it over a graph of the contours of the other.

The drawing ensures that the function shall satisfy the right boundary conditions, shall have any necessary discontinuities and shall elsewhere be "smooth." The testing by perimeter and volume integration and consequent adjustment ensures that it shall satisfy the particular differential equation obtaining throughout the body. The accuracy of the result may be improved, in the case of certain differential equations, by converting it into a table of numbers and then making successive arithmetical approximations.\* Further, whenever it has been decided to attack a problem by this arithmetical method, it will generally be quickest first to improve the preliminary assumption by graphs and perimeter integration.

Errors in graphs of integrals of  $\nabla^2\phi=0$  are easily seen because the shape of the chequers into which the contours of  $\phi$  and the orthogonal curves divide the field is a known function of their position and direction†. When, however, we have to deal with rather more complicated equations such as  $\nabla^2\phi=\text{constant}$ ,  $\nabla^2\phi=k^2\phi$  or  $\nabla^4\phi=0$ , the shape of the chequers is no longer a known function of their position and direction, but, instead, certain relations have to be satisfied by the differences of the ratio of length to breadth of adjoining chequers, relations which are usually too subtle for the eye to perceive. Instead of attempting to perceive them one may use contour integration in the manner described. The same may also, of course, be applied to  $\nabla^2\phi=0$ .

An account of Green's functions connected with equation  $\nabla^{2m}\phi=0$ , where  $m$  is an integer, will be found in two Papers by T. Boggio‡. He shows, among other things, that

$$0 = \int (u \nabla^{2m} v - v \nabla^{2m} u) d(\text{volume}) \\ - \sum_1^m \int \left\{ \nabla^{2i-2} u \frac{d \nabla^{2m-2i} v}{dn} - \nabla^{2m-2i} v \frac{d \nabla^{2i-2} u}{dn} \right\} d(\text{surface}), \quad (1)$$

where  $dn$  is an element of the outwardly directed normal.

\* L. F. Richardson, "Phil. Trans." A (1910).

† This statement applies only when  $\phi$  is a function of two co-ordinates, these co-ordinates being of a certain common kind; see L. F. Richardson, "Phil. Mag.," February, 1908.

‡ Accademia Reale delle Scienze di Torino, 1900. Rendiconti del Circolo Matematico di Palermo, 1905.

§ 2.1. Let us consider more particularly the equation

$$\nabla_1^4 \gamma = \{\partial^4 / \partial x^4 + 2 \partial^4 / \partial x^2 \partial y^2 + \partial^4 / \partial y^4\} \gamma = 0, \dots \quad (2)$$

which is the one satisfied by the stress function in plain strain, \* by the stream function in a viscous liquid moving under certain conditions, and by the displacement in a plane plate bent by forces at its edge.

Putting  $m=2$  the formula (1) becomes

$$\begin{aligned} & \int (u \nabla^4 v - v \nabla^4 u) d(\text{volume}) \\ &= \int \left( u \frac{d \nabla^2 v}{dn} - \nabla^2 v \frac{du}{dn} + \nabla^2 u \frac{dv}{dn} - v \frac{d \nabla^2 u}{dn} \right) d(\text{surface}). \quad (3) \end{aligned}$$

Then, if  $\nabla^4 v = 0$  everywhere except at one point P, where it becomes infinite in such a way that the integral of it over any volume, including this point, is +1, then

$$\int u \nabla^4 v d(\text{volume}) = \text{the value of } u \text{ at P.}$$

If also  $v$  and  $\frac{\partial v}{\partial n}$  vanish on the boundary then

$$u \text{ at P} = \int \left( u \frac{d \nabla^2 v}{dn} - \frac{du}{dn} \nabla^2 v \right) d(\text{surface}) + \int v \nabla^4 u d(\text{volume}). \quad (4)$$

Here  $u$  is supposed to be the integral which we require and  $v$  a known function which is used to find  $u$ .

It is assumed that we are dealing with cases in which  $\nabla^4 u$  either vanishes or is a given function of  $u$  and of the co-ordinates, so that the volume integral can easily be estimated by adding up values at a number of points.

