VII.—On the Adelphic Integral of the Differential Equations of Dynamics. By Professor E. T. Whittaker, F.R.S.

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§ 1. *Ordinary and singular periodic solutions of a dynamical system.*—The present paper is concerned with the motion of dynamical systems which possess an integral of energy. To fix ideas, we shall suppose that the system has two degrees of freedom, so that the equations of motion in generalised co-ordinates may be written in Hamilton's form

$$\frac{dq_1}{dt}=\frac{\partial H}{\partial p_1},\quad \frac{dq_2}{dt}=\frac{\partial H}{\partial p_2},\quad \frac{dp_1}{dt}=-\frac{\partial H}{\partial q_1},\quad \frac{dp_2}{dt}=-\frac{\partial H}{\partial q_2} \quad . \quad . \quad (1)$$

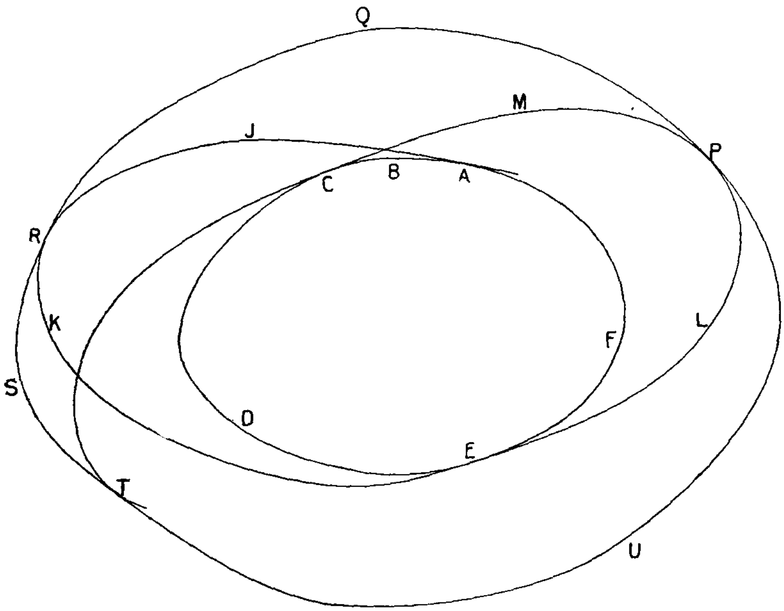
where (q_1, q_2) are the generalised co-ordinates, (p_1, p_2) are the generalised momenta, and where H is a function of (q_1, q_2, p_1, p_2) which represents the sum of the kinetic and potential energies.

The successive states of the system may be illustrated by the motion of a point whose co-ordinates referred to the axes are (q_1, q_2): the curve described by such a point is called a *trajectory*. Particular interest attaches to those trajectories which are *closed* curves: these are known as *periodic solutions*.

I wish to draw attention in the first place to a distinction which should be made in regard to these periodic solutions; the matter may perhaps be elucidated most readily by considering a particular problem, namely, that of the motion of a particle on the surface of an ellipsoid under no external forces. The particle describes a geodesic on the surface, so the periodic solutions are simply those geodesics which are closed curves. Now for a geodesic on an ellipsoid we have Joachimstal's equation

$$pd = \text{constant},$$

where p denotes the perpendicular from the centre of the ellipsoid on the tangent-plane at the point, and d is the diameter parallel to the tangent to the geodesic at the point. The same equation holds for the lines of curvature on the ellipsoid; so that every geodesic may be associated with a line of curvature, namely, that line of curvature for which pd has the same value as it has for the geodesic. We shall speak of the geodesic as "belonging to" the line of curvature. There is only one line of curvature having a prescribed value for pd, but there is an infinite number of geodesics having this value for pd, so that an infinite number of geodesics "belong to" each line of curvature. Now the line of curvature consists of two closed curves on the ellipsoid (being in fact the intersection of the ellipsoid with a confocal quadric): the region between these two portions of the line of curvature is a belt extending round the ellipsoid: and all the geodesics which belong to this line of curvature are comprised within this belt,* and touch the two portions of the line of curvature alternately. The matter is represented schematically in the diagram, where ABCDEF and PQRSTU are the two portions of the line of curvature, and AJRKELPMCT is an arc of one of the geodesics belonging to it, touching one of the



portions of the line of curvature at A, C, E, and touching the other portion at R, P, T.

In order that the geodesic may be closed, it is necessary (as in all poristic problems) that a certain parameter (depending in this case on the value of the constant pd of the line of curvature) should be a rational number: the geodesic is unclosed if this parameter is an irrational number. If it is closed, then there are ∞^1 other geodesics which belong to the same line of curvature and which are also closed; but if it is not closed, then no other geodesic belonging to this particular line of curvature can be a closed geodesic.†

* Ignoring the exceptional case of those geodesics which pass through an umbilicus.

† This is obvious in the case when the ellipsoid is of revolution: for then the two portions of the line of curvature are parallel circles on the surface, and the ∞^1 geodesics which belong to this line of curvature are obtained from each other by mere rotation about the axis of symmetry.

Now consider the connection between the ∞^1 members of the family of geodesics which belong to the same line of curvature. It is known * that if

$$\phi\ (q_1, q_2, p_1, p_2) = \text{Constant}$$

is an integral of a dynamical system, then the infinitesimal contact-transformation which is defined by the equations

$$\delta q_1 = \epsilon \frac{\partial \phi}{\partial p_1}, \quad \delta q_2 = \epsilon \frac{\partial \phi}{\partial p_2}, \quad \delta p_1 = -\epsilon \frac{\partial \phi}{\partial q_1}, \quad \delta p_2 = -\epsilon \frac{\partial \phi}{\partial q_2}$$

(where ϵ is a small constant) transforms any trajectory into an adjacent curve which is also a trajectory. If we apply this theorem to the motion on the ellipsoid, we find without much difficulty † that the infinitesimal transformation which corresponds to the integral

$$pd = \text{Constant}$$

transforms any geodesic into another geodesic which belongs to the same line of curvature.

Summing up, we see that *the* ∞^2 *geodesics on an ellipsoid may be classified into* ∞^1 *families, each family consisting of* ∞^1 *geodesics: the members of any one family are either all closed or all unclosed: and a certain continuous group of transformations, which is closely associated with the integral* pd = *Constant, transforms any geodesic into all the geodesics which belong to the same family.*

Besides these geodesics which can be arranged in families, there are on the ellipsoid three other closed geodesics, namely, the three principal sections of the ellipsoid. These have quite a different character: they are solitary, instead of belonging to families: and the infinitesimal transformation which has just been mentioned tranforms them not into other geodesics but into themselves—that is, they are invariant under the transformation. This last property suggests a resemblance with the theory of "singular solutions" of ordinary differential equations of the first order: for if a differential equation of the first order admits a particular infinitesimal transformation, then this infinitesimal transformation changes the ordinary integral-curves into each other, but it leaves invariant the singular integral-curve. On account of this resemblance I propose to call a periodic solution (of a dynamical system with two degrees of freedom) *ordinary* if

* *Cf.*, *e.g.*, my *Analytical Dynamics*, § 144.

† As this problem of motion on an ellipsoid is only a special case of the general theory which is given later, I do not give the analysis relating to it in detail.

it belongs to a continuous family of ∞^1 periodic solutions for which the constant of energy has the same value, and which are transformed into each other by the infinitesimal transformation belonging to a certain integral (this is specified more closely later on); but a periodic solution is to be called *singular* if there is no periodic solution adjacent to it which corresponds to the same value of the constant of energy: the above-mentioned infinitesimal transformation leaves the singular periodic solutions invariant.

It should be noticed that we have inserted the condition "for which the constant of energy has the same value." If we suppose the constant of energy to vary, an "ordinary" periodic solution is in general a member of a continuous family of ∞^2 periodic solutions, whereas a "singular" periodic solution is a member of a family of ∞^1 periodic solutions.*

There are marked differences between the properties of "ordinary" and those of "singular" periodic solutions. For instance, the "asymptotic solutions" of Poincaré † can exist only in connection with *singular* periodic solutions, and not in connection with ordinary periodic solutions; an illustration of this is again afforded by the theory of geodesics on quadrics; for the only asymptotic solutions among the geodesics of quadrics are those geodesics which wind round and round the hyperboloid of one sheet, becoming ultimately asymptotic to the principal elliptic section of the hyperboloid: and this elliptic section is a singular periodic solution.

