

## ON THE REDUCTION OF SETS OF INTERVALS

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1. It has been already pointed out that, before sets of intervals were studied for their own sake, various writers had had occasion to make use of them, and had in this way virtually obtained the Heine-Borel theorem without, however, enunciating it. In particular this is true of Cantor, Heine, Dini, and Darboux, in their proofs of the theorem that the property of uniform continuity, first considered by Heine, is possessed by every continuous function.

We do not intend to enter further into these matters here.\* It is, however, not only interesting but of some importance to notice that a slight modification of Heine's proof enables us to obtain a theorem, including the Heine-Borel theorem as a special case. We have indeed only to retain that which is essential in Heine's argument and reject that which is accessory.

The new theorem† so obtained is not only more general than the Heine-Borel theorem; it leads to results unobtainable by application of the Heine-Borel theorem alone, and in particular to a number of theorems relating to intervals which have been at the base of much of the work in the modern theory of derivatives and their integrals. The new theorem accordingly may well be destined to introduce order into the somewhat heterogeneous collection of theorems of the type of the Heine-Borel theorem which have been formulated and employed from time to time by writers on the Theory of Functions of a Real Variable.

We propose in the present paper to state the theorem in question, and to modify Heine's reasoning in such a way as to render it applicable for the purposes of proof. For the convenience of readers, we give Heine's own brief demonstration *verbatim*. We then pass to the deduction as corollaries of the theorems relating to sets of intervals above referred to. It will be noticed that nowhere are transfinite numbers employed.

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\* The reader may refer to a footnote in a paper entitled "Note on Overlapping Regions," by W. H. Young (1912), *Messenger of Mathematics*, pp. 126, 127.

† See below, § 3.

2. As is well known, a continuous function  $f(x)$  has the property of uniform continuity; that is to say, given any positive quantity  $e$ , we can find a positive quantity  $d$ , so that, in any interval of length  $\leq d$  in the fundamental segment, the oscillation of the function  $\leq 2e$ ; in symbols it comes to the same thing if we say

$$|f(x+h) - f(x)| \leq e,$$

provided only

$$-d \leq h \leq d.$$

We proceed to reproduce Heine's proof of this property in the original.\*

“Bezeichnet  $3e$  eine beliebige Grösse, so existiert eine solche Zahl, dass von  $x = a$  bis zu ihr hin  $f(x) - f(a)$  absolut  $\leq 3e$  ist. Ein Wert der dies leistet, ist der grösste und macht zugleich  $f(x) - f(a) - 3e = 0$ . Dieser Wert sei  $x_1$ . In ähnlicher Art findet man eine Zahl  $x_2$  als die grösste, welche bewirkt, dass von  $x = x_1$  bis  $x = x_2$  immer  $|f(x) - f(x_1)| \leq 3e$  bleibt. So fährt man fort; kommt man nach einer endlichen Anzahl  $n$  von Operationen zu  $x_n = b$ , oder findet, dass  $|f(x) - f(x_{n-1})|$  von  $x = x_{n-1}$  bis  $x = b$  noch nicht  $3e$  überschreitet, so ist der Satz bewiesen.

“Es bleibt noch der Fall übrig, dass kein  $n$  existiert, dass also die Grössen  $x_1, x_2, \dots$  eine unendliche Reihe von wachsenden Grössen bilden, die unter  $b$  liegen. Diese Reihe wäre dann eine Zahlenreihe, deren Zahlzeichen  $X$  sei: hervorzuheben ist ihre Eigenschaft, nach der für jedes  $n$  die Gleichung besteht,  $f(x_{n+1}) - f(x_n) = 3e$ . Nun sei  $\eta_0$  von der Beschaffenheit, dass  $f(X)$  sich von  $f(X - \eta)$  um weniger als  $e$  unterscheidet, so lange  $\eta < \eta_0$ . Zwischen die Zahlen  $X - \eta_0$  und  $X$  mögen von der Zahlenreihe  $x_n, x_{n+1}, \dots$ , etc., fallen, so dass  $|f(x_{n+1}) - f(x_n)|$  kleiner als  $3e$  wäre, während andererseits es  $3e$  sein müsste. Die zu Grunde liegende Annahme ist daher unmöglich, und die Funktion ist eine gleichmässig continuirliche.”