The special case which we chiefly require is that of a circle with P at the centre. Let the radius be  $a$ , then

$$v = \frac{1}{8\pi} \left[ r^2 \log \frac{r}{a} + \frac{1}{2} (a^2 - r^2) \right]^\dagger. \dots \quad (5)$$

Forming  $\nabla^2 v$  and  $\frac{\partial}{\partial r} \nabla^2 v$  at  $r=a$  it is found that, if  $\nabla^4 \gamma = 0$ , then

$$\begin{aligned} \gamma \text{ at centre of circle} &= (\text{mean value of } \gamma \text{ on circumference}) \\ &- (\text{half radius}) \times \left( \text{mean value of } \frac{\partial \gamma}{\partial r} \text{ on circumference} \right). \quad (5A) \end{aligned}$$

To determine these mean values on a graph one divides the circumference into a number of parts and adds up the values at their middle points arithmetically.

\* Love, "Theory of Elasticity," edition 1906, p. 201.

† Love, "Elasticity," 1906 edition, § 314.

Another special case which may be required is that where the boundary is an infinite straight line  $y=0$  and P is at  $x=0$ ,  $y=a$ . Then

$$v = \frac{1}{16\pi} [x^2 + (y+a)^2] [\log \{x^2 + (y+a)^2\} - \log \{x^2 + (y-a)^2\}] * \\ + \text{an arbitrary linear function :}$$

it follows that  $\gamma$  at  $x=0$ ,  $y=a$  is equal to

$$\int_{-\infty}^{+\infty} \left[ (\gamma)_{y=0} \frac{2a^3}{\pi(x^2+a^2)^2} - \left( \frac{\partial \gamma}{\partial n} \right)_{y=0} \frac{a^2}{\pi(x^2+a^2)} \right] dx.$$

The forms of  $v$  are known for many more general shapes of boundary and with P anywhere. Thus O. Venske (Göttingen Nachrichten, 1891) gives  $v$  for the circle, circular ring, space between intersecting straight lines, parallel straight lines and the space bounded by radii and concentric circles. But these will seldom be required for the graphic method.

§ 2.2. *Example.*—A thin elastic plate of the shape shown in Fig. 1 is stretched by forces applied uniformly over its ends. Required to make a graph of the stress function. The plate is 24 units long, 10 wide, and the semicircular notches have a radius of 2.1 and are placed centrally †. If  $x$  and  $y$  are co-ordinates in the plane of the plate then the stresses are  $X_x = \frac{\partial^2 \gamma}{\partial y^2}$ ,  $Y_y = \frac{\partial^2 \gamma}{\partial x^2}$ ,  $X_y = Y_x = -\frac{\partial^2 \gamma}{\partial x \partial y}$ , where  $\gamma$  is the stress-function satisfying  $(\partial^4/\partial x^4 + 2 \partial^4/\partial x^2 \partial y^2 + \partial^4/\partial y^4) \gamma = 0$  ‡. Effect of the thickness of the plate is here neglected. The  $y$  axis is taken parallel to the length of the plate. Over the ends  $Y_y = \text{constant}$ , therefore,  $\frac{\partial^2 \gamma}{\partial x^2} = \text{constant}$ . Now, arbitrary values may be given to  $\gamma$ ,  $\frac{\partial \gamma}{\partial x}$  and  $\frac{\partial \gamma}{\partial y}$  at any one point without affecting the stresses.

So let  $\gamma$ ,  $\frac{\partial \gamma}{\partial x}$ ,  $\frac{\partial \gamma}{\partial y}$  vanish at the centre of one end. Then, over this end  $\gamma = Ax^2$ , where A is a constant and  $x$  is measured from the centre of the end. Further,  $X_y$  vanishes over the end, and, therefore, by integrating  $X_y = -\frac{\partial^2 \gamma}{\partial x \partial y}$ , it follows that  $\frac{\partial \gamma}{\partial y}$  vanishes also. Now, it may be shown § that when the boundary stresses

\* Love, "Elasticity," 1906 edition, § 314.

† This problem is taken from a note by G. H. Gulliver, "Nature," April 14, 1910, Figs. 5 and 6.

‡ Michell, "Proc." Lond. Math. Soc., xxi. Love, "Theory of Elasticity," ed. 1906.

