

ON THE NEWTONIAN POTENTIAL DUE TO A SURFACE DISTRIBUTION HAVING A DISCONTINUITY OF THE SECOND KIND.

By **G. Prasad** (Calcutta).

Adunanza dell'11 giugno 1916.

The object of the following lines is chiefly to point out a case in which the Newtonian potential, due to a surface distribution having a discontinuity of the second kind, is such that the normal derivate does *not* tend to a limit as the boundary is approached. It is believed that such a case ¹⁾ has never been discussed by any of the previous writers on the subject from Prof. HÖLDER to Prof. PICARD. About the end of the paper, I have given a simple case ²⁾ in which the derivate tends to a limit, although the surface distribution has a discontinuity of the second kind.

1. In order to avoid unnecessary complications, I take the surface to be a sphere of unit radius. Let C be the centre of the sphere and O any point on it. Then, taking OC as the z -axis and the tangent-plane at O as the z -plane, the potential V at any point $(0, 0, z)$ on OC may be taken to be $u + U$, where u is the potential due to a small element S of the sphere bounded by the circle QOQ' and U that due to the remaining part of the sphere. It is obvious that $\lim_{z=0} \frac{\partial U}{\partial z}$ exists; also

$$\frac{\partial u}{\partial z} = -2z\pi \int_0^a \frac{\sigma r dr}{(z^2 + r^2)^{\frac{3}{2}}}$$

where σ is the surface density, $OQ = a$, and the curvature of the element S is neglected, it being taken to be a plane circle with centre O and radius a .

2. I proceed to prove that $\lim_{z=0} \frac{\partial u}{\partial z}$ is non-existent provided that $\sigma = \cos \log \frac{1}{r}$.

¹⁾ For the logarithmic potential due to a linear distribution, I have given a similar case in the Bulletin of the Calcutta Mathematical Society, vol. V (1914).

²⁾ For the logarithmic potential due to a linear distribution, Dr. G. PUCCIANO has given a similar case: G. PUCCIANO, *Studio sui potenziali logaritmici di strato lineare semplice e doppio, e delle loro derivate prime* [Rendiconti del Circolo Matematico di Palermo, t. XXIII (1° semestre 1907), pp. 374-393].

Here, putting $\log \frac{1}{r} = v$, we have

$$\begin{aligned}
 -\frac{1}{2\pi\chi} \frac{\partial u}{\partial \chi} &= \int_0^a \frac{r \cos \log \frac{1}{r} dr}{(\chi^2 + r^2)^{\frac{3}{2}}} \\
 &= \int_{\log \frac{1}{a}}^{\infty} \frac{e^{-2v} \cos v dv}{(\chi^2 + e^{-2v})^{\frac{3}{2}}} \\
 &= \int_{\log \frac{1}{a}}^{\log \frac{1}{\chi}} \frac{e^{-2v} \cos v dv}{(\chi^2 + e^{-2v})^{\frac{3}{2}}} + \int_{\log \frac{1}{\chi}}^{\infty} \frac{e^{-2v} \cos v dv}{(\chi^2 + e^{-2v})^{\frac{3}{2}}} \\
 &= \int_{\log \frac{1}{a}}^{\log \frac{1}{\chi}} dv e^v \cos v \left\{ 1 - \frac{3}{2} \chi^2 e^{2v} + \frac{3 \cdot 5}{2 \cdot 4} \chi^4 e^{4v} - \dots \right\} \\
 &\quad + \int_{\log \frac{1}{\chi}}^{\infty} \frac{dv e^{-2v} \cos v}{\chi^3} \left\{ 1 - \frac{3}{2} \frac{e^{-2v}}{\chi^2} + \frac{3 \cdot 5}{2 \cdot 4} \frac{e^{-4v}}{\chi^4} - \dots \right\} \\
 &= \left[\frac{e^v (\sin v + \cos v)}{1 + 1^2} - \frac{3}{2} \frac{\chi^2 e^{3v} (\sin v + 3 \cos v)}{1 + 3^2} + \dots \right]_{\log \frac{1}{a}}^{\log \frac{1}{\chi}} \\
 &\quad + \left[\frac{e^{-2v} (\sin v - 2 \cos v)}{\chi^3 (1 + 2^2)} - \frac{3}{2} \frac{e^{-4v} (\sin v - 4 \cos v)}{\chi^5 (1 + 4^2)} + \dots \right]_{\log \frac{1}{\chi}}^{\infty} \\
 &= \frac{1}{\chi} \sin \log \frac{1}{\chi} \left\{ \left(\frac{1}{1 + 1^2} - \frac{3}{2} \frac{1}{1 + 3^2} + \frac{3 \cdot 5}{2 \cdot 4} \frac{1}{1 + 5^2} - \dots \right) \right. \\
 &\quad \left. + \left(\frac{1}{1 + 2^2} - \frac{3}{2} \frac{1}{1 + 4^2} + \frac{3 \cdot 5}{2 \cdot 4} \frac{1}{1 + 6^2} - \dots \right) \right\} \\
 &\quad + \frac{1}{\chi} \cos \log \frac{1}{\chi} \left\{ \left(\frac{1}{1 + 1^2} - \frac{3}{2} \frac{3}{1 + 3^2} + \frac{3 \cdot 5}{2 \cdot 4} \frac{5}{1 + 5^2} - \dots \right) \right. \\
 &\quad \left. + \left(\frac{2}{1 + 2^2} - \frac{3}{2} \frac{4}{1 + 4^2} + \frac{3 \cdot 5}{2 \cdot 4} \frac{6}{1 + 6^2} - \dots \right) \right\} \\
 &\quad - \frac{1}{a} \frac{\sin \log \frac{1}{a} + \cos \log \frac{1}{a}}{1 + 1^2} + \text{terms which vanish with } \chi.
 \end{aligned}$$