The beginning of this proof contains the germ of a very powerful mode of treating sets of intervals. Heine has, corresponding to each point  $x$  of the closed interval  $(a, b)$ , a set of intervals on the right and on the left of  $x$ , with  $x$  as common end-point; these are the intervals for which

$$-3e \leq f(x+h) - f(x) \leq 3e \quad (0 < h)$$

on the right, and

$$-3e \leq f(x) - f(x-h) \leq 3e \quad (0 < h)$$

on the left.

These intervals, which, for fixed  $x$ , form a set of potency  $c$ , since they include all the intervals with  $x$  as end-point inside any one of them,

\* H. Heine, Oct., 1871, *Jour. für reine und angewandte Mathematik*, Vol. LXXIV, p. 188.

Heine replaces by an unique interval on each side of the point  $x$ , namely the smallest interval with  $x$  as end-point containing all the given intervals on the appropriate side. We may call these intervals  $r_x$  and  $l_x$ . These intervals have the property that some interval with  $x$  as end-point, whose other end-point is outside  $r_x$  but as near to it as we please, is not an interval of the given set associated with the point  $x$ ; and similarly on the left there is such an interval, not associated with  $x$  and containing  $l_x$ .

Having so reduced our intervals to an unique pair at each point, Heine starts with the left-hand point of the interval, the point  $a$ , and takes the corresponding interval  $r_a$ . Let  $x_1$  be its right-hand end-point; he then takes  $r_{x_1}$ , which abuts at  $x_1$  with  $r_a$ , and so on. He points out that, if this process comes to an end after a finite number of stages, that is, if we arrive at a point  $x_n$  which is either  $b$  or lies to the right of  $b$ , the theorem is proved. We have therefore only to discuss the possibility of the alternative hypothesis, that no such number  $n$  exists.

Suppose this is the case, then the abutting intervals  $r_a, r_{x_1}, r_{x_2}, \dots$  form an infinite set of abutting intervals, and their end-points  $a, x_1, x_2, \dots$  form a monotone sequence, proceeding towards the right. They therefore define uniquely a limiting point  $X$ , to the right of them all.

So far Heine's reasoning has been perfectly general and of a masterly character. At this point, however, he descends to the artifice of a proof *ad hoc*. The real point is hidden. This point consists in the fact that inside any interval whatever with  $X$  as right-hand end-point there is an infinite number of the points  $x_i$ , say

$$x_n, x_{n+1}, x_{n+2}, \dots;$$

and therefore there is certainly an interval  $(x_n, y_n)$ , where  $y_n$  lies between  $x_{n+1}$  and  $x_{n+2}$ , which is *not* an interval of the given set for either of its end-points  $x_n$  and  $y_n$ . Indeed it has been already pointed out that this is a consequence of the characteristic of the unique interval  $r_{x_n}$ .

On the other hand, there is an interval with  $X$  as right-hand end-point which is such that every interval inside it is an interval of the given set. This is not in general a property of sets of intervals associated with the points of an interval in the way we have been considering. It is a special property of the set of intervals which Heine was contemplating. It belongs, in fact, to the greatest interval throughout which

$$-e \leq f(X) - f(X-h) \leq e,$$

in virtue of the fact that

$$f(X-h_n) - f(X-h_{n+1}) = f(X) - f(X-h_{n+1}) - f(X) + f(X-h_n).$$

In fact there was no occasion to take  $3\epsilon$ , Heine might have taken  $2\epsilon$ ; he took  $3\epsilon$  because he was using the special property that the increment over any one of his unique intervals  $r_x$  and  $l_x$  was exactly  $3\epsilon$ , a property due to the continuity of  $f(x)$ . The introduction of this special property instead of the general one that as near as we please to the point  $x_{n+1}$  there is a point  $y_n$  such that the increment of  $f(x)$  over  $(x_n, y_n)$  is *greater* than  $3\epsilon$ , must, from our present point of view, be regarded as an artistic fault in Heine's proof.

The property which we have assumed in this latter part of Heine's proof is then that there is one of the given intervals on the left of  $X$  such that every interval inside it belongs to the given set. A particular consequence of this property is that the interval  $r_x$  corresponding to any point  $x'$  in this neighbourhood of the point  $X$ , reaches at least to  $X$ ; and this consequence is alone needed in the proof.