§ "Phil. Trans.," A Vol., 210, pp. 330—331.

and the external forces, such as gravity acting on the internal matter, vanish, then  $\frac{\partial \gamma}{\partial x}$  and  $\frac{\partial \gamma}{\partial y}$  are constant round the boundary.

These conditions are satisfied in the present case except over the ends of the plate. Consequently, if we imagine a length,  $\gamma$ , set up normal to the plane  $x, y$  so as to form a surface, then on the longer edges of the strip and round the curves of the neck this surface is the plane which touches the surface  $\gamma = Ax^2$  at the extremity of the end. The distribution of  $\gamma$  is obviously symmetrical about the lines through the centre of the plate parallel to the straight sides and ends. Take the origin at the centre. The following is a record of the actual process by which the results to be given at the end were obtained.

Short pieces of the contours of the surface  $\gamma = Ax^2$  were first marked on the ends of the boundary of the drawing, and over the rest of the boundary were marked the ends of the contours of the plane which has  $\gamma = A(5)^2$  at the straight parts of the longer edges and which has  $\frac{\partial \gamma}{\partial x} = 2A \cdot 5$ ;  $\frac{\partial \gamma}{\partial y} = 0$ . In the drawing  $A$  was taken to be  $2/5$ . Consequently the equal and opposite forces acting on the ends of the bar, being each equal to

$$\int_{x=-5}^{x=+5} \frac{\partial^2 \gamma}{\partial x^2} dx, \quad \text{are} \quad \left[ 4x/5 \right]_{-5}^{+5} = 8.$$

This completed the boundary conditions. Next, the ends of the contours were joined across by lines drawn at a guess in Fig. 1A. The value of  $\gamma$  at the centre of the plate was unknown, so that the dotted line  $\gamma = 1$  was particularly uncertain. The only places at which  $\gamma$  vanishes are the points marked 0. Now, the large circle was drawn and divided into equal segments by dots. The values of  $\gamma$  at the centres of these segments, starting from the left, were read off, thus—

0.9, 5.0, 9.0, 9.0, 4.0, 1.5, average 4.90.

In reading off values of  $\frac{\partial \gamma}{\partial r}$  it is a help to have a piece of tracing paper with a line ruled on it, which is placed as tangent to the circle. The values of  $\gamma$  at two dots a half unit on either

side of the tangent are read off and differenced. In other places it is more convenient to measure the distance along a radius between two contours, and to use for  $\frac{\partial \gamma}{\partial r}$  the reciprocal

of this distance multiplied by the difference in  $\gamma$  between the said contours. Again, at or very near the boundary,  $\frac{\partial\gamma}{\partial x}$  and  $\frac{\partial\gamma}{\partial y}$  are given, and it is then often most accurate to take the direction cosines of the radius from a book of tables and so calculate  $\frac{\partial\gamma}{\partial r}$ . By these methods the following values of  $\frac{\partial\gamma}{\partial r}$  were quickly found at the centres of the segments, reading from the left

0.15, 2.00, 3.8, 3.8, 2.35, zero, mean 2.01.

Therefore, by equation (5),

$$\gamma \text{ at centre} = 4.9 - \frac{5}{2} \times 2.01 = -0.12,$$

instead of about +0.6, as indicated directly by the figure. Accordingly a new drawing was made in the quarter of Fig. 1 marked B, and in this  $\gamma$  was made negative in the centre of the plate. It hardly seemed necessary to alter  $\gamma = 2, 4, 6, 8, 10$ . The mean values of  $\gamma$  and  $\frac{\partial\gamma}{\partial r}$  were thereby changed so as to make  $\gamma = -0.5$  at the centre of the large circle, and this was also the value given directly by the graph as near as one could tell.

