



XXII. The coefficient of end-correction.—Part II

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of the above results. A small quantity of argon, present in helium when a discharge is sent through a Geissler tube, may be easily detected spectroscopically; but when some helium is present in argon its quantity must be considerably greater in order to give a distinct enough spectrum.

With regard to the electronegative gases, gases in which electrons lose their energy at every collision with the molecules, the analogous considerations are probably far more complicated.

Petrograd,
Physical Institute of the University.

XXII. *The Coefficient of End-Correction.*—Part II. By P. J. DANIELL, B.A., Assistant Professor in Applied Mathematics, The Rice Institute, Houston, Texas*.

§ 1. **I**F an electrical current passes through a long cylindrical tube of conducting material, and then out into a large hemispherical volume of the same, the total resistance is proportional to the total length of the tube plus a certain multiple of the radius. This multiple is the coefficient of end-correction which we require to find.

Rayleigh in his 'Theory of Sound' found that

$$\cdot785 < \text{this coefficient } k < \cdot845.$$

In the previous paper with the same title the author assumed the normal current at the open end to be of a form

$$A + B(1 - \varpi^2) + C(1 - \varpi^2)^{-1/3}.$$

Thus it was found that

- (1) if B is neglected, the approximate value of k is $\cdot82171$;
- (2) considering B, the approximate value of k is $\cdot82168$.

Then the method shows that the real value of $k < \cdot82168$.

In this paper the author states a method by which an approximate value of k can be found which is less than the real value. In fact

$$\cdot82141 < \text{real value } k.$$

Thus k is confined between the narrow limits

$$\cdot82141 < k < \cdot82168.$$

The method by which Rayleigh found the lower limit $\cdot785$ for k , was to assume a value for the potential V at the

* Communicated by the Author.

open end. The author has amplified this and assumed that at the open end V is of the form

$$A + B(1 - \varpi^2) + C(1 - \varpi^2)^2 + D(1 - \varpi^2)^3.$$

§ 2. As before, let us use cylindrical coordinates ϖ, z and let us take the radius of the tube to be 1, and its length to be L . Let us divide the whole space into two parts: first, the hemisphere $\varpi \geq 0, z \geq 0$, and $\varpi^2 + z^2 \leq R^2$ where R is large; secondly, the cylinder $0 \leq \varpi \leq 1, -L \leq z \leq 0$.

Let V be the potential at any point, then a solution is given by Dirichlet's condition that $\int V \frac{\partial V}{\partial n} dS$ is a minimum, where S_n is an element of normal drawn outwards from the region over whose surface the integral is taken. But in this case we shall assume, not a given value of the current, but of the potentials at $z = -L$ and at $z = +\infty$.

In the region II. a solution of Laplace's equation will be given by

$$V = -lz + E + \sum_r \alpha_r e^{k_r z} J_0(k_r \varpi).$$

This satisfies the proper conditions; for near the end $z = -L$ we have

$$V = -lz + E,$$

while at the boundary $\varpi = 1$,

$$\frac{\partial V}{\partial n} = 0,$$

$$\text{or} \quad J_1(k_r) = 0.$$

Thus the k_r 's must be chosen so as to satisfy this equation.

Then

$$\begin{aligned} V_{z=0} &= E + \sum_r \alpha_r J_0(k_r \varpi) \\ &= A + B(1 - \varpi^2) + C(1 - \varpi^2)^2 + D(1 - \varpi^2)^3. \end{aligned}$$

Multiply by $J_0(k_r \varpi) \varpi$ and integrate from 0 to 1. Then

$$\begin{aligned} \alpha_r \frac{1}{2} J_0^3(k_r) &= \int_0^1 A J_0(k_r \varpi) \varpi d\varpi + \int_0^1 B(1 - \varpi^2) J_0(k_r \varpi) \varpi d\varpi \\ &\quad + \int_0^1 C(1 - \varpi^2)^2 J_0(k_r \varpi) \varpi d\varpi + \int_0^1 D(1 - \varpi^2)^3 J_0(k_r \varpi) \varpi d\varpi. \end{aligned}$$

But *

$$\int_0^1 J_0(k \varpi) (1 - \varpi^2)^{\nu-1} \varpi d\varpi = 2^{\nu-1} \Gamma(\nu) \frac{J_\nu(k)}{k^\nu}.$$

* Schafheitlin, 'Bessel Functions,' p. 31.