3. We are thus led to enunciate the following theorem.\* The proof of the theorem is taken almost verbatim from Heine, as will be seen by comparison with the above extract. For convenience of reference and classification the theorem, though more general than any contemplated or required by Heine, is therefore called the Heine-Young theorem, and a similar nomenclature is adopted for the simpler corollaries from it.

It should be noticed that the proof of the theorem and of its first corollary is independent of the principle of arbitrary choice, and that it is shown how in a unique manner the set of intervals required is to be constructed.

**THEOREM** (*The Heine-Young Theorem*).—*If with each point of a closed segment  $(a, b)$  we have associated a pair<sup>†</sup> of intervals  $r_x$  and  $l_x$ , such that*

- (i)  *$x$  is the left-hand end-point of  $r_x$  and the right-hand end-point of  $l_x$ ;*
- (ii) *if  $x'$  is an internal point of  $l_x$ , then  $x$  is an internal or end-point of  $r_x$ ;*

\* This theorem was first enunciated by W. H. Young in his course of lectures on "Integration and the Theory of Sets of Points," 1913, at the University of Liverpool. It is here published for the first time. The second corollary, which is here called the Heine-Lusin theorem, has recently been used explicitly by Lusin—indicating that it may be proved by Lebesgue's device (see below, § 11)—for the purpose of proving the important result that *a continuous function cannot have at every point of a set of positive content an infinite differential coefficient*. Lusin remarks that his theorem is not a special case of the Heine-Borel theorem, even in its generalised form. He does not, however, observe that it is virtually this theorem, rather than the Heine-Borel theorem, which is proved, but not enunciated, by Heine.

† In the case of the end-points  $a$  and  $b$ , the intervals  $r_a$  and  $l_b$  are not used, and need not exist.

then we can find a finite number of the intervals  $r_x$ , abutting end-to-end, and covering over the whole segment  $(a, b)$ .\*

By hypothesis there is an interval  $r_a$ , with  $a$  as left-hand end-point; let its right-hand end-point be  $x_1$ . Similarly there is an interval  $r_{x_1}$ , with  $x_1$  as left-hand end-point. Thus we proceed; if after a finite number  $n$  of these operations, we arrive at  $x_n = b$ , or if we find that on arriving at  $x_{n-1}$  the corresponding interval  $r_{x_{n-1}}$  contains  $b$ , the theorem is proved.

There remains only the case in which no such integer  $n$  exists, so that the points  $x_1, x_2, \dots$  form an infinite monotone sequence each lying to the right of the preceding, and all to the left of  $b$ . Let  $X$  be the limiting point. There is then an interval  $l_X$  with  $X$  as right-hand end-point. Inside  $l_X$  there will fall all but a finite number of the points of the sequence, since  $X$  is the limiting point and the sequence is monotone and on the left of  $X$ . Let  $x_n, x_{n+1}, \dots$  lie inside  $l_X$ . Then, since  $x_n$  is an internal or end-point of  $l_X$ ,  $r_{x_n}$  must reach at least as far as  $X$ , by the hypothesis (2). But this is not the case, since it reaches only to  $x_{n+1}$ . Thus the original supposition is untenable, and the succession of points  $x_1, x_2, \dots$  cannot be infinite. This proves the theorem.

**COR. 1** (*The Heine-Young Theorem for a half-open interval*).—If the interval  $(a, b)$  is a half open interval, open on the right, that is, if the conditions of the theorem hold for every point except the point  $b$ , so that the interval  $l_b$  does not exist, the theorem is still true if we substitute a countably infinite number of the intervals  $r_x$  for a finite number of them.

In fact the reasoning of the above proof shows that the point  $X$  can in this case only be the point  $b$ .

4. We can now prove the following theorem, due to Lusin, as an immediate corollary.

**COR. 2** (*The Heine-Lusin† Theorem*).—If associated with every point of a closed interval  $(a, b)$  we have all the intervals with  $x$  as end-point in a certain neighbourhood of the point  $x$  on both sides,‡ then a finite

\* It should be noticed that, though the conditions of the theorem refer to both  $l_x$  and  $r_x$ , the filling up of the segment is effected by the intervals  $r_x$  alone. The existence of the intervals  $l_x$  constitutes a restriction on the generality of the set of intervals  $r_x$ .