Next a circle with centre at the origin and radius 2.9 was taken and divided into segments as before. At the centres of these segments

$$\begin{aligned} \gamma &= -0.5, +0.8, +1.7, \text{ mean } +0.67, \\ \frac{\partial\gamma}{\partial r} &= +1.9, +0.95, +0.15, \text{ mean } 1.00. \end{aligned}$$

Therefore, by equation (5),

$$\gamma \text{ at centre of the plate} = +0.67 - \frac{2.9}{2} \times 1.00 = -0.78.$$

Next a circle with radius 2.5 and centre at  $y=4, x=2.5$  was taken. The value of  $\gamma$  at the centre of this, as given directly by the figure is 1.6. On the circumference, in counterclockwise order from the right, were found

$$\gamma = 2.7, 7.0, 9.5, 9.4, 7.0, 3.6, 1.2, -0.1, -0.5, -0.7, -0.4, +0.5, \text{ mean } 3.27,$$

$$\begin{aligned} \frac{\partial\gamma}{\partial r} &= 0.9, 2.9, 3.85, 3.85, 2.5, 0.85, -0.15, -0.4, -0.1, -0.1, \\ &\quad -0.4 - 0.4, \text{ mean } 1.11; \end{aligned}$$

whence, by equation (5),  $\gamma$  at the centre  $= 3.27 - \frac{2.5}{2} \times 1.11 = 1.88$  instead of 1.6, as found directly. The graph was adjusted to

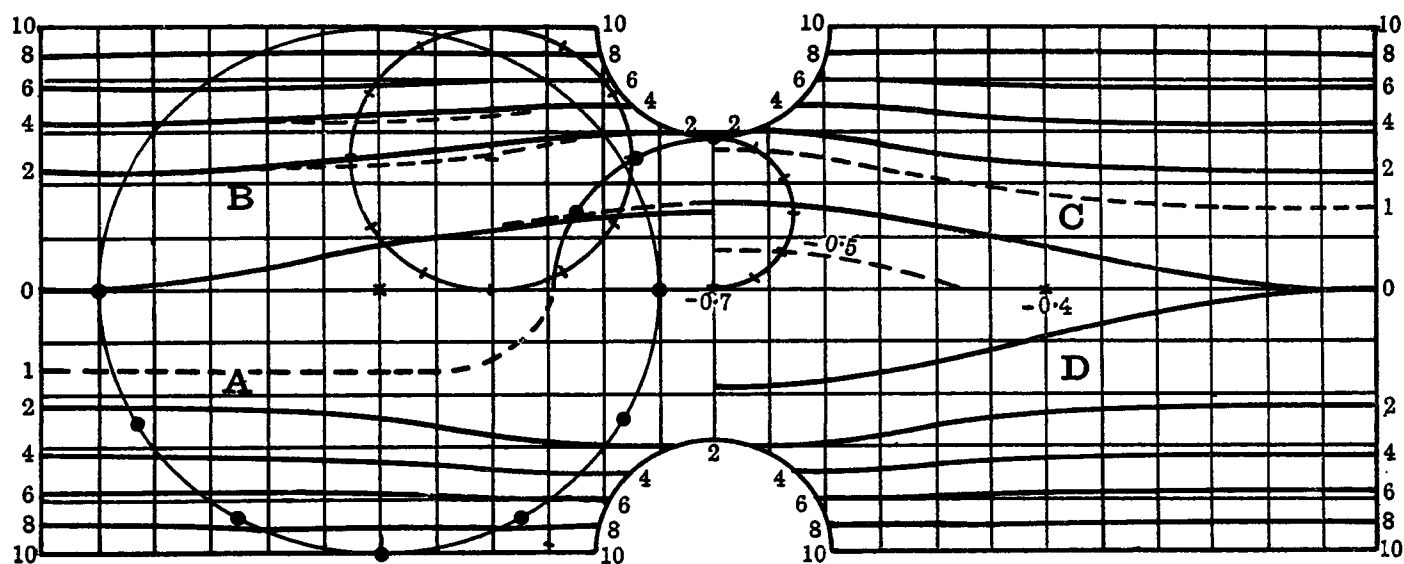


FIG. 1.