Then

$$\alpha_r \frac{J_0^2(k_r)}{2} = A \frac{J_1(k_r)}{k_r} + B \frac{2J_2(k_r)}{k_r^2} + C \frac{2^3 J_3(k_r)}{k_r^3} + D 2^{4.3} \frac{J_4(k_r)}{k_r^4}.$$

But $J_1(k_r) = 0$,

$$\text{then } J_2(k_r) = \frac{2}{k_r} J_1(k_r) - J_0(k_r) = -J_0(k_r),$$

$$J_3(k_r) = \frac{4}{k_r} J_2(k_r) - J_1(k_r) = -\frac{4}{k_r} J_0(k_r),$$

$$J_4(k_r) = \frac{6}{k_r} J_3(k_r) - J_2(k_r) = -J_0(k_r) \left(\frac{24}{k_r^2} - 1 \right).$$

Then

$$-\alpha_r \frac{J_0(k_r)}{2} = \frac{2B}{k_r^2} + \frac{32C}{k_r^4} + \frac{48D}{k_r^4} \left(\frac{24}{k_r^2} - 1 \right).$$

Again,

$$\left(\frac{\partial V}{\partial z} \right)_{z=0} = -l + \sum_r \alpha_r k_r J_0(k_r \varpi).$$

Therefore $\frac{1}{2\pi} \int V \frac{\partial V}{\partial n} dS$ at end $z=0$ of II.

$$= \int_0^1 V_{z=0} \left(\frac{\partial V}{\partial z} \right)_{z=0} \varpi d\varpi$$

$$= -\frac{1}{2} l E + \sum_r \alpha_r k_r \frac{J_0^2(k_r)}{2}.$$

At the other end

$$V = lL + E, \quad -\frac{\partial V}{\partial z} = \frac{\partial V}{\partial n} = l,$$

so that

$$\frac{1}{2\pi} \int V \frac{\partial V}{\partial n} dS = \frac{1}{2} (l^2 L + lE).$$

Then

$$\frac{1}{2\pi} \int_I V \frac{\partial V}{\partial n} dS = \frac{1}{2} l^2 L + \sum_r \alpha_r k_r \frac{J_0^2(k_r)}{2}$$

$$\left. \begin{aligned} &= \frac{1}{2} l^2 L + (2B)^2 \sum_r \frac{2}{k_r^3} + 2(2B)(32C) \sum_r \frac{2}{k_r^5} + (32C)^2 \sum_r \frac{2}{k_r^7} \\ &\quad + 2(2B)(48D) \sum_r \frac{2}{k_r^5} \left(\frac{24}{k_r^2} - 1 \right) + 2(32C)(48D) \sum_r \frac{2}{k_r^7} \left(\frac{24}{k_r^2} - 1 \right) \\ &\quad + (48D)^2 \sum_r \frac{2}{k_r^7} \left(\frac{24}{k_r^2} - 1 \right)^2. \end{aligned} \right\} \quad (i.)$$

Again, since

$$\begin{aligned} V_{z=0} &= E + \sum_r \alpha_r J_0(k_r \varpi) \\ &= A + B(1 - \varpi^2) + C(1 - \varpi^2)^2 + D(1 - \varpi^2)^3, \end{aligned}$$

if we multiply by ϖ and integrate from 0 to 1,

$$E = A + \frac{1}{2}B + \frac{1}{3}C + \frac{1}{4}D. \quad \dots \dots (ii.)$$

In the region I. we have to make V have a definite value over $z=0$, $\varpi < 1$ and $V=0$ when $z = +\infty$ and $\frac{\partial V}{\partial z} = 0$ when $z=0$, $\varpi > 1$.

Introduce * new variables ξ, ζ which satisfy the relations

$$z = \xi\zeta, \quad \varpi = \sqrt{(1 - \xi^2)(1 + \zeta^2)}.$$

Then the solution of Laplace's equation which makes $V=0$ at infinity and finite when $\varpi = 0$ is

$$V = \sum_n A_n P_n(\xi) Q_n(i\xi),$$

where P_n is the first, Q_n the second Legendre function, and i is the imaginary $\sqrt{-1}$.