† See footnote \* on preceding page. N. Lusin, "Sur un théorème fondamentale du calcul intégral," 1911, *Recueil de la Société mathématique de Moscou*, Vol. xxviii, 2, in Russian.

‡ As before, the neighbourhoods of  $a$  and  $b$  may be taken to be only on that side of the point in question which is towards the interval  $(a, b)$ .

*number of these intervals can be found, abutting end-to-end, and covering in the segment  $(a, b)$ .*

In fact we only have to take for  $r_x$  (as Heine does), the smallest interval with  $x$  as left-hand end-point containing all those of the given intervals which have  $x$  for end-point, whether originally associated with  $x$  or not, and for  $l_x$  the given neighbourhood on the left of  $x$ . The conditions of the theorem are then satisfied.

Therefore there is a finite number of the intervals  $r_x$  covering up  $(a, b)$ , and abutting end-to-end. Let the points of division be

$$a, x_1, \dots, x_{n-1}, b.$$

We may evidently assume that  $x_n$  coincides with  $b$ , for we may suppose that none of the given neighbourhoods reach further than  $b$ ; the intervals on the right associated with  $b$  do not enter into the reasoning, and may be non-existent.

We may therefore choose an interval of the given set with  $x_{n-1}$  as left-hand end-point reaching either to  $b$  or to within the given neighbourhood of  $b$  on the left. Thus, if this interval does not reach  $b$ , we may choose in addition the interval abutting with it and reaching to  $b$ , since this is one of the given intervals associated with the point  $b$ . Similarly we proceed in each of the  $n$  divisions, and so obtain the required finite set of the given intervals.

It is to be observed that it is only in the latter part of this proof that the principle of arbitrary choice is assumed, and this only a finite number of times.

5. COR. 3 (*The Heine-Borel\* Theorem*).—*If corresponding to each point  $x$  of a closed segment  $(a, b)$  we are given one or more intervals containing the point  $x$  as internal point, then we can find a finite number of these intervals, so that each point of the closed segment  $(a, b)$  is an internal point (not an end-point) of one of these intervals.*

In fact if we replace the given intervals corresponding to the point  $x$  by all the intervals with  $x$  as end-point contained each in one of the given intervals, we shall have the conditions of Cor. 1. Having chosen a finite number of these new intervals in accordance with that corollary, we only

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\* E. Borel enunciated and proved this theorem in a note at the end of his "Thèse," re-published in the *Ann. de l'École Norm.* (3), Vol. XII (1895), pp. 50 seq. It has been variously proved since both by the author and others.

need to replace each by one of the original intervals containing it, to obtain that which was required.

This again only requires a finite number of arbitrary choices.

The next corollary shows the power of the original theorem, as more general than any required or tacitly proved by Heine.

It is by means of this Lemma, or Lemmas of the same form, that we are enabled to deal with the question of the integration of derivates, as well as that of the points at which the derivates are infinite, whereas the Heine-Lusin and Heine-Borel theorem are inadequate for this purpose, except in the case when a differential coefficient exists everywhere.

*COR. 4 (Young's First Lemma for a Closed Segment).—If with each point\* of a closed interval (a, b) as left-hand end-point we are given an interval, or several intervals forming a finite or infinite set, we can find a finite number of these, nowhere overlapping, and such that the content of the complementary intervals is less than any pre-assigned positive quantity e.*

Let  $G$  denote the content of the whole set of intervals, and let the integer  $n$  be so chosen that

$$2G/n < e. \quad (1)$$

Let us elongate each of the given intervals  $(x, x+h)$  on the left by one  $n$ -th of its length. Now let us define as  $r_x$  the smallest interval with  $x$  as left-hand end-point containing all the original intervals which in their elongated form† contain  $x$  as internal point. Let us define  $l_x$  as the interval with  $x$  as right-hand end-point of length  $k_x/n$ , where  $k_x$  denotes the upper bound of the lengths of the original intervals with  $x$  as left-hand end-point.