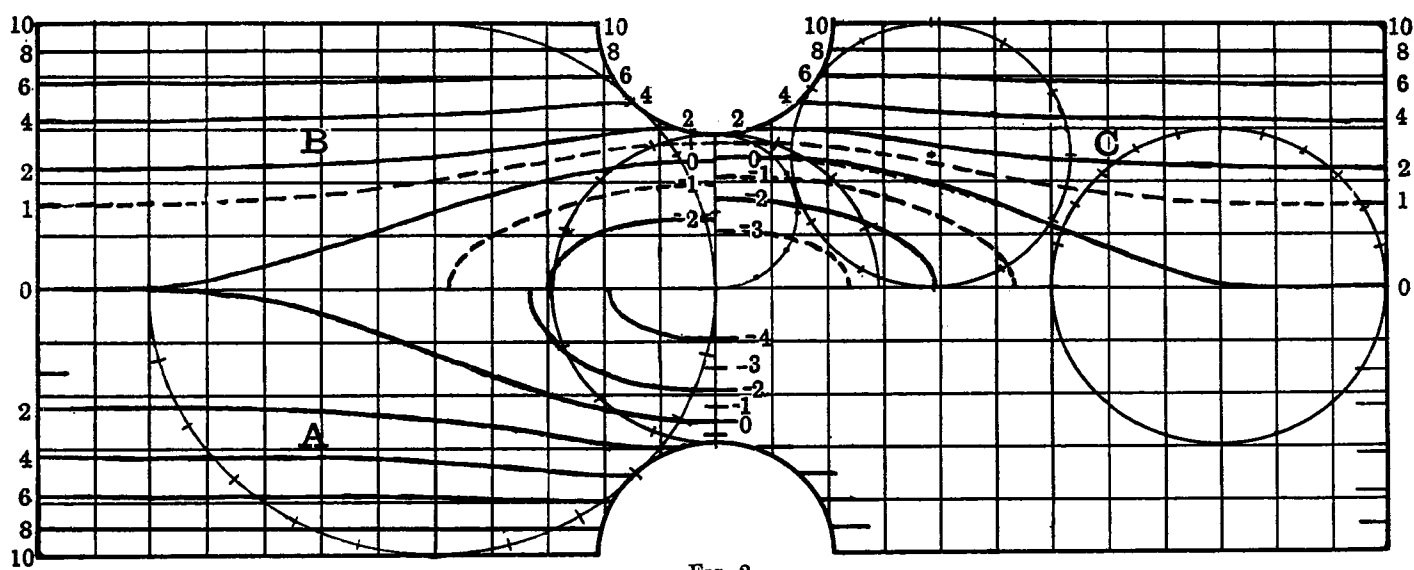


FIG. 2.

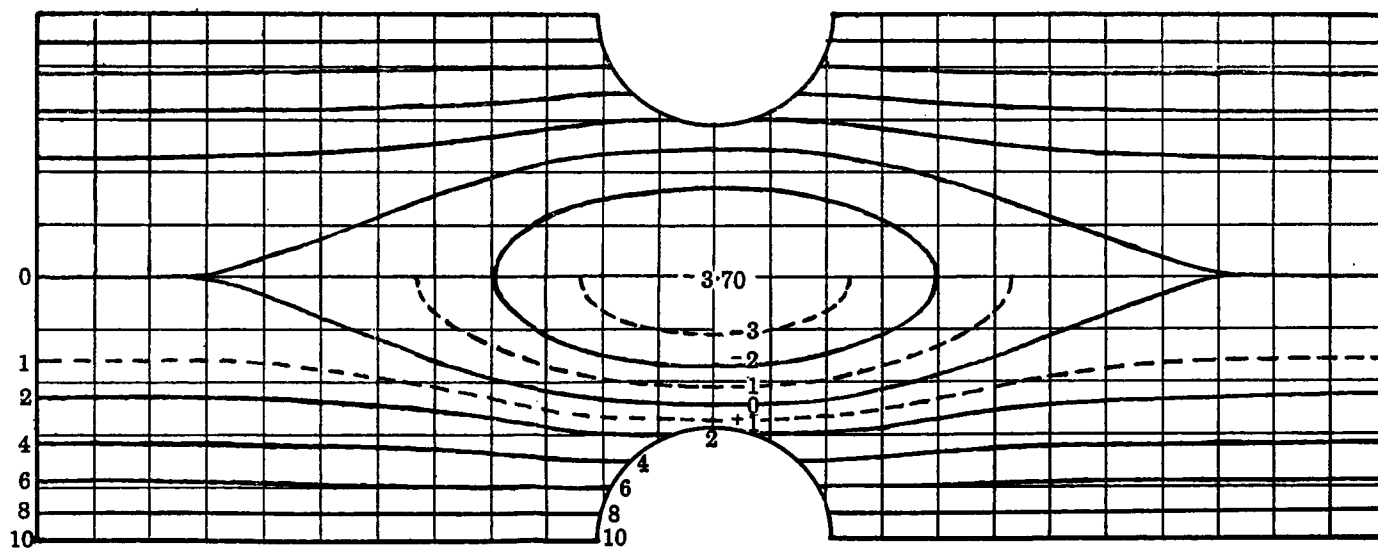


FIG. 3.