We must now examine into the existence of families of "ordinary" periodic solutions in the general dynamical system with two degrees of freedom. For this purpose we recall that in the solution of such systems by infinite trigonometric series,‡ the generalised co-ordinates (q_1, q_2) are ultimately expressed in the following way: each co-ordinate is a sum of terms like

$$a_{mn} \cos (m\beta_1 + n\beta_2)$$

where m and n are integers (positive, negative, or zero); the coefficients a_{mn} are functions of two of the constants of integration, α_1 and α_2 only; and the angles β_1 and β_2 are defined by equations

$$\beta_1 = \mu_1 t + \epsilon_1 , \quad \beta_2 = \mu_2 t + \epsilon_2 ,$$

* The case of geodesic problems is exceptional, as in them the value of the constant of energy is immaterial.

† *Nouvelles Méth. de la Méc. Cel.*, i (1892), iii (1899).

‡ *Cf.*, *e.g.*, chap. xvi of my *Analytical Dynamics.*

where μ_1 and μ_2 are functions of a_1 and a_2 only, and ϵ_1 and ϵ_2 are the two remaining constants of integration.

Periodic solutions evidently arise when the constants a_1 and a_2 are such that μ_1 is commensurable with μ_2: the period of the solution is then $2\pi/\nu$, where ν is the largest quantity of which μ_1 and μ_2 are integer multiples.

Suppose then that a_1 and a_2 have such values. Then if the constant ϵ_1 be varied continuously, we obtain a family of periodic solutions, each having the same period (since this does not depend on ϵ_1). The constant of energy depends only on a_1 and a_2, and is therefore the same for each of these periodic solutions. The family is therefore a family of "ordinary" periodic solutions.

It might hastily be supposed that by varying ϵ_2 as well as ϵ_1 we should get a family of ∞^2 periodic solutions. But it is easily seen that the transformation which is obtained by varying ϵ_2 may be obtained by combining the transformation which consists in varying ϵ_1 with that which consists in adding a small constant to t. Now this latter transformation merely transforms every orbit into itself (each point being displaced in the direction of the tangent to the orbit), and so may be disregarded. The ϵ_1 and ϵ_2 transformations are therefore to be regarded as not distinct from each other.*

Singular periodic solutions are found chiefly in domains where the solution by purely trigonometric series is not possible.

§ 2. *Definition of the adelphic integral.*—Having now distinguished the "ordinary" and "singular" periodic solutions of a dynamical system, we shall consider those infinitesimal transformations which change each trajectory of the system into an adjacent trajectory, *in such a way that every ordinary periodic solution is changed into an adjacent periodic solution of the same family, i.e.* having the same period and the same constant of energy. In the notation we have just been using, this transformation corresponds to a small change in ϵ_1. This transformation will be called the *adelphic* transformation.† The adelphic transformation changes any solution of the dynamical system, whether periodic or not, into one of ∞^1 other solutions which stand in a particularly close relation to it, being in fact derived from it by a change of the constant ϵ_1 only.

* The only case of exception is when *all* the orbits of the system are periodic.

† From ἀδελφικός, *brotherly*, because these orbits stand in very close relation to each other, and also because the integral corresponding to the transformation stands in a much closer relation to the integral of energy than do the other integrals of the system.

To the adelphic transformation there corresponds an integral of the dynamical system: this integral we shall call the *adelphic integral* of the system.

As there is only one really distinct adelphic transformation of a given dynamical system with two degrees of freedom, so there is only one really distinct adelphic integral: all other adelphic integrals may be obtained from this by combining it in various ways with the integral of energy.*

In practically all the known soluble problems of dynamics with two degrees of freedom, the integral which enables us to effect the solution is an adelphic integral. Thus, when the trajectories are the geodesics on an ellipsoid, the adelphic integral is the equation pd = Constant. When the problem is that of two centres of gravitation, the adelphic integral is Euler's integral of that problem. When the solubility of the problem is due to the presence of an ignorable co-ordinate, say q_2, the corresponding integral (namely p_2 = Constant) is adelphic.

In the present paper we shall find the adelphic integral for the general dynamical system with two degrees of freedom, and make this the basis from which to complete the integration of the system. It will appear that by this procedure we are enabled to overcome the difficulty formulated in Poincaré's celebrated theorem, that "the series of Celestial Mechanics, if they converge at all, cannot converge uniformly for all values of the time on the one hand, and on the other hand for all values of the constants comprised between certain limits." This unsatisfactory feature of the usual series springs from peculiarities which are deep-seated in the nature of the problem, and which are difficult to discern by the methods of solution employed in Celestial Mechanics. By fixing our attention in the first place on a single integral of the dynamical system, rather than attempting at once a complete solution, we shall find what these peculiarities are; for they manifest themselves very clearly in connection with the adelphic integral, and (as we shall see) may be so taken account of in its determination, that they no longer remain to trouble us in the final stages of the complete integration of the dynamical system.

§ 3. *The form of the Hamiltonian function.*—We now proceed to inquire how the adelphic integral of a dynamical system with two degrees of freedom may be determined.

* The integral of energy corresponds to that infinitesimal transformation which changes every orbit into itself, each point of an orbit being displaced in the direction of the tangent to the orbit.

The differential equations will be taken in the Hamiltonian form

$$\frac{dq_1}{dt}=\frac{\partial H}{\partial p_1},\quad \frac{dq_2}{dt}=\frac{\partial H}{\partial p_2},\quad \frac{dp_1}{dt}=-\frac{\partial H}{\partial q_1},\quad \frac{dp_2}{dt}=-\frac{\partial H}{\partial q_2} \qquad (1)$$

where

$$H(q_1, q_2, p_1, p_2)=\text{Constant} \qquad (2)$$

is the integral of energy.

In general—at any rate in the problems of practical importance—it is possible* to choose the generalised co-ordinates in such a way that H can be expanded as an infinite series proceeding in powers of $\sqrt{q_1}$ and $\sqrt{q_2}$, and in trigonometric functions of multiples of p_1 and p_2: that is to say, in terms of the type

$$q_1^{\frac{1}{2}m} q_2^{\frac{1}{2}n} \cos(ip_1+jp_2)$$

where m and n are integers (positive or zero) and i and j are integers (positive or negative or zero): moreover, if we call $(m+n)$ the "order" of a term, the terms of lowest order are linear in q_1 and q_2 and free from p_1 and p_2, so that they may be written $(s_1q_1+s_2q_2)$, where s_1 and s_2 are constants. In most cases we find also the condition that $m-|i|$ is zero or an even integer, and $n-|j|$ is also zero or an even integer.

The Hamiltonian function H may therefore be expanded in the form

$$H=s_1q_1+s_2q_2+H_3+H_4+H_5+\ldots \qquad (3)$$

where H_r denotes the terms of order r, so we may write

$$\begin{aligned} H_3 = {} & q_1^{\frac{3}{2}}(U_1\cos p_1+U_2\cos 3p_1)+q_1q_2^{\frac{1}{2}}\{U_3\cos p_2+U_4\cos(2p_1+p_2)+U_5\cos(2p_1-p_2)\} \\ & +q_1^{\frac{1}{2}}q_2\{U_6\cos p_1+U_7\cos(2p_2+p_1)+U_8\cos(2p_2-p_1)\}+q_2^{\frac{3}{2}}\{U_9\cos p_2+U_{10}\cos 3p_2\}, \end{aligned}$$

and

$$\begin{aligned} H_4 = {} & q_1^2(X_1+X_2\cos 2p_1+X_3\cos 4p_1) \\ & +q_1^{\frac{3}{2}}q_2^{\frac{1}{2}}\{X_4\cos(p_1+p_2)+X_5\cos(p_1-p_2)+X_6\cos(3p_1+p_2)+X_7\cos(3p_1-p_2)\} \\ & +q_1q_2\{X_8+X_9\cos 2p_1+X_{10}\cos 2p_2+X_{11}\cos(2p_1+2p_2)+X_{12}\cos(2p_1-2p_2)\} \\ & +q_1^{\frac{1}{2}}q_2^{\frac{3}{2}}\{X_{13}\cos(p_1+p_2)+X_{14}\cos(p_1-p_2)+X_{15}\cos(p_1+3p_2)+X_{16}\cos(p_1-3p_2)\} \\ & +q_2^2\{X_{17}+X_{18}\cos 2p_2+X_{19}\cos 4p_2\}, \end{aligned}$$

the coefficients $U_1, U_2, \ldots U_{10}, X_1, X_2, \ldots X_{19}$ being constants.

It will appear that it is necessary to distinguish three cases: in each case an adelphic integral exists and will be determined, but the form of the adelphic integral is different in each of the three cases.

* *Cf.*, *e.g.*, *Analytical Dynamics*, §§ 184–6.

CASE I. The ratio s_1/s_2 is an irrational number.

CASE II. The ratio s_1/s_2 is a rational number, say equal to m/n (where m and n are integers and the fraction m/n is in its lowest terms) and terms in $\cos(np_1 - mp_2)$ are absent from H_3.