Now the infinite series in the above expression are all convergent.

Therefore, as χ tends to zero, $\frac{\partial u}{\partial \chi}$ behaves as $A \cos \log \frac{1}{\chi} + B \sin \log \frac{1}{\chi}$, where A and B are known constants.

Thus it is proved that $\lim_{\chi=0} \frac{\partial u}{\partial \chi}$ and, consequently, $\lim_{\chi=0} \frac{\partial V}{\partial \chi}$ are non-existent.

3. It is easily seen that, if the distribution be

$$\sigma = \sum_1^{\infty} \frac{f(\theta - \omega_n)}{n^2},$$

where θ is the angle made by any radius CP with a fixed diameter, $\{\omega_n\}$ is the aggregate of rational numbers between 0 and π , and $f(x)$ is $\cos \log \frac{1}{x}$ or $\cos \log \frac{1}{-x}$ according as x is positive or negative, the normal derivate at any point Q on the radius CP makes infinite number of fluctuations (without tending to a limit) as Q approaches P , provided that θ be a member of the aggregate.

4. I proceed now to prove that $\lim_{\chi=0} \frac{\partial u}{\partial \chi}$ and, consequently, $\lim_{\chi=0} \frac{\partial V}{\partial \chi}$ are existent provided that $\sigma = \cos \frac{1}{r}$.

Putting $\frac{1}{r} = v$, we have

$$-\frac{1}{2\chi\pi} \frac{\partial u}{\partial \chi} = \int_0^a \frac{r \cos \frac{1}{r} dr}{(\chi^2 + r^2)^{\frac{3}{2}}} = \int_{\frac{1}{a}}^{\infty} \frac{\cos v dv}{(1 + v^2 \chi^2)^{\frac{3}{2}}}.$$

Now $\frac{1}{(1 + v^2 \chi^2)^{\frac{3}{2}}}$ is always positive and its greatest value is $\frac{1}{\left(1 + \frac{\chi^2}{a^2}\right)^{\frac{3}{2}}}$. The-

refore

$$\left| \int_{\frac{1}{a}}^{\infty} \frac{\cos v dv}{(1 + v^2 \chi^2)^{\frac{3}{2}}} \right| < \frac{1}{\left(1 + \frac{\chi^2}{a^2}\right)^{\frac{3}{2}}}.$$

Hence $\frac{\partial u}{\partial \chi}$ is numerically less than $2\pi\chi$. Therefore $\lim_{\chi=0} \frac{\partial u}{\partial \chi} = 0$.

Calcutta, 10 May 1916.

G. PRASAD.