accord with this value by moving the contour  $\gamma=2$  into the position of the dotted line, so that  $\gamma$  was increased by 0.30. Then, to preserve the smoothness of the function  $\gamma$ , the contour  $\gamma=4$  had to be given a similar but smaller displacement. These

displacements reduced the mean value of  $\frac{\partial\gamma}{\partial r}$  on the circumference by only about 0.01. They increased the mean value of  $\gamma$  by about 0.02. The value at the centre, by perimeter

integration, was therefore changed by  $0.02 + \frac{r}{2}(0.01) = 0.03$ . So

it is seen that the slight displacement of the contour passing nearest to the centre of the circle, with accompanying changes to preserve smoothness, made much less change in the value of  $\gamma$  found by perimeter integration than in the value read off directly, the ratio being as 0.03 to 0.30. The motion of the contours was stable. This property, when it exists, allows improvements to be made rapidly. As a result of the last change we have now, at  $x=2.5$ ,  $y=4$ ,  $\gamma=1.90$  directly and  $\gamma=1.91$  by perimeter integration. Next, if the contour  $\gamma=0$  be displaced into the dotted position the effects on the circle which we have just considered will be slight, the change in  $\gamma$  compensating that in  $\frac{\partial\gamma}{\partial r}$ . This was done.

The value at the origin by perimeter integration became  $-0.72$ .

The graph now appeared to be fairly satisfactory. Some further tests were proposed to confirm it, and had they been applied they would doubtless have revealed what was now noticed—namely, that the stress  $Y_y$  across the neck of the plate is probably fairly uniform. If quite uniform, then  $\gamma$  is there of the second degree in  $x$ . But we have the values of  $\gamma$  and  $\frac{\partial\gamma}{\partial x}$  at

the boundary  $x=2.9$ , and so find at the centre of the plate  $\gamma=-4.75$  if  $Y_y$  is uniform, instead of  $-0.7$  given by perimeter integration. It was at once clear that either  $Y_y$  was not uniform or else the figure was seriously out. To decide which, the latest graph was copied into the quarter C of Fig. 1. An auxiliary diagram, Fig. 4, showing  $\gamma$  as a function of  $x$  along  $y=0$  was used to find the points where  $\gamma=1$  and  $-0.5$ , and the contours through these points were then marked with dots in Fig. 1c.

A circle was taken with centre at  $x=1.45$ ,  $y=0$  passing through the origin and touching the boundary. At its centre perimeter integration gave  $-0.32$ , whereas the value on the graph is there  $-0.1$ . Suppose that to correct this discrepancy

the zero line were shifted into the new position shown in Fig. 1D. This shift would cause, on the circumference of the circle which we have just considered, an increase in  $\frac{\partial\gamma}{\partial r}$  and a decrease in  $\gamma$ , and would, therefore, create a demand for a further displacement in the same direction. The motion of the contours would be unstable.

It was, therefore, necessary to make a large displacement, preferably one overshooting the position of stable equilibrium of the contours. A fresh start was accordingly made from the assumption of uniform stress  $Y_y$  across  $y=0$ . (See Fig. 2A.) Here the contours  $\gamma=2, 4, 6, 8$  were simply copied from Fig. 1c. The others were guessed. Perimeter integrations were then made around two circles in the same way as in the examples above and with the following results:—

Co-ordinates of centre.		Radius.	Value at centre by perimeter integration.	Value at centre directly from the graph.
$x$	$y$			
0	0	2.9	-4.30	-4.75
0	5	5.0	-1.34	-0.8