CASE III. The ratio s_1/s_2 is a rational number, say equal to m/n, and terms in $\cos(np_1 - mp_2)$ are present in H_3.

We shall now determine the adelphic integral in each of these cases in turn.

§ 4. *Determination of the adelphic integral in Case I.*—Let us then first suppose that the Hamiltonian function is expanded as in § 3, and that the ratio s_1/s_2 is an irrational number. We shall now show how to set up formally a series which, if it converges, is an integral of the system.

If $\phi(q_1, q_2, p_1, p_2) = \text{Constant}$ is an integral, we must have (from the equations of motion)

$$\frac{\partial \phi}{\partial q_1}\frac{\partial H}{\partial p_1} + \frac{\partial \phi}{\partial q_2}\frac{\partial H}{\partial p_2} - \frac{\partial \phi}{\partial p_1}\frac{\partial H}{\partial q_1} - \frac{\partial \phi}{\partial p_2}\frac{\partial H}{\partial q_2} = 0 \quad . \quad . \quad . \quad . \quad (4)$$

an equation which we may write $(\phi, H) = 0$.

Let us see if this equation can be satisfied formally by a series proceeding in ascending powers of $\sqrt{q_1}$ and $\sqrt{q_2}$ and trigonometric functions of p_1 and p_2 (like the series for H), whose terms of lowest order are $(s_1 q_1 - s_2 q_2)$: so that we may write

$$\phi \equiv s_1 q_1 - s_2 q_2 + \phi_3 + \phi_4 + \phi_5 + \ . \ . \ .$$

where ϕ_r denotes the terms which are of degree r in $\sqrt{q_1}$ and $\sqrt{q_2}$.

Substituting in equation (4), and equating to zero the terms of lowest order, we have

$$s_1\frac{\partial \phi_3}{\partial p_1} + s_2\frac{\partial \phi_3}{\partial p_2} = s_1\frac{\partial H_3}{\partial p_1} - s_2\frac{\partial H_3}{\partial p_2}.$$

This evidently implies that to any term $A\cos(mp_1 + np_2)$ in H_3, there corresponds a term $\dfrac{s_1 m - s_2 n}{s_1 m + s_2 n} A\cos(mp_1 + np_2)$ in ϕ_3: so the value of ϕ_3 may be written down at once. Having thus determined ϕ_3, we equate to zero the terms in equation (4) which are of order 4 in $\sqrt{q_1}$ and $\sqrt{q_2}$: this gives the equation

$$s_1\frac{\partial \phi_4}{\partial p_1} + s_2\frac{\partial \phi_4}{\partial p_2} = s_1\frac{\partial H_4}{\partial p_1} - s_2\frac{\partial H_4}{\partial p_2} + (\phi_3, H_3).$$

As the quantities on the right-hand side are all known, we can solve this equation for ϕ_4 in the same way as the preceding equation was solved for ϕ_3: and thus combining our results we obtain for our integral-series ϕ

$$\text{Constant} = \phi \equiv s_1 q_1 - s_2 q_2 + q_1^{\frac{3}{2}}(U_1 \cos p_1 + U_2 \cos 3p_1)$$

$$+ q_1 q_2^{\frac{1}{2}}\left\{ - U_3 \cos p_2 + \frac{2s_1 - s_2}{2s_1 + s_2} U_4 \cos(2p_1 + p_2) + \frac{2s_1 + s_2}{2s_1 - s_2} U_5 \cos(2p_1 - p_2) \right\}$$

$$+ q_1^{\frac{1}{2}} q_2 \left\{ U_6 \cos p_1 + \frac{s_1 - 2s_2}{s_1 + 2s_2} U_7 \cos(2p_2 + p_1) + \frac{s_1 + 2s_2}{s_1 - 2s_2} U_8 \cos(2p_2 - p_1) \right\}$$

$$+ q_2^{\frac{3}{2}} \left\{ - U_9 \cos p_2 - U_{10} \cos 3p_2 \right\}$$

$$+ q_1^2 \left[\left\{ \frac{1}{2s_1 + s_2} U_3 U_4 + \frac{1}{2s_1 - s_2} U_3 U_5 + X_2 \right\} \cos 2p_1 + \left\{ \frac{s_2}{(2s_1 + s_2)(2s_1 - s_2)} U_4 U_5 + X_3 \right\} \cos 4p_1 \right]$$

$$\begin{aligned}
+ q_1^{\frac{3}{2}} q_2^{\frac{1}{2}} \Bigg[& \frac{\cos(p_1 + p_2)}{s_1 + s_2} \left\{ - \frac{2s_1}{s_1 + 2s_2} U_3 U_7 - \frac{6s_1 s_2}{(2s_1 - s_2)(s_1 - 2s_2)} U_5 U_8 + U_3 U_6 \right. \\
& \qquad \left. + \frac{s_2}{2s_1 + s_2} U_4 U_6 - U_1 U_3 + \frac{4s_2}{2s_1 + s_2} U_1 U_4 + \frac{6s_2}{2s_1 - s_2} U_2 U_5 + (s_1 - s_2) X_4 \right\} \\
& + \frac{\cos(p_1 - p_2)}{s_1 - s_2} \left\{ - \frac{6s_1 s_2}{(2s_1 + s_2)(s_1 + 2s_2)} U_4 U_7 + \frac{2s_1}{s_1 - 2s_2} U_3 U_8 - U_3 U_6 \right. \\
& \qquad \left. + \frac{s_2}{2s_1 - s_2} U_5 U_6 - U_1 U_3 - \frac{4s_2}{2s_1 - s_2} U_1 U_5 - \frac{6s_2}{2s_1 + s_2} U_2 U_4 + (s_1 + s_2) X_5 \right\} \\
& + \frac{\cos(3p_1 + p_2)}{3s_1 + s_2} \left\{ \frac{10 s_1 s_2}{(2s_1 - s_2)(s_1 + 2s_2)} U_5 U_7 + \frac{s_2}{2s_1 + s_2} U_4 U_6 - 3U_2 U_3 \right. \\
& \qquad \left. + \frac{2s_2}{2s_1 + s_2} U_1 U_4 + (3s_1 - s_2) X_6 \right\} \\
& + \frac{\cos(3p_1 - p_2)}{3s_1 - s_2} \left\{ \frac{10 s_1 s_2}{(2s_1 + s_2)(s_1 - 2s_2)} U_4 U_8 + \frac{s_2}{2s_1 - s_2} U_5 U_6 - 3U_2 U_3 \right. \\
& \qquad \left. - \frac{2s_2}{2s_1 - s_2} U_1 U_5 + (3s_1 + s_2) X_7 \right\} \Bigg]
\end{aligned}$$

$$\begin{aligned}
+ q_1 q_2 \Bigg[& \cos 2p_1 \left\{ - \frac{2}{2s_1 + s_2} U_4 U_9 + \frac{2}{2s_1 - s_2} U_5 U_9 + \frac{8s_2}{(s_1 + 2s_2)(s_1 - 2s_2)} U_7 U_8 \right. \\
& \qquad \left. - \frac{2}{2s_1 + s_2} U_3 U_4 - \frac{2}{2s_1 - s_2} U_3 U_5 + X_9 \right\} \\
& + \cos 2p_2 \left\{ \frac{2}{s_1 + 2s_2} U_6 U_7 - \frac{2}{s_1 - 2s_2} U_6 U_8 + \frac{2}{s_1 + 2s_2} U_1 U_7 + \frac{2}{s_1 - 2s_2} U_1 U_8 \right. \\
& \qquad \left. + \frac{8s_1}{(2s_1 - s_2)(2s_1 + s_2)} U_4 U_5 - X_{10} \right\} \\
& + \frac{\cos(2p_1 + 2p_2)}{2s_1 + 2s_2} \left\{ \frac{-2s_1}{2s_1 + s_2} U_4 U_9 + \frac{6s_1}{2s_1 - s_2} U_5 U_{10} + \frac{4s_2}{s_1 + 2s_2} U_6 U_7 \right. \\
& \qquad \left. + \frac{2s_2}{s_1 + 2s_2} U_1 U_7 + \frac{6s_2}{s_1 - 2s_2} U_2 U_8 - \frac{4s_1}{2s_1 + s_2} U_3 U_4 + (2s_1 - 2s_2) X_{11} \right\} \\
& + \frac{\cos(2p_1 - 2p_2)}{2s_1 - 2s_2} \left\{ \frac{2s_1}{2s_1 - s_2} U_5 U_9 - \frac{6s_1}{2s_1 + s_2} U_4 U_{10} + \frac{4s_2}{s_1 - 2s_2} U_6 U_8 \right. \\
& \qquad \left. - \frac{2s_2}{s_1 - 2s_2} U_1 U_8 - \frac{6s_2}{s_1 + 2s_2} U_2 U_7 - \frac{4s_1}{2s_1 - s_2} U_3 U_5 + (2s_1 + 2s_2) X_{12} \right\} \Bigg]
\end{aligned}$$