Then, if  $x$  is any internal point of  $l_x$ ,

$$n(x-x') < k_x.$$

Therefore there is one of the original intervals with  $x$  as left-hand end-point which in its elongated form reaches beyond the point  $x'$ . Thus  $r_{x'}$  contains  $x$  as internal point.

We may therefore apply the Heine-Young theorem, and obtain a finite

\* An exceptional closed set of content zero clearly makes no difference.

† It is hardly necessary to point out that intervals, if any, which originally contained  $x$  as internal point are not here included; such an interval could, of course, not lie in  $r_x$ , whose left-hand end-point is  $x$ .

number of the intervals  $r_x$ , abutting end-to-end and covering up the whole of  $(a, b)$ .

Let the points of division be

$$a, x_1, x_2, \dots, x_{n-2}, x_{n-1}, x_n.$$

Then, since  $x_n$  is  $b$  or lies to the right of it, we can by the definition of  $r_{x_{n-1}}$ , which is  $(x_{n-1}, x_n)$ , find one of the original intervals, which, in its elongated form enclosed  $x_{n-1}$ , and whose right-hand end-point lies to the right of  $b$  or at most at a distance  $(x_n - x_{n-1})/n$  to the left of  $b$ . This interval therefore covers up the whole interval  $(x_{n-1}, b)$ , excepting perhaps two complementary intervals, whose length is less than  $2(x_n - x_{n-1})/n$ .

Similarly we choose an interval in each of the  $n$  segments between the points  $a, x_1, \dots, x_n$ . Thus we get  $n$  chosen intervals, nowhere overlapping, and whose complementary intervals have a content less than

$$\frac{2}{n} \{ (x_1 - a) + (x_2 - x_1) + \dots \} = \frac{2}{n} (x_n - a),$$

which is less than  $G/n$ , that is, less than  $\epsilon$ . This proves the corollary.

7. It will be noted that in the Heine-Borel theorem we are able to replace the given sets of intervals corresponding to the points  $x$  by the set of *all* the completely open intervals containing the point  $x$  and lying in a certain neighbourhood of the point  $x$ , namely the interval made up of  $r_x$  and  $l_x$ . Such sets of intervals were considered by W. H. Young.\* He introduced the terms "tile" and "point of attachment" for such an interval and its corresponding point, and, in the case in which to the point of attachment we have corresponding *all* the intervals in a certain neighbourhood of it, he speaks of a "tile which may be chipped as much as we please." Chipping a tile is, of course, removing a sub-interval containing an end-point and not containing the point of attachment.

The first theorem given by W. H. Young for sets of tiles was originally called an extension of the Heine-Borel theorem;† in virtue of the second theorem given by the same author it may be called "The Tile

\* "On an Extension of the Heine-Borel Theorem," 1904, *Messenger of Mathematics*, "The Tile Theorem," 1904, *Proc. London Math. Soc.*, Ser. 2, Vol. 2, pp. 67-69.

† See next footnote.

Theorem for a closed Segment," and is again a corollary from our theorem :—

COR. 5 (*The Tile Theorem for a Closed Segment*).—Given a set of tiles, which may be chipped as much as we please, and whose points of attachment fill up a closed segment  $(a, b)$ , then we can determine a finite number of the tiles having the following properties :—

- (i) The length of each tile  $d_{P_i}$  is less than  $e$ .
- (ii) Each point of  $(a, b)$  is covered by one or more of the chosen tiles  $d_{P_i}$ .
- (iii) The point of attachment  $P_i$  of the tile  $d_{P_i}$  is not covered by any other of the chosen tiles, but only by its own tile  $d_{P_i}$ .
- (iv) The sum of the tiles differs from the content of  $(a, b)$  by less than  $e'$ .

Here  $e$  and  $e'$  are any assigned positive quantities.

In order to satisfy the condition (i) we omit at once all tiles of length  $\geq e$ .

Let us take for  $r_x$  the smallest interval with  $x$  as left-hand end-point, containing all those parts of tiles to the right of the point  $x$ , such that the tile contains  $x$ , and its point of attachment is  $x$  or lies to the right of  $x$ .