These two discrepancies can be simultaneously reduced by making  $\gamma$  less negative at the centre of the plate. The amount of change required was not apparent, but for trial a value half way between that in Figs. 1c and 2A was taken—namely,  $\frac{1}{2}(-4.7-0.7)=-2.7$ . The corresponding graph is Fig. 2B. Taking again the large circle with centre at  $x=0, y=5$  and  $r=5$ , it was found on its circumference that the changes from Fig. 2A increase the mean of  $\gamma$  by 0.28 and the mean of  $\frac{\partial\gamma}{\partial r}$  by -0.01. Therefore,  $\gamma$  at the centre should be increased from -1.34 to  $-1.34+0.28+2.5\times 0.01=-1.04$ . Actually it is -0.9. Next from the circle with centre at the origin and radius 2.9 it was found that  $\gamma$  should be -3.13 instead of -2.7 as assumed. Both these tests on Fig. 2B point to the true value at the origin being more negative than -2.7. Accordingly -3.7 was next tried. Fig. 2c is the corresponding graph. In making it the curve marked 1c in Fig. 4 was first drawn to look suitable and was then used to determine the ends of the contours in Fig. 2c along the  $y$  axis. The lines  $\gamma=4, 6, 8$  are the same as in the preceding graph;  $\gamma=2$  has been slightly straightened. Around the circumference of the large circle

with centre  $x=0, y=5$  and radius 5 the distribution in Fig. 2c is simply the average of those in Figs. 2A and 2B; so  $\gamma$  at its centre was made equal to the average of the values found by perimeter integration from Figs. 2A and 2B—namely, 1.19. Next the following tests were applied :—

Centre of circle.		Radius.	Angle between values used in summations.	$\gamma$ at centre.		
$x$	$y$			By perimeter integration.	Directly.	Diff.
0	0	2.9	22°.5	-3.71	-3.70	-0.01
1.45	0	1.45	30°	-2.46	-2.40	-0.06
2.5	3.85	2.5	30°	+1.35	+1.40	-0.05
0	9.0	3.0	{ 30° 15°	+0.09	0.00	+0.09
				+0.035		+0.035

The differences are all less than 0.1, which is 0.73 per cent. of the total range of  $\gamma$  in the figure.

Enough has now been done to illustrate the process. Fig. 4

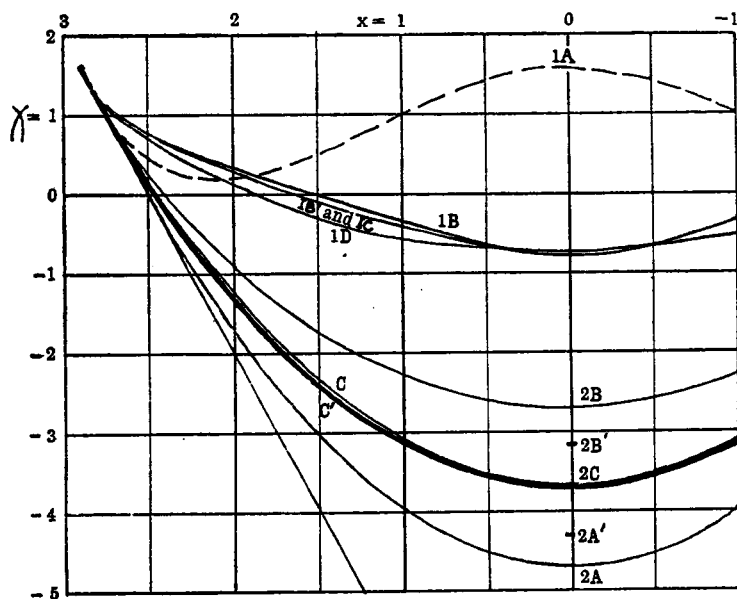


FIG. 4.

shows the successive stages at a glance. The final contours have been copied from Fig. 2c to Fig. 3, where they can be seen

more clearly. In copying it was found desirable, in order to improve the smoothness, to shift the line  $\gamma=0$  into the position shown dotted in Fig. 2c. The effect of this on the errors tabulated above will be less than 0.1 of  $\gamma$ .