$$
\begin{aligned}
&+q_1^{\frac{1}{2}}q_2^{\frac{3}{2}}\Bigg[\frac{\cos(p_1+p_2)}{s_1+s_2}\Big\{U_6U_9-\frac{4s_1}{s_1+2s_2}U_7U_9+\frac{6s_1}{s_1-2s_2}U_8U_{10}-U_3U_6-\frac{s_1}{s_1+2s_2}U_3U_7\\
&\qquad\qquad+\frac{2s_2}{2s_1+s_2}U_4U_6+\frac{6s_1s_2}{(2s_1-s_2)(s_1-2s_2)}U_5U_8+(s_1-s_2)X_{13}\Big\}\\
&\quad+\frac{\cos(p_1-p_2)}{s_1-s_2}\Big\{-U_6U_9-\frac{6s_1}{s_1+2s_2}U_7U_{10}+\frac{4s_1}{s_1-2s_2}U_8U_9-U_3U_6\\
&\quad-\frac{s_1}{s_1-2s_2}U_3U_8-\frac{6s_1s_2}{(s_1+2s_2)(2s_1+s_2)}U_4U_7-\frac{2s_2}{2s_1-s_2}U_5U_6+(s_1+s_2)X_{14}\Big\}\\
&\quad+\frac{\cos(p_1+3p_2)}{s_1+3s_2}\Big\{3U_6U_{10}-\frac{2s_1}{s_1+2s_2}U_7U_9-\frac{s_1}{s_1+2s_2}U_3U_7\\
&\qquad\qquad+\frac{10s_1s_2}{(s_1-2s_2)(2s_1+s_2)}U_4U_8+(s_1-3s_2)X_{15}\Big\}\\
&\quad+\frac{\cos(p_1-3p_2)}{s_1-3s_2}\Big\{-3U_6U_{10}+\frac{2s_1}{s_1-2s_2}U_8U_9-\frac{s_1}{s_1-2s_2}U_3U_8\\
&\qquad\qquad-\frac{10s_1s_2}{(s_1+2s_2)(2s_1-s_2)}U_5U_7+(s_1+3s_2)X_{16}\Big\}\Bigg]\\
&+q_2^2\left[\left\{\frac{1}{s_1+2s_2}U_6U_7+\frac{1}{s_1-2s_2}U_6U_8-X_{18}\right\}\cos 2p_2+\left\{\frac{s_1}{(s_1-2s_2)(s_1+2s_2)}U_7U_8-X_{19}\right\}\cos 4p_2\right]\\
&+\text{terms of the 5th and higher orders in }\sqrt{q_1}\text{ and }\sqrt{q_2}\qquad . \quad . \quad . \quad . \quad . \quad (5)
\end{aligned}
$$

The terms of higher order in the series may be determined in the same way as the terms in ϕ_3 and ϕ_4, and we thus obtain the complete expansion of ϕ.

We may note that instead of assuming $(s_1q_1-s_2q_2)$ as the lowest term of our integral, we might have assumed q_1, or q_2, or any linear function of q_1 and q_2; the integral then obtained would be merely a linear combination of our integral (5) with the integral of energy, whose lowest terms are $(s_1q_1+s_2q_2)$.

We may further note that in the above process, when finding ϕ_4, we may if we please add to ϕ_4 any terms of the form $\alpha q_1^2+\beta q_1q_2+\gamma q_2^2$, where α, β, γ are constants; for these terms are annulled by the operator $\left(s_1\frac{\partial}{\partial p_1}+s_2\frac{\partial}{\partial p_2}\right)$, and therefore ϕ_4 satisfies its differential equation just as well when these terms are present as when they are absent. The introduction of these terms into ϕ_4 will cause changes in the terms of higher order—in ϕ_5, ϕ_6, etc.: and the sum total of all the changes will merely amount to adding to our function ϕ a quadratic function of the two integrals which we know, namely, the integral of energy and the integral (5) itself.

Similarly we may add any terms of the form $(\alpha q_1^3+\beta q_1^2q_2+\gamma q_1q_2^2+\delta q_2^3)$ to ϕ_6: the ultimate effect is merely to add to our integral a cubic function of itself and the integral of energy. There is evidently nothing to be gained by doing this, and we may therefore omit these arbitrary terms in $\phi_4, \phi_6, \phi_8, \ldots$

§ 5. *An example of the integral found in § 4, with remarks on its convergence.*—As an example, consider the dynamical system which is specified by the Hamiltonian function

$$\mathrm{H} = 2^{\frac{1}{2}}q_1 \sin^2 p_1 + q_2 \sin^2 p_2 - \frac{1 + 3.2^{\frac{1}{4}}q_1^{\frac{1}{2}} \cos p_1}{3(1 + 2^{\frac{5}{4}}q_1^{\frac{1}{2}} \cos p_1 + 2^{\frac{1}{2}}q_1 \cos^2 p_1 + 2q_2 \cos^2 p_2)^{\frac{3}{2}}},$$

or expanding,

$$\begin{aligned} \mathrm{H} = 2^{\frac{1}{2}}q_1 + q_2 + 2^{\frac{7}{4}}q_1^{\frac{3}{2}}(-\cos p_1 - \tfrac{1}{3}\cos 3p_1) \\ + 2^{-\frac{3}{4}}q_1^{\frac{1}{2}}q_2\{-2\cos p_1 - \cos(p_1 + 2p_2) - \cos(p_1 - 2p_2)\} + \ldots \end{aligned} \quad (6)$$

The corresponding integral, obtained by substituting in formula (5) is

$$\begin{aligned} \text{Constant} = \phi \equiv 2^{\frac{1}{2}}q_1 - q_2 + 2^{\frac{7}{4}}q_1^{\frac{3}{2}}(-\cos p_1 - \tfrac{1}{3}\cos 3p_1) \\ + 2^{-\frac{3}{4}}q_1^{\frac{1}{2}}q_2\{-2\cos p_1 + (1 - \sqrt{2})^2 \cos(p_1 + 2p_2) + (1 + \sqrt{2})^2 \cos(p_1 - 2p_2)\} + \ldots \end{aligned} \quad (7)$$

Now it may be verified readily by differentiation that this dynamical system possesses the integral

$$\begin{aligned} \text{Constant} = (q_2^{\frac{1}{2}} \sin p_2 + 2^{\frac{1}{4}}q_1^{\frac{1}{2}}q_2^{\frac{1}{2}} \sin p_2 \cos p_1 - 2^{\frac{3}{4}}q_1^{\frac{1}{2}}q_2^{\frac{1}{2}} \sin p_1 \cos p_2)^2 \\ - \frac{1 + 2^{\frac{1}{4}}q_1^{\frac{1}{2}} \cos p_1}{(1 + 2^{\frac{5}{4}}q_1^{\frac{1}{2}} \cos p_1 + 2^{\frac{1}{2}}q_1 \cos^2 p_1 + 2q_2 \cos^2 p_2)^{\frac{1}{2}}}, \end{aligned}$$

which when expanded takes the form

$$\begin{aligned} \text{Constant} = q_2 + 2^{-\frac{1}{4}}(1 - \sqrt{2})q_1^{\frac{1}{2}}q_2 \cos(p_1 + 2p_2) \\ - 2^{\frac{1}{4}}(1 + \sqrt{2})q_1^{\frac{1}{2}}q_2 \cos(p_1 - 2p_2) + \ldots \end{aligned} \quad (8)$$

It is evident, on comparing the series, that the series (7) is what would be obtained by subtracting twice the series (8) from the series (6), which represents the integral of energy. This shows that for the particular dynamical system we are considering, the ϕ-series (5) is identical with the expansion, formed by ordinary algebraic and trigonometric processes under conditions which ensure convergence, of a known integral: and the convergence of the series (5), for sufficiently small values of $\sqrt{q_1}$ and $\sqrt{q_2}$, is thereby established for this particular system.

It is by considering particular dynamical systems such as this, in which the convergence of the series can be proved, that I have formed the opinion that the series (5) is in general convergent, for sufficiently small values of q_1 and q_2, so long as the ratio s_1/s_2 is an irrational number. A general proof of its convergence would probably be very difficult, and I have not as yet succeeded in obtaining one. But the following considerations may be adduced in support of the opinion of convergence.