But when $z=0$,

$$\varpi > 1, \quad \xi = 0, \quad \frac{\partial V}{\partial z} = 0$$

and therefore $\frac{\partial V}{\partial \xi} = 0$ when $\xi = 0$.

Then only even integers n may be taken, $n = 2r$ say. When $z=0$, $\varpi < 1$, $\zeta = 0$, so that

$$V_{z=0} = \sum_r A_{2r} P_{2r}(\xi) Q_{2r}(0).$$

Let $A_{2r} Q_{2r}(0) = b_{2r}$.

Then, since when $\zeta = 0$, $1 - \varpi^2 = \xi^2$ it follows that

$$\sum_r b_{2r} P_{2r}(\xi) = V_{z=0} = A + B\xi^2 + C\xi^4 + D\xi^6.$$

This equation will yield the coefficients b_{2r} .

When $z=0$, $\varpi < 1$ or $\zeta = 0$,

$$\begin{aligned} - \frac{\partial V}{\partial z} &= - \frac{1}{\xi} \frac{\partial V}{\partial \zeta} \\ &= - \frac{1}{\xi} \sum_r A_{2r} P_{2r}(\xi) \left[\frac{\partial}{\partial \zeta} Q_{2r}(i\zeta) \right]_{\zeta=0}. \end{aligned}$$

* Jeans, 'Electricity and Magnetism,' p. 252.

But

$$Q_{2r}(i\xi) = i^{-(2r+1)} \int_0^\infty (\xi + \sqrt{\xi^2 + 1} \cosh u)^{-(2r+1)} du$$

or

$$Q_{2r}(0) = i^{-(2r+1)} \int_0^\infty \cosh u^{-(2r+1)} du.$$

Also

$$\begin{aligned} \frac{\partial}{\partial \xi} Q_{2r}(i\xi) &= i^{-(2r+1)} \int_0^\infty (\xi + \sqrt{\xi^2 + 1} \cosh u)^{-(2r+2)} (-2r-1) \\ &\quad \times \left(1 + \frac{\xi}{\sqrt{\xi^2 + 1}} \cosh u \right) du, \end{aligned}$$

or when $z=0$ this

$$= i^{-(2r+1)} (-2r-1) \int_0^\infty \cosh u^{-(2r+2)} du.$$

Then

$$\left(-\frac{\partial V}{\partial z} \right)_{z=0} = \frac{1}{\xi} \sum_r b_{2r} P_{2r}(\xi) C_{2r},$$

where
$$\begin{aligned} C_{2r} &= (2r+1) \int_0^\infty (\cosh u)^{-(2r+2)} du / \int_0^\infty (\cosh u)^{-(2r+1)} du \\ &= 2 \left\{ \frac{\Gamma(r+1)}{\Gamma(r+\frac{1}{2})} \right\}^2. \end{aligned}$$

$$\begin{aligned} \int_0^1 V_{z=0} \left(-\frac{\partial V}{\partial z} \right)_{z=0} \varpi d\varpi &= \int_0^1 \left[\sum_r b_{2r} P_{2r}(\xi) \right] \left[\sum_r b_{2r} C_{2r} P_{2r}(\xi) \right] \frac{1}{\xi} \cdot \xi d\xi \\ &= \sum_r b_{2r}^2 C_{2r} \frac{1}{4r+1}. \end{aligned}$$

Between the regions I. and II. the boundary of non-conducting material is continuous, that is to say there is no chance of any current escaping between the regions I. and II. except through the open end itself. The total current passing into the region I must then be the same as πl , the total current in the region II. This total current is

$$2\pi \int_0^1 \left(-\frac{\partial V}{\partial z} \right)_{z=0} \varpi d\varpi = 2\pi b_0 c_0 = 4b_0$$

and $=\pi l$ on the other hand.