Let  $l_x$  be defined as the smallest interval with  $x$  as right-hand end-point containing all those parts of tiles to the left of the point  $x$  whose point of attachment is  $x$ . Then, if  $x'$  is internal to  $l_x$ , there is a tile with  $x$  as point of attachment containing  $x'$ : hence, since  $x$  lies to the right of  $x'$ ,  $r_x$  contains this tile, and therefore contains  $x$ .

The conditions of the Heine-Young theorem are thus satisfied.

There is therefore a finite number of the intervals  $r_x$  covering up  $(a, b)$  and abutting end-to-end. Let the points of division be

$$a, x_1, \dots, x_{n-1}, x_n.$$

Now, since  $r_{x_{n-1}}$  is  $(x_{n-1}, x_n)$ , and contains  $b$  as internal point, or as the end-point  $x_n$ , we can find in  $(x_{n-1}, x_n)$  a tile containing  $x_{n-1}$  and whose point of attachment is either  $x_{n-1}$  or lies in the completely open interval  $(x_{n-1}, x_n)$  (but cannot coincide with  $x_n$ ), and which reaches as near as we please to  $x_n$ . Thus, if  $b$  does not coincide with  $x_n$ , this tile covers both  $x_{n-1}$  and  $b$ . But if  $b$  coincides with  $x_n$ , we can make this tile reach so near to  $b$  that the abutting interval required to reach  $b$  is part of one of the tiles associated with  $b$ . Thus we get one or two abutting tiles covering  $x_{n-1}$  and  $b$ .

Again, since  $r_{x_{n-2}}$  is  $(x_{n-2}, x_{n-1})$ , there is in it a tile whose point of attachment is either  $x_{n-2}$  or lies in the completely open interval  $(x_{n-2}, x_{n-1})$ , but does not coincide with  $x_{n-1}$ , and which reaches so near to  $x_{n-1}$  that the abutting interval required to reach  $x_n$  is part of the tile first chosen. We may then, by suitably chipping off from the first tile a part to the left of  $x_{n-1}$ , and therefore to the left of its point of attachment, ensure that the first tile does not contain the point of attachment of the second tile, and that the common part is less than  $e'/n$ .

Similarly we treat the remaining segments in order, and so get a finite number of the given tiles, the sum of whose lengths differs from that of  $(a, b)$  by less than  $e'$ , covering over the whole of  $(a, b)$ , and satisfying the condition (iii). This proves the theorem.\*

8. The theorems which we have been discussing for a closed interval might equally well have been stated for a closed set; the reasoning requires no sensible alteration, only we have on occasion to prove that certain points belong to the set, *e.g.*, the point  $X$  in the proof of the first theorem, and we have to remember that instead of abutting intervals, we have intervals which do not overlap, and whose complementary intervals are free of points of the closed set, that is, are the whole or parts of black intervals of the closed set.

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\* The proof originally given of the Tile Theorem is imperfect, owing to the fact that, though a tile may be chipped as much as we please, we must not chip off the point of attachment; in consequence the chipping demanded on p. 68, *loc. cit.*, lines 20, 21, for instance, is not always allowable. This remark refers to both the papers quoted. In fact the proof depended in the first instance on the Heine-Borel theorem, which, without previous preparation of the tiles, is certainly insufficient for the purpose in hand. It is only by the use of the more powerful theorem here given that the result has actually been obtained. In fact, if the closed segment be  $(0, 1)$  and the given tiles are such that they can be reduced by means of the Heine-Borel theorem to the two tiles  $(-\frac{1}{5}, \frac{2}{5})$  with  $\frac{2}{5}$  as point of attachment, and  $(\frac{1}{5}, \frac{3}{5})$  with  $\frac{1}{5}$  as point of attachment, no chipping will cause these tiles to have the property (iii).

The theorem here given (Tile Theorem for a closed Segment) was originally used by W. H. Young to prove that the upper integral of the sum of two upper semi-continuous functions was the sum of their upper integrals. This was thus proved *before* it was shown that these upper integrals are, in fact, the generalised integrals in this case. The general Tile Theorem, a proof of which is given in the present communication, was applied by the same author ("General Theory of Integration," 1905, *Phil. Trans. (A)*, Vol. 204, p. 235) to prove that the upper integral of a function with respect to a measurable set  $S$  was the lower bound of the corresponding upper summations. The proof there given seems not to utilise the property (iii), given by the Tile Theorem; this is, however, only due to a small error of calculation at the bottom of p. 235, "less than" being used for "greater than"; the correction will afford no difficulty to the careful reader.