Since the ratio s_1/s_2 is an irrational number, none of the denominators $(s_1 + s_2)$, $(s_1 - s_2)$, $(2s_1 + s_2)$, $(2s_1 - s_2)$, $(s_1 + 2s_2)$, $(3s_1 + s_2)$, ... can vanish, and therefore no term of the series can be infinite. The series is a power-series in $\sqrt{q_1}$ and $\sqrt{q_2}$, and it has been derived from another absolutely convergent power-series in $\sqrt{q_1}$ and $\sqrt{q_2}$, namely, the series for H, by operations which are of an ordinary algebraical and trigonometrical combinatory character, except as regards the operation of introducing the divisors of the type (ms_1+ns_2) (where m and n are positive or negative integers) in the integrations. We may therefore expect that the series will converge for sufficiently small values of $\sqrt{q_1}$ and $\sqrt{q_2}$, unless the smallness of some of these divisors causes the series to diverge for all values of $\sqrt{q_1}$ and $\sqrt{q_2}$, however small. Now the values of the integers m and n may indeed be so chosen that the divisor (ms_1+ns_2) may be as small as we please: but $|m|$ and $|n|$ are then large, and since $|m|$ and $|n|$ are not greater than the order of the term, this small divisor can occur only in a term of high order, where it will be more or less neutralised by the high powers of $\sqrt{q_1}$ and $\sqrt{q_2}$: and it was in fact shown many years ago by Bruns * that this state of things is consistent with the absolute convergence of a series. The example given by Bruns was the series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{q_1{}^m q_2{}^n}{m-n\mathrm{A}}$$

where q_1 and q_2 are proper fractions and A is a positive irrational number, which is an algebraic number, *i.e.* a root of an irreducible algebraic equation

$$\mathrm{A}^s+\mathrm{G}_1\mathrm{A}^{s-1}+\mathrm{G}_2\mathrm{A}^{s-2}+\ \ldots\ +\mathrm{G}_n=0$$

with integer coefficients G. If we multiply the numerator and denominator of any term in Bruns' series by

$$(m-n\mathrm{A}')(m-n\mathrm{A}'')\ \ldots$$

where A′, A″, . . . are the other roots of the algebraic equation, then the denominator becomes a polynomial in m and n with integer coefficients: and as it is never zero, it must be at least equal to unity: while in the numerator we now have a polynomial in m and n of degree $(s-1)$: whence it follows at once that Bruns' series converges.

The series (5) is much more complicated than Bruns' series: and although the analogy so far as it goes is favourable to the convergence of (5), yet our opinion must rest mainly on the undoubted convergence of (5) in the case of particular systems where a test is possible.

§ 6. *Use of the integral found in* § 4 *in order to complete the integration of the system.*—Still restricting ourselves to Case I, in which the ratio s_1/s_2 is an irrational number, we now know two integrals of the dynamical system, namely, the integral of energy (which is obtained by

* *Astr. Nach.* **109** (1884), p. 215.

equating the Hamiltonian function to a constant) and the integral expressed by equation (5). But it is known * that if, in any conservative holonomic dynamical system with two degrees of freedom, we know one integral in addition to the integral of energy, the system can be completely integrated, *i.e.* we can find expressions for the co-ordinates and momenta (q_1, q_2, p_1, p_2) in terms of the time and 4 arbitrary constants of integration. We shall now perform this process, which incidentally will show that the integral (5) is the adelphic integral of the system.

If we add the integral of energy to the integral (5), and divide throughout by $2s_1$, we obtain

$$\begin{aligned} l_1 = q_1 &+ q_1^{\frac{3}{2}}\left\{\frac{1}{s_1}U_1 \cos p_1 + \frac{1}{s_1}U_2 \cos 3p_1\right\} \\ &+ q_1 q_2^{\frac{1}{2}}\left\{\frac{2}{2s_1+s_2}U_4 \cos(2p_1+p_2) + \frac{2}{2s_1+s_2}U_5 \cos(2p_1-p_2)\right\} \\ &+ q_1^{\frac{1}{2}} q_2\left\{\frac{1}{s_1}U_6 \cos p_1 + \frac{1}{s_1+2s_2}U_7 \cos(2p_2+p_1) + \frac{1}{s_1-2s_2}U_8 \cos(2p_2-p_1)\right\} \\ &+ \text{terms of the 4th and higher orders,} \end{aligned}$$

where l_1 denotes an arbitrary constant.

Similarly by subtracting the integral (5) from the integral of energy, and dividing by s_2, we obtain

$$\begin{aligned} l_2 = q_2 &+ q_1 q_2^{\frac{1}{2}}\left\{\frac{1}{s_2}U_3 \cos p_2 + \frac{1}{2s_1+s_2}U_4 \cos(2p_1+p_2) - \frac{1}{2s_1-s_2}U_5 \cos(2p_1-p_2)\right\} \\ &+ q_1^{\frac{1}{2}} q_2\left\{\frac{2}{s_1+2s_2}U_7 \cos(2p_2+p_1) - \frac{2}{s_1-2s_2}U_8 \cos(2p_2-p_1)\right\} \\ &+ q_2^{\frac{3}{2}}\left\{\frac{1}{s_2}U_9 \cos p_2 + \frac{1}{s_2}U_{10} \cos 3p_2\right\} + \text{terms of the 4th and higher orders,} \end{aligned}$$

where l_2 represents a second arbitrary constant.

It is an easy matter to obtain q_1 and q_2 from these equations in terms of (l_1, l_2, p_1, p_2) by successive approximation: in fact, the first approximation gives $q_1 = l_1$, $q_2 = l_2$, and the second approximation gives

$$\begin{aligned} q_1 = l_1 &- l_1^{\frac{3}{2}}\left\{\frac{1}{s_1}U_1 \cos p_1 + \frac{1}{s_2}U_2 \cos 3p_1\right\} \\ &- l_1 l_2^{\frac{1}{2}}\left\{\frac{2}{2s_1+s_2}U_4 \cos(2p_1+p_2) + \frac{2}{2s_1-s_2}U_5 \cos(2p_1-p_2)\right\} \\ &- l_1^{\frac{1}{2}} l_2\left\{\frac{1}{s_1}U_6 \cos p_1 + \frac{1}{s_1+2s_2}U_7 \cos(2p_2+p_1) + \frac{1}{s_1-2s_2}U_8 \cos(2p_2-p_1)\right\} \\ &+ \text{terms of the 4th and higher order in } \sqrt{l_1} \text{ and } \sqrt{l_2}, \end{aligned}$$

* *Cf.*, *e.g.*, *Analytical Dynamics*, § 121.

$$q_2 = l_2 - l_1 l_2^{\frac{1}{2}}\left\{\frac{1}{s_2}\mathrm{U}_3\cos p_2 + \frac{1}{2s_1+s_2}\mathrm{U}_4\cos(2p_1+p_2) - \frac{1}{2s_1-s_2}\mathrm{U}_5\cos(2p_1-p_2)\right\}$$
$$- l_1^{\frac{1}{2}} l_2\left\{\frac{2}{s_1+2s_2}\mathrm{U}_7\cos(2p_2+p_1) - \frac{2}{s_1-2s_2}\mathrm{U}_8\cos(2p_2-p_1)\right\}$$
$$- l_2^{\frac{3}{2}}\left\{\frac{1}{s_2}\mathrm{U}_9\cos p_2 + \frac{1}{s_2}\mathrm{U}_{10}\cos 3p_2\right\}$$
$$+ \text{terms of the 4th and higher order in } \sqrt{l_1} \text{ and } \sqrt{l_2}.$$

We know from the general theory of Dynamics * that the expressions thus found for q_1 and q_2 must be the partial differential coefficients with respect to p_1 and p_2 of some function of (l_1, l_2, p_1, p_2): and, in fact, we have obviously

$$q_1 = \frac{\partial \mathrm{W}}{\partial p_1}, \quad q_2 = \frac{\partial \mathrm{W}}{\partial p_2},$$

where

$$\mathrm{W} = l_1 p_1 + l_2 p_2 - l_1^{\frac{3}{2}}\left(\frac{1}{s_1}\mathrm{U}_1\sin p_1 + \frac{1}{3s_1}\mathrm{U}_2\sin 3p_1\right)$$
$$- l_1 l_2^{\frac{1}{2}}\left\{\frac{1}{s_2}\mathrm{U}_3\sin p_2 + \frac{1}{2s_1+s_2}\mathrm{U}_4\sin(2p_1+p_2) + \frac{1}{2s_1-s_2}\mathrm{U}_5\sin(2p_1-p_2)\right\}$$
$$- l_1^{\frac{1}{2}} l_2\left\{\frac{1}{s_1}\mathrm{U}_6\sin p_1 + \frac{1}{2s_2+s_1}\mathrm{U}_7\sin(2p_2+p_1) + \frac{1}{2s_2-s_1}\mathrm{U}_8\sin(2p_2-p_1)\right\}$$
$$- l_2^{\frac{3}{2}}\left\{\frac{1}{s_2}\mathrm{U}_9\sin p_2 + \frac{1}{3s_2}\mathrm{U}_{10}\sin 3p_2\right\}$$
$$+ \text{terms of the 4th and higher orders in } \sqrt{l_1} \text{ and } \sqrt{l_2} \quad . \quad . \quad . \quad (9)$$