Then $b_0 = \frac{\pi}{4} l.$

Now $1 = P_0,$

$$\xi^2 = \frac{2}{3} \left(P_2 + \frac{1}{2} P_0 \right),$$

$$\xi^4 = \frac{8}{35} \left(P_4 + \frac{5}{2} P_2 + \frac{7}{8} P_0 \right),$$

$$\xi^6 = \frac{16}{231} \left(P_6 + \frac{9}{2} P_4 + \frac{55}{8} P_2 + \frac{33}{16} P_0 \right).$$

But

$$b_0 P_0 + b_2 P_2 + b_4 P_4 + b_6 P_6 = A + B \xi^2 + C \xi^4 + D \xi^6.$$

Then

$$b_0 = A + \frac{1}{3} B + \frac{1}{5} C + \frac{1}{7} D,$$

$$b_2 = \frac{2}{3} B + \frac{4}{7} C + \frac{10}{21} D,$$

$$b_4 = \frac{8}{35} C + \frac{24}{77} D,$$

$$b_6 = \frac{16}{231} D.$$

But

$$b_0 = \frac{\pi}{4} l = A + \frac{1}{3} B + \frac{1}{5} C + \frac{1}{7} D$$

and

$$E = A + \frac{1}{2} B + \frac{1}{3} C + \frac{1}{4} D \quad \text{by (ii.)}$$

so that

$$E = \frac{\pi}{4} l + \frac{1}{6} B + \frac{2}{15} C + \frac{3}{28} D.$$

The potential V at the end $z = -L$ is $lL + E$ and is assumed to be fixed.

Then $l \left(L + \frac{\pi}{L} \right) + \frac{1}{6} B + \frac{2}{15} C + \frac{3}{28} D$ is fixed. . (iii.)

In the region I. at the boundary $\varpi^2 + z^2 = R^2$, the potential V is proportional to $\frac{1}{R}$ and $\frac{\partial V}{\partial n}$ to $\frac{1}{R^2}$, so that the integral $\int V \frac{\partial V}{\partial n} dS$ can be made as small as we please by increasing R sufficiently.

Thus

$$\begin{aligned} \frac{1}{2\pi} \int_I V \frac{\partial V}{\partial n} dS &= \int_0^1 V_{z=0} \left(-\frac{\partial V}{\partial z} \right)_{z=0} \varpi d\varpi \\ &= \sum_r b_{2r}^2 c_{2r} \frac{1}{4r+1} \\ &= \frac{1}{5} \cdot \frac{8}{\pi} \cdot \left(\frac{2}{3} B + \frac{4}{7} C + \frac{10}{21} D \right)^2 + \frac{1}{9} \cdot \frac{8}{\pi} \cdot \frac{16}{9} \cdot \left(\frac{8}{35} C + \frac{24}{77} D \right)^2 \\ &\quad + \frac{1}{13} \cdot \frac{8}{\pi} \cdot \frac{16}{9} \cdot \frac{36}{25} \left(\frac{16}{231} D \right)^2. \quad \dots \dots \dots \text{(iv.)} \end{aligned}$$

Combining (i.) and (iv.) and using the calculated values of the coefficients Dirichlet's condition becomes that

$$\begin{aligned} \frac{l^2}{2} \left(L + \frac{\pi}{4} \right) &+ B^2[182024 + \cdot 226354] + 2BC[164180 + \cdot 194017] \\ &+ C^2[171548 + \cdot 192580] + 2BD[13972 + \cdot 16168] \\ &+ 2CD[15863 + \cdot 17442] + D^2[15483 + \cdot 16675] \\ &\hspace{15em} \text{is minimum,} \end{aligned}$$

$$\text{while} \quad l \left(L + \frac{\pi}{4} \right) + \frac{1}{6} B + \frac{2}{15} C + \frac{3}{28} D \quad \text{is fixed.}$$

§ 3. The problem then is to find the minimum value of

$$\begin{aligned} \frac{l^2}{2} \left(L + \frac{\pi}{4} \right) &+ B^2(\cdot 408379) + 2BC(\cdot 358197) + C^2(\cdot 364128) \\ &+ 2BD(\cdot 30140) + 2CD(\cdot 33305) + D^2(\cdot 32159), \end{aligned}$$

when

$$l \left(L + \frac{\pi}{4} \right) + \frac{1}{6} B + \frac{2}{15} C + \frac{3}{28} D = \text{fixed in value.} \quad \dots \quad \text{(v.)}$$