[N.B.—D. Mirimanoff has just shown me a very elegant proof of the Tile Theorem founded on the reasoning in the original paper, and involving an ingenious preparation of the tiles. He is going to publish this.—G. E. C. Y. (Written after completion of paper.)]

It is, however, still more clear if we deduce these theorems from those stated for the closed intervals, using the artifice of associating with each point not belonging to the closed set a set of intervals of the same form as that associated in the enunciation with each point of the closed set, but lying in the black interval containing the point in question.

We thus have the following theorems:—

**THEOREM.**—*If with each point  $x$  of a closed set  $S$  we have associated a pair\* of intervals  $r_x$  and  $l_x$ , such that*

- (i)  *$x$  is the left-hand end-point of  $r_x$  and the right-hand end-point of  $l_x$ ;*
- (ii) *if  $x'$  is a point of  $S$  internal to  $l_x$ , then  $x$  is an internal or end-point of  $r_x$ ;*

*then we can find a finite number of the intervals  $r_x$ , non-overlapping and containing every point of  $S$  as internal or end-point.*

**COR. 1** (*Generalised Heine-Lusin Theorem*†).—*If associated with each point of a closed set  $S$  we have all the intervals with  $x$  as end-point in a certain neighbourhood of the point  $x$  on both sides, then a finite number of these intervals can be found, non-overlapping, and containing all the points of the closed set  $S$  as internal or end-points.*

**COR. 2** (*The Generalised Heine-Borel Theorem*).—*If corresponding to each point  $x$  of a closed set  $S$  we are given one or more intervals containing the point  $x$  as internal point, then we can find a finite number of these intervals, so that each point of the closed set  $S$  is an internal point (not an end-point) of one of these intervals.*

**COR. 3** (*Young's First Lemma*‡).—*If with each point of a closed set  $S$  as left-hand end-point we are given an interval, or several intervals, forming a finite or infinite set, we can find a finite number of these nowhere overlapping, and such that the sub-set of  $S$  in the complementary intervals can be enclosed in a finite number of intervals whose content is less than any pre-assigned positive quantity  $\epsilon$ .*

\* In the case of the end-points of the black intervals only one of these intervals need exist, viz., that on the contrary side to the black interval, if it is uniquely determined by the point (as will be the case if  $S$  is perfect), and when  $x$  is an end-point of two black intervals (i.e.,  $x$  is an isolated point of  $S$ ), it is immaterial whether it is  $l_x$  or  $r_x$ , which exists.

† N. Lusin, *loc. cit.*

‡ W. H. Young and Grace Chisholm Young, "On the Existence of a Differential Co-efficient," 1910, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 325-335.

COR. 4 (*The Tile Theorem for a Closed Set*).—Given a set of tiles, which may be chipped as much as we please, and whose points of attachment fill up a closed set  $S$ , then we can determine a finite number of the tiles having the following properties :—

- (i) *The length of each tile  $d_p$  is less than  $e$  ;*
  - (ii) *each point of  $S$  is covered by one or more of the chosen tiles  $d_p$  ;*
  - (iii) *the point of attachment of any chosen tile is only covered by that tile, and not by any other of the chosen tiles ;*
  - (iv) *the sum of the tiles differs from the content of  $S$  by less than  $e'$  ;*
- here  $e$  and  $e'$  are any assigned positive quantities.

9. The theorems hitherto obtained involve, as has been pointed out, the principle of arbitrary choice at most a finite number of times. The following theorems involve it a countably infinite number of times.

THEOREM (*Of the Equivalent Countable Set of Overlapping Intervals*).<sup>\*</sup>—Given a set of overlapping intervals, we can determine a countable set from among them having the same internal points, the same external points and the same isolated end-points as the given set.

The points not internal to the given intervals form a closed set, whose black intervals, considered as completely open intervals, consist precisely of the internal points of the given set of overlapping intervals. Since these black intervals are countable, it is only necessary to prove the theorem in each of them separately, and it will follow that the theorem, as stated, is true.