The terms in which p_1 and p_2 occur otherwise than in the arguments of trigonometric functions are

$$p_1(l_1 + \text{terms of the 4th and higher order in } \sqrt{l_1} \text{ and } \sqrt{l_2})$$
$$+ p_2(l_2 + \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad)$$

Denote the coefficients of p_1 and p_2 in this expression by a_1 and a_2 respectively: express l_1 and l_2 in terms of a_1 and a_2 by reversion of series, and replace l_1 and l_2 throughout in the series (9) by these values in terms of a_1 and a_2; so that we now have

$$\mathrm{W} = a_1 p_1 + a_2 p_2 - a_1^{\frac{3}{2}}\left(\frac{1}{s_1}\mathrm{U}_1\sin p_1 + \frac{1}{3s_1}\mathrm{U}_2\sin 3p_1\right)$$
$$- a_1 a_2^{\frac{1}{2}}\left\{\frac{1}{s_2}\mathrm{U}_3\sin p_2 + \frac{1}{2s_1+s_2}\mathrm{U}_4\sin(2p_1+p_2) + \frac{1}{2s_1-s_2}\mathrm{U}_5\sin(2p_1-p_2)\right\}$$
$$- a_1^{\frac{1}{2}} a_2\left\{\frac{1}{s_1}\mathrm{U}_6\sin p_1 + \frac{1}{2s_2+s_1}\mathrm{U}_7\sin(2p_2+p_1) + \frac{1}{2s_2-s_1}\mathrm{U}_8\sin(2p_2-p_1)\right\}$$
$$- a_2^{\frac{3}{2}}\left\{\frac{1}{s_2}\mathrm{U}_9\sin p_2 + \frac{1}{3s_2}\mathrm{U}_{10}\sin 3p_2\right\}$$
$$+ \text{terms of the 4th and higher orders in } \sqrt{a_1} \text{ and } \sqrt{a_2} \quad . \quad . \quad . \quad (10)$$

* *Cf.*, *e.g.*, *Analytical Dynamics*, § 121.

and now p_1 and p_2 do not occur except in the arguments of trigonometric functions and in the terms $(\alpha_1 p_1 + \alpha_2 p_2)$.

Now the equations

$$q_1 = \frac{\partial W}{\partial p_1}, \quad q_2 = \frac{\partial W}{\partial p_2}, \quad \beta_1 = \frac{\partial W}{\partial \alpha_1}, \quad \beta_2 = \frac{\partial W}{\partial \alpha_2}$$

define a contact-transformation from the variables (q_1, q_2, p_1, p_2) to the variables $(\alpha_1, \alpha_2, \beta_1, \beta_2)$: so in terms of these new variables the differential equations take the form

$$\frac{d\alpha_1}{dt} = \frac{\partial H}{\partial \beta_1}, \quad \frac{d\alpha_2}{dt} = \frac{\partial H}{\partial \beta_2}, \quad \frac{d\beta_1}{dt} = -\frac{\partial H}{\partial \alpha_1}, \quad \frac{d\beta_2}{dt} = -\frac{\partial H}{\partial \alpha_2} \qquad (11)$$

But we know that $\alpha_1 = \text{Constant}$ and $\alpha_2 = \text{Constant}$ are two of the integrals of the system, since l_1 and l_2 are constant: and therefore

$$\frac{\partial H}{\partial \beta_1} = 0, \quad \frac{\partial H}{\partial \beta_2} = 0,$$

so when H is expressed in terms of $(\alpha_1, \alpha_2, \beta_1, \beta_2)$, it will be found to involve α_1 and α_2 only: and then the second pair of equations (11) give

$$\left.\begin{aligned} \beta_1 &= -\frac{\partial H(\alpha_1, \alpha_2)}{\partial \alpha_1} t + \epsilon_1 \\ \beta_2 &= -\frac{\partial H(\alpha_1, \alpha_2)}{\partial \alpha_2} t + \epsilon_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where ϵ_1 and ϵ_2 are arbitrary constants.

Thus we have the complete solution of the dynamical system expressed by the equations

$$\frac{\partial W}{\partial p_1} = q_1, \quad \frac{\partial W}{\partial p_2} = q_2,$$
$$\frac{\partial W}{\partial \alpha_1} = -\frac{\partial H(\alpha_1, \alpha_2)}{\partial \alpha_1} t + \epsilon_1, \quad \frac{\partial W}{\partial \alpha_2} = -\frac{\partial H(\alpha_1, \alpha_2)}{\partial \alpha_2} t + \epsilon_2,$$

where W is given by equation (10), and the four arbitrary constants of integration are $(\alpha_1, \alpha_2, \epsilon_1, \epsilon_2)$. On referring to the form of W, we see that these equations enable us to express q_1 and q_2 as purely trigonometric series, the arguments of the trigonometric functions being of the form

$$m\beta_1 + n\beta_2,$$

where m and n are integers (positive, negative, or zero) and where β_1 and β_2 are linear functions of the time, given by equations (12). *We have thus obtained expressions for the co-ordinates in terms of the time, by means of series in which the time occurs only in the arguments of trigonometric functions.*

It is moreover evident that a change in ϵ_1, in which the other constants of integration $(\epsilon_2, \alpha_1, \alpha_2)$ are left unaltered, does not affect either of the constants l_1 and l_2 (since these depend only on a_1 and a_2) and therefore does not affect the constant of the integral (5) or the constant of energy: this shows that all the orbits, which differ from each other only in having different values of the constant ϵ_1, have the same values for the constant of the integral (5) and the constant of energy: and hence that the infinitesimal transformation which corresponds to the integral (5) transforms these orbits into each other: that is to say, *the integral* (5) *is the adelphic integral of the dynamical system.*

§ 7. *Determination of the adelphic integral in Case II.*—We now proceed to the discussion of "Case II," in which the ratio s_1/s_2 is a rational number (say equal to m/n), but no term in cos $(np_1 - mp_2)$ is present among the third-order terms in the Hamiltonian function H. Certain terms of the series (5) now contain in their denominators the factor $(ns_1 - ms_2)$, which vanishes since $s_1/s_2 = m/n$: and therefore the series (5) as it stands cannot converge in Case II, unless the terms which have zero denominators have numerators which also vanish. We have here come upon the real root of the principal difficulty of Celestial Mechanics: by removing it here, so as to obtain a valid adelphic integral in Cases II and III, we shall be enabled to remove it from the whole subject.

To fix ideas, we shall suppose that $s_1 = 2$, $s_2 = 1$, so that s_1/s_2 has the rational value 2, and the denominator $(s_1 - 2s_2)$, which occurs frequently in the series (5), is zero.

In this case the equation for ϕ_3 becomes

$$2\frac{\partial\phi_3}{\partial p_1} + \frac{\partial\phi_3}{\partial p_2} = 2\frac{\partial \mathrm{H}_3}{\partial p_1} - \frac{\partial \mathrm{H}_3}{\partial p_2},$$

and indeed the equation for any one of the functions ϕ_3, ϕ_4, ϕ_5, . . . takes the form

$$2\frac{\partial\phi_r}{\partial p_1} + \frac{\partial\phi_r}{\partial p_2} = \text{ a known sum of terms of the type } q_1^{\frac{1}{2}m'}q_2^{\frac{1}{2}n'}\sin(kp_1 + lp_2).$$

Now in integrating the differential equations for ϕ_3, ϕ_4 . . . in § 4, we used only the "particular integral," which corresponds term-by-term to the known function on the right-hand side of the equation: so that, *e.g.*, the integral of the equation

$$s_1\frac{\partial\phi_3}{\partial p_1} + s_2\frac{\partial\phi_3}{\partial p_2} = q_1^{\frac{3}{2}}\sin p_1$$

would be taken to be

$$\phi_3 = -\frac{q_1^{\frac{3}{2}}}{s_1}\cos p_1.$$

The reason for this was that the "complementary function," or arbitrary part of the solution of the differential equation, is a function of $(s_2p_1 - s_1p_2)$, and so does not contain terms of the type appropriate to ϕ_3. But when $s_1=2$, $s_2=1$, the arbitrary part of the solution of the differential equation *does* contain terms of the type proper to ϕ_3, and these must be taken account of; so that we must take the integral of the equation

$$2\frac{\partial\phi_3}{\partial p_1}+\frac{\partial\phi_3}{\partial p_2}=q_1^{\frac{3}{2}}\sin p_1$$

to be

$$\phi_3=-\tfrac{1}{2}q_1^{\frac{3}{2}}\cos p_1+aq_1^{\frac{1}{2}}q_2\cos(p_1-2p_2),$$

where a is an arbitrary constant. *In this way we obtain terms with arbitrary coefficients in ϕ_3, ϕ_4, ϕ_5, . . .: and these arbitrary coefficients must be chosen in such a way as to remove terms with vanishing denominators from the subsequently determined part of ϕ.* This principle enables us to obtain, in Case II, an adelphic integral free from vanishing denominators.