By the method of indeterminate multipliers,

$$B(\cdot 408379) + C(\cdot 358197) + D(\cdot 30140) = \frac{1}{6} \lambda,$$

$$B(\cdot 358197) + C(\cdot 364128) + D(\cdot 33305) = \frac{2}{15} \lambda,$$

$$B(\cdot 30140) + C(\cdot 33305) + D(\cdot 32159) = \frac{3}{28} \lambda,$$

$$\frac{l}{2} \left(L + \frac{\pi}{4} \right) = \left(L + \frac{\pi}{4} \right) \lambda,$$

$$\text{or} \quad \lambda = \frac{1}{2} l.$$

Then

$$B[.408379 \times .364128 - (.358197)^2] = \lambda \left[\frac{1}{6} \text{ of } .364128 - \frac{2}{15} \text{ of } .358197 \right] \\ - D[.30140 \times .364128 - .33305 \times .358197],$$

or $B(.020397) = \lambda(.0129284) + D(.00955).$

Also

$$C[.408379 \times .364128 - (.358197)^2] = \lambda \left[\frac{2}{15} \text{ of } .408379 - \frac{1}{6} \text{ of } .358197 \right] \\ - D[.33305 \times .408379 - .30140 \times .358197],$$

or $C(.020397) = -\lambda(.005249) - D(.02805),$

or $B = \lambda(.63384) + D(.4682),$

$C = -\lambda(.25734) - D(1.3752).$

Then

$$D(.30140 \times .4682 - .33305 \times 1.3752 + .32159) \\ = \lambda \left(\frac{3}{28} - .30140 \times .63384 + .33305 \times .25734 \right)$$

or $D(.00470) = \lambda(.00181)$

or $D = \lambda(.385).$

The fixed value of the linear expression (ν) is equal to the potential at the end $z = -L$, that is $= l(L+k)$ where k is the coefficient required. But $L+k$ is proportional to

$$\frac{(V_{z=-L})^2}{\int_I V \frac{\partial V}{\partial n} dS + \int_{II} V \frac{\partial V}{\partial n} dS}$$

that is to say, making

$$\int_I V \frac{\partial V}{\partial n} dS + \int_{II} V \frac{\partial V}{\partial n} dS$$

a minimum for a fixed $V_{z=-L}$ corresponds to finding the maximum k .

Each term introduced will increase this maximum and therefore the k we obtain is less than the proper value, contrary to the case where we gave the current a fixed value

when the value of k obtained was greater than the proper value. Here we find

$$l(L+k) = l\left(L + \frac{\pi}{4}\right) + \frac{1}{6}B + \frac{2}{15}C + \frac{3}{28}D,$$

or since $\lambda = \frac{1}{2}l$,

$$\begin{aligned} k &= \frac{\pi}{4} + \frac{1}{12}(\cdot63384) - \frac{1}{15}(\cdot25734) \\ &\quad + \frac{1}{2}(\cdot385)\left(\frac{1}{6} \text{ of } \cdot4682 - \frac{2}{15} \text{ of } 1\cdot3752 + \frac{3}{28}\right) \\ &= \frac{\pi}{4} + \cdot035664 + \cdot000348. \end{aligned}$$

Then neglecting the term $D(1-\varpi^2)^3$ we obtain

$$\begin{aligned} k_1 &= \frac{\pi}{4} + \cdot035664 \\ &= \cdot821062, \end{aligned}$$

and introducing the term $D(1-\varpi^2)^3$

$$\begin{aligned} k_2 &= \cdot821062 + \cdot000348 \\ &= \cdot821410. \end{aligned}$$

Then using the result of the previous paper,

$$\cdot82141 < k < \cdot82168.$$

The change produced in k by introducing a further term is about $\cdot00035$ in this paper. In the previous paper it was about $\cdot000030$. Let us assume that the actual errors are proportional to these changes.

Then k will probably differ very slightly from

$$\cdot82168 - \frac{3}{35+3}(\cdot82168 - \cdot82141)$$

or from $\cdot82166$.

Then certainly

$$\cdot82141 < k < \cdot82168$$

and probably

$$k = \text{about } \cdot82166.$$