From Fig. 3 the following relations between the stresses in the neck have been deduced. As they depend on second differences they are necessarily very inaccurate, and are to be regarded merely as suggestive. If a greater accuracy were required it would be best either to work with a larger diagram (Figs. 1, 2, 3 are reproductions of drawings 10 cm.  $\times$  24 cm.), or else to read from the graph a table of numbers and treat them by successive arithmetical approximations \*. To return to the stresses in the neck. In the following table  $\delta^2$  is the operator which takes the second difference of the tabulated numbers. Along  $y=0$  :—

at $x=$	-1.45	0	+1.45	+2.90
$\gamma=$	-2.46	-3.70	-2.46	+1.60
Here the value $\gamma = -2.46$ was found by perimeter integration.				
See table on opposite page.				
Therefore $\delta^2\gamma=$	...	+2.48	+2.82	...
hence $\frac{\delta^2\gamma}{\delta x^2}=$	...	+1.18	+1.34	...

But the mean value of the stress  $\frac{\partial^2\gamma}{\partial x^2} = Y_y$  is the total force 8. (see p. 79) divided by the width of the neck 5.9—that is, 1.355.

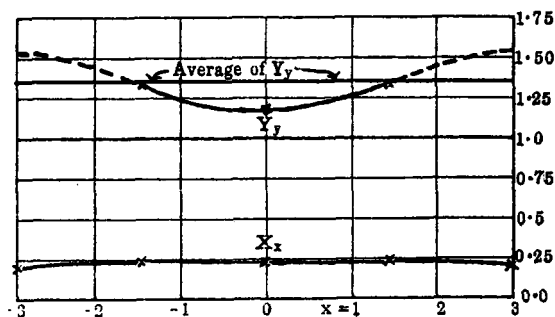


FIG. 5.

To give this mean value  $Y_y$  must increase as the edge of the plate is approached. This is indicated in Fig. 5. The shear  $Y_x$ .

\* L. F. Richardson, "Phil. Trans.", A., 1910, §3.2.

vanishes by symmetry along  $y=0$ . The principal stress  $X_x = \frac{\partial^2 \gamma}{\partial y^2}$ , at the origin, must be to  $\frac{\partial^2 \gamma}{\partial x^2}$  approximately inversely as the squares of the axes of the ellipse  $\gamma = -3$  on Fig. 3. By measuring the ellipse its semi-axes were found to be 1.07 and 2.38. Therefore,  $X_x = 1.18 \times \left(\frac{1.07}{2.38}\right)^2 = 0.24$  at the origin.

Again along  $y=0$ , since  $\frac{\partial \gamma}{\partial y} = 0$ ,

$$\frac{\partial^2 \gamma}{\partial y^2} = (\text{curvature of line } \gamma = \text{const.}) \times \frac{\partial \gamma}{\partial x}.$$

Hence at $x =$	1.45	1.90
$X_x = \frac{\partial^2 \gamma}{\partial y^2} =$	0.24	0.20

In conclusion, I have pleasure in thanking Prof. A. E. H. Love for references to Papers about  $\nabla^4 \gamma = 0$  and Dr. C. Chree for reading and criticising the manuscript.

#### ABSTRACT.

By boundary problems are here meant problems in which an unknown function of position—hereinafter called the integral—has to be found throughout a given region from the knowledge that it satisfies a given differential equation—hereinafter called the body equation—throughout the region, and that the integral and (if necessary) its space-rates are given on the bounding surface in such a way as to make it determinate throughout the region.

The method described applies to a wide range of body equations, and to any shape of boundary which can be represented satisfactorily by an outline drawn on paper. The discussion is for simplicity confined to integrals, which can be represented by a single graph showing a family of lines, along successive members of which the integral has successive constant values.

The method of solution depends on a knowledge of the value of the integral at a point inside some simple figure—say at the centre of a circle—as a function of the integral and (if necessary) its space-rates on the perimeter of the figure.

A drawing is made showing the boundary, and the contours are sketched in so as to satisfy the boundary equations, but elsewhere merely by guesswork. This graph is then tested by circles of various sizes and variously centred, and is adjusted accordingly. The testing and adjustment are repeated until sufficient accuracy is obtained or until improvement becomes too slow to be worth while. Some skill in freehand drawing is required.

As an example a rough determination is made of the stress-function in a notched plate, of special shape, when in tension.