§ 8. *Study of a particular dynamical system, as an illustration of the method of* § 7.—We shall now illustrate the working of this principle by an example. Consider the dynamical system which is specified by the Hamiltonian function

$$\left.\begin{aligned}\mathrm{H}=2q_1\sin^2 p_1+q_2\sin^2 p_2+\frac{1}{2(1+2q_1^{\frac{1}{2}}\cos p_1+q_1\cos^2 p_1+2q_2\cos^2 p_2)^2}\\ -\frac{1+q_1^{\frac{1}{2}}\cos p_1}{(1+2q_1^{\frac{1}{2}}\cos p_1+q_1\cos^2 p_1+2q_2\cos^2 p_2)^{\frac{3}{2}}}\end{aligned}\right\}\quad(13)$$

If this be expanded in ascending powers of $\sqrt{q_1}$ and $\sqrt{q_2}$, we obtain

$$\begin{aligned}\mathrm{H}=2q_1+q_2+q_1^{\frac{3}{2}}(-\tfrac{9}{2}\cos p_1-\tfrac{3}{2}\cos 3p_1)+q_1^2(\tfrac{75}{16}+\tfrac{25}{4}\cos 2p_1+\tfrac{25}{16}\cos 4p_1)\\ +q_1q_2\{-3-3\cos 2p_1-3\cos 2p_2-\tfrac{3}{2}\cos(2p_1+2p_2)-\tfrac{3}{2}\cos(2p_1-2p_2)\}\\ +q_2^2\{-\tfrac{9}{16}-\tfrac{3}{4}\cos 2p_2-\tfrac{3}{16}\cos 4p_2\}+\text{terms of the 5th and higher order}\\ \text{in }\sqrt{q_1}\text{ and }\sqrt{q_2},\end{aligned}$$

so that in this case $s_1=2$, $s_2=1$.

As explained at the end of § 4, we may assume that the lowest term of the adelphic integral is simply q_2. Then if we write

$$\phi=q_2+\phi_3+\phi_4+\phi_5+\ .\ .\ .$$

the equation to determine ϕ_3 is

$$2\frac{\partial\phi_3}{\partial p_1}+\frac{\partial\phi_3}{\partial p_2}=0,$$

so by § 7,

$$\phi_3=aq_1^{\frac{1}{2}}q_2\cos(p_1-2p_2),$$

where a is an arbitrary constant.

The equation for ϕ_4 now becomes

$$\begin{aligned}2\frac{\partial\phi_4}{\partial p_1}+\frac{\partial\phi_4}{\partial p_2}=q_1q_2\left\{\left(6+\frac{9a}{2}\right)\sin 2p_2+\left(3+\frac{9a}{4}\right)\sin(2p_1+2p_2)-\left(3+\frac{9a}{4}\right)\sin(2p_1-2p_2)\right\}\\ +q_2^2(\tfrac{3}{2}\sin 2p_2+\tfrac{3}{2}\sin 4p_2)\end{aligned}$$

of which the integral is

$$\phi_4 = q_1 q_2\left\{-\left(3+\frac{9\alpha}{4}\right)\cos 2p_2 - \left(\tfrac{1}{2}+\frac{3\alpha}{8}\right)\cos(2p_1+2p_2) + \left(\tfrac{3}{2}+\frac{9\alpha}{8}\right)\cos(2p_1-2p_2)\right\}$$
$$+ q_2^2\left(-\tfrac{3}{4}\cos 2p_2 - \tfrac{3}{16}\cos 4p_2\right).$$

The equation to determine ϕ_5 is now

$$2\frac{\partial\phi_5}{\partial p_1} + \frac{\partial\phi_5}{\partial p_2} = \frac{\partial H_5}{\partial p_2} + (\phi_3, H_4) + (\phi_4, H_3),$$

and we have to choose α so as to annul the terms in $\sin(p_1 - 2p_2)$ on the right-hand side of this equation. On calculating these terms, we find

$\left(\text{from } \frac{\partial H_5}{\partial p_2}\right)$	$\frac{39}{2} q_1^{\frac{3}{2}} q_2 \sin(p_1 - 2p_2)$
(from (ϕ_4, H_3))	$-\frac{45}{2}\left(1+\frac{3\alpha}{4}\right) q_1^{\frac{3}{2}} q_2 \sin(p_1 - 2p_2)$
(from (ϕ_3, H_4))	$+\frac{123}{8}\alpha q_1^{\frac{3}{2}} q_2 \sin(p_1 - 2p_2)$.

The quantity α must therefore satisfy the equation

$$\tfrac{39}{2} - \tfrac{45}{2}\left(1+\frac{3\alpha}{4}\right) + \tfrac{123}{8}\alpha = 0,$$

which gives

$$\alpha = -2.$$

Substituting this value of α in ϕ_3 and ϕ_4 our integral becomes

$$\text{Constant} = q_2 - 2q_1^{\frac{1}{2}} q_2 \cos(p_1 - 2p_2)$$
$$+ q_1 q_2\left\{\tfrac{3}{2}\cos 2p_2 + \tfrac{1}{4}\cos(2p_1+2p_2) - \tfrac{3}{4}\cos(2p_1-2p_2)\right\} + q_2^2\left(-\tfrac{3}{4}\cos 2p_2 - \tfrac{3}{16}\cos 4p_2\right)$$
$$+ \text{terms of the 5th and higher orders in } \sqrt{q_1} \text{ and } \sqrt{q_2} \quad . \quad . \quad . \quad . \quad (14)$$

Now it may be verified by differentiation that the dynamical system specified by equation (13) possesses the integral

$$\text{Constant} = \tfrac{1}{2}\{\sqrt{2q_2}\sin p_2 + q_1^{\frac{1}{2}}\sqrt{2q_2}\cos p_1 \sin p_2 - 2\sqrt{2q_1q_2}\sin p_1 \cos p_2\}^2$$
$$- \frac{1 + q_1^{\frac{1}{2}}\cos p_1}{(1 + 2q_1^{\frac{1}{2}}\cos p_1 + q_1\cos^2 p_1 + 2q_2\cos^2 p_2)^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad (15)$$

and this integral is adelphic, as may be shown by completing the solution, or more simply by remarking that the integral (15) is a function of the variables $(\sqrt{q_1}, \sqrt{q_2}, p_1, p_2)$ which is one-valued and free from singularities for a certain range of values, and therefore the infinitesimal transformation corresponding to it will also be one-valued and free from singularities, and so must transform closed orbits into closed orbits.

But *on expanding this integral* (15) *in ascending powers of* $\sqrt{q_1}$ *and* $\sqrt{q_2}$ *by the multinomial theorem, we arrive at the series* (14). This shows that, *for the dynamical system we are considering, the series obtained by the process of* § 7 *converges for all real values of* p_1 *and* p_2 *so long as* $|q_1|$

and $|q_2|$ *are inferior to certain fixed quantities, and that the series represents the adelphic integral of the dynamical system.*

§ 9. *Determination of the adelphic integral in Case III.*—The principle for the removal of vanishing divisors from the adelphic integral, which was explained in § 7 and illustrated in § 8, is not sufficient for the purpose if the Hamiltonian function contains, among its third-order terms, a term in $\cos(s_2p_1 - s_1p_2)$: for this term gives rise to a vanishing divisor in ϕ_3, and the arbitrary terms which are used in order to annul terms with vanishing divisors do not come into operation early enough to remove vanishing divisors from ϕ_3.

In this "Case III" we must make use of another principle (concurrently with the principle of § 7) which may be explained thus: Suppose that an integral of a system of differential equations in variables (q_1, q_2, p_1, p_2) is of the form

$$f(q_1, q_2, p_1, p_2) + \frac{g(q_1, q_2, p_1, p_2)}{\mu} = \gamma$$

where γ is the *arbitrary* constant and μ is a *definite* constant formed of quantities occurring in the differential equations. The integral in this form ceases to have a meaning when μ tends to zero. But we may derive from it an integral which has a meaning when $\mu \to 0$, by merely supposing first that μ is different from zero, and multiplying the equation throughout by μ, so that it becomes

$$\mu f(q_1, q_2, p_1, p_2) + g(q_1, q_2, p_1, p_2) = \mu\gamma$$

and then making $\mu \to 0$; the equation becomes

$$g(q_1, q_2, p_1, p_2) = c,$$

where c denotes $\mathrm{Lt}_{\mu\to 0}(\mu\gamma)$. This is the desired form of the integral when μ is zero.

Our case is not so simple as this, since the vanishing divisor occurs not only in the inverse first power, but in an infinite series containing all the inverse powers. The method we follow, which will be illustrated in the next article, is really equivalent to using the principle of § 7 in order to remove all inverse powers of the small divisor except the first, and then using the principle of this article in order to remove this inverse first power.

§ 10. *Example of the principle of* § 9.—We shall now show by considering a particular dynamical system how the principle just mentioned is applied in order to obtain an adelphic integral free from vanishing divisors in "Case III."

Consider the dynamical system whose Hamiltonian function is

$$\mathrm{H} = q_1 - 2q_2 + q_1^{\frac{3}{2}}\mathrm{U}_1 \cos p_1 + q_1 q_2^{\frac{1}{2}}\mathrm{U}_4 \cos(2p_1 + p_2) \qquad (16)$$

Now if the Hamiltonian function is

$$H = s_1 q_1 + s_2 q_2 + q_1^{\frac{3}{2}} U_1 \cos p_1 + q_1 q_2^{\frac{1}{2}} U_4 \cos (2p_1 + p_2),$$

where s_1 and s_2 are arbitrary, the adelphic-integral series to which we are led by the method of § 4 is

$$\begin{aligned}
\text{Constant} = {} & s_1 q_1 - s_2 q_2 + q_1^{\frac{3}{2}} U_1 \cos p_1 + \frac{2s_1 - s_2}{2s_1 + s_2} q_1 q_2^{\frac{1}{2}} U_4 \cos (2p_1 + p_2) \\
& + \frac{s_2}{2s_1 + s_2} U_1 U_4 q_1^{\frac{3}{2}} q_2^{\frac{1}{2}} \left\{ \frac{4 \cos (p_1 + p_2)}{s_1 + s_2} + \frac{2 \cos (3p_1 + p_2)}{3s_1 + s_2} \right\} \\
& + U_1^2 U_4 q_1^2 q_2^{\frac{1}{2}} \frac{s_2}{2s_1 + s_2} \left\{ - \frac{3}{3s_1 + s_2} \frac{\cos (4p_1 + p_2)}{4s_1 + s_2} - \frac{6}{3s_1 + s_2} \frac{\cos (2p_1 + p_2)}{2s_1 + s_2} \right. \\
& \qquad \left. - \frac{6}{s_1 + s_2} \frac{\cos p_2}{s_2} \right\} \\
& + U_1 U_4^2 q_1^{\frac{3}{2}} q_2 \frac{s_2}{2s_1 + s_2} \left\{ \frac{2(9s_1 + s_2)}{(s_1 + s_2)(3s_1 + s_2)} \frac{\cos p_1}{s_1} + \frac{4}{s_1 + s_2} \frac{\cos (3p_1 + 2p_2)}{3s_1 + 2s_2} \right\} \\
& + U_1 U_4^2 q_1^{\frac{5}{2}} \frac{s_2}{2s_1 + s_2} \cdot \frac{5s_1 + s_2}{(3s_1 + s_2)(s_1 + s_2)} \cdot \frac{\cos p_1}{s_1} \\
& + \text{terms of the 6th and higher orders in } \sqrt{q_1} \text{ and } \sqrt{q_2} \quad . \quad . \quad (17)
\end{aligned}$$

In our problem $s_1 = 1$, $s_2 = -2$, so $2s_1 + s_2$ is a vanishing denominator. This denominator makes its appearance in the fourth term of the above expression, and occurs in every subsequent term, being squared in the coefficient of the fifth-order term $q_1^2 q_2^{\frac{1}{2}} \cos (2p_1 + p_2)$. We must now modify this series (17) so as to obtain an integral which has no vanishing denominators.

In the first place, we apply the principle of § 9: the lowest term which is affected by the vanishing denominator is the term

$$\frac{2s_1 - s_2}{2s_1 + s_2} q_1 q_2^{\frac{1}{2}} U_4 \cos (2p_1 + p_2):$$

we therefore try to form an integral whose lowest term (discarding the non-essential factors $(2s_1 - s_2)$ and U_4) shall be

$$q_1 q_2^{\frac{1}{2}} \cos (2p_1 + p_2).$$

If then we suppose this integral to be

$$\text{Constant} = \phi \equiv q_1 q_2^{\frac{1}{2}} \cos (2p_1 + p_2) + \phi_4 + \phi_5 + \phi_6 + \ \ldots$$

where ϕ_r denotes the terms of degree r in $\sqrt{q_1}$ and $\sqrt{q_2}$, and substitute in the equation $(\phi, H) = 0$, we find on equating to zero the terms of order 4 that ϕ_4 is to be determined from the equation

$$\frac{\partial \phi_4}{\partial p_1} - 2\frac{\partial \phi_4}{\partial p_2} = q_1^{\frac{3}{2}} q_2^{\frac{1}{2}} U_1 \{ 2 \sin (p_1 + p_2) + \sin (3p_1 + p_2) \}$$

The integral of this is

$$\phi_4 = q_1^{\frac{3}{2}} q_2^{\frac{1}{2}} U_1 \{ 2 \cos (p_1 + p_2) - \cos (3p_1 + p_2) \}$$

to which, however, we may add terms of the type

$$\alpha q_1^2 + \beta q_1 q_2 + \gamma q_2^2 \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where α, β, γ are arbitrary constants, since these terms satisfy the differential equation and are of the type proper to ϕ_4. It should be noticed that these terms are not now superfluous, as they were in the general case studied in § 4; for in the general case the addition of such terms to ϕ_4 would merely be equivalent to adding on an arbitrary quadratic function of the integral of energy and the adelphic integral: but in our present case the adelphic integral does not begin with terms linear in q_1 and q_2, and therefore a quadratic function of it does not account for terms like those in (18). The arbitrary constants in (18) are to be determined in such a way as to make terms with vanishing denominators disappear from the higher-order terms of ϕ. Thus, writing now

$$\phi_4 = q_1^{\frac{3}{2}} q_2^{\frac{1}{2}} U_1 \{2 \cos(p_1 + p_2) - \cos(3p_1 + p_2)\} + \alpha q_1^2$$

and substituting in the differential equation satisfied by ϕ_5 which is

$$\frac{\partial \phi_5}{\partial p_1} - 2\frac{\partial \phi_5}{\partial p_2} = (\phi_4, H_3) \quad . \quad . \quad . \quad . \quad . \quad (19)$$

we find that on the right-hand side of (19) the terms involving $\sin(2p_1 + p_2)$ (which would lead to vanishing denominators on integration) are

$$-3q_1^2 q_2^{\frac{1}{2}} U_1^2 \sin(2p_1 + p_2) - 4\alpha q_1^2 q_2^{\frac{1}{2}} U_4 \sin(2p_1 + p_2)$$

and these will collectively vanish provided

$$\alpha = -\frac{3}{4}\frac{U_1^2}{U_4}$$

In this way, by repeated application of the principle of § 7, we are able to remove all terms with vanishing denominators and obtain an adelphic integral free from them.

§ 11. *Completion of the integration of the dynamical system in Cases II and III.*—Having now in §§ 7–10 overcome the difficulty caused by the presence of terms with vanishing divisors in the adelphic integral in Cases II and III, we can use this integral in order to integrate the dynamical system completely, just as was done for Case I in § 6. We thus obtain expansions for the co-ordinates in terms of the time in all cases: but these expansions are completely different in form, according as the dynamical system falls under Case I, II, or III. This result supplies the underlying explanation of Poincaré's theorem that the series of Celestial Mechanics cannot converge uniformly over any continuous range of values of the constants: for the series to which he was referring were of the kind which we have classified under Case I, and we have seen that when the constants s_1, s_2 are continuously varied, these series must be replaced by the series appropriate to Case II or Case III, whenever the ratio s_1/s_2 passes from an irrational to a rational value. The advantage of solving by means of the adelphic integral is that the forms of the adelphic integral corresponding to the three cases can be readily determined: and thus the difficulty is removed before the adelphic integral is used in order to obtain the complete expressions for the co-ordinates in terms of the time.

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