Let then  $(a, b)$  be any one of these black intervals and  $M$  its middle point. Corresponding to each point  $x$  of the completely open interval  $(a, b)$ , we define the pair of intervals  $l_x$  and  $r_x$  as follows :—

$l_x$  is the smallest interval with  $x$  as right-hand end-point containing the left-hand end-points of all those of the given intervals which contain  $x$  as internal point ;

$r_x$  is defined in like manner, interchanging left and right.

Now, if  $x'$  is a point of  $l_x$ , there is one of the given intervals containing both  $x$  and  $x'$  as internal points ; therefore  $x$  is internal to  $r_x$ .

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<sup>\*</sup> W. H. Young, "Overlapping Intervals," 1902 (not 1903), *Proc. London Math. Soc.*, Ser. 1, Vol. xxxv, pp. 384–386. Subsequent proofs have been supplied by the author and others. The present proof is a fresh one.

Thus we may apply the Heine-Young theorem for a half-open interval (Cor. 1, § 3 *supra*) to the interval  $(M, b)$ .

Interchanging left and right, we may apply the same theorem to the half-open interval  $(a, M)$ , using the intervals  $l_x$  instead of  $r_x$  to cover it up.

We have therefore the whole interval  $(a, b)$  covered up by a countably infinite set of the abutting intervals  $l_x$  and  $r_x$ . But each of these can be covered by *two* of the given intervals; for instance, if  $(c, d)$  is an interval  $r_x$ , there is an interval of the given set covering  $c$  and reaching so near to  $d$  that it overlaps with  $l_d$ , and therefore with one of the given intervals containing  $d$ . This therefore proves the theorem in so far as the internal points are concerned.

That such a set has also the same isolated end-points, external points and semi-external points as the given set is easily seen. Indeed an end-point of a set of intervals is a point which, without being an internal point of the intervals, is the extremity at least on one side of an interval consisting entirely of such points; thus the identity of internal points enforces also the identity of end-points. If the end-point is an isolated one, it is the common extremity of two completely open intervals, one on each side of it, one at least of which consists entirely of internal points of the intervals, while the other either does the same, or contains none of these internal points; a semi-external point, however, has no such interval on one side of the point, while on the other there is the interval required to define the point to be an end-point. In both these cases, therefore, the identity of internal points renders the defining property invariant.

This proves the theorem.

It should be noticed that the construction here given is such that, *in any closed interval containing only internal points of the set there is only a finite number of the given intervals.*

10. The proof of the Tile Theorem in its general form, which also involves the principle of arbitrary choice a countably infinite number of times, will now be given by means of the Heine-Young theorem.

**THEOREM (The Tile Theorem).**—*Given a set of tiles, each of which may be chipped as much as is convenient, whose points of attachment fill up a measurable set  $S$ ; we can determine a finite or countably infinite number of the tiles, having the following properties:—*

- (i) *the linear dimensions of each of the chosen tiles  $d_{P_i}$  are less than  $\epsilon$ ;*
- (ii) *each point of  $S$  is covered by one or more of the chosen tiles;*

(iii) *the point of attachment  $P_i$  of the chosen tile  $d_{P_i}$  is not covered by any other of the chosen tiles, but only by its own tile  $d_{P_i}$ ;*

(iv) *the sum of the tiles differs from the content of  $S$  by less than  $e'$ ; here  $e$  and  $e'$  are any assigned small positive quantities.*

By the definition of the content of a measurable set, we can find a set of non-overlapping intervals of content lying between  $S$  and  $S + \frac{1}{2}e'$ , and having all the points of  $S$  as internal points. Since the tiles may be chipped, and their points of attachment are all inside these intervals, we may so chip them that they themselves all lie inside these intervals. Whatever set of the tiles we then choose will certainly have content less than  $S + \frac{1}{2}e'$ . We shall suppose all the tiles chipped so as to satisfy the condition (i). If we now show how to choose the tiles, so chipped, so as to satisfy the condition (iii), it is evident that we can further chip the tiles so that the sum of their overlapping parts, which contain no point of attachment, is less than  $\frac{1}{2}e'$ ; the sum of the chosen tiles will then differ from their content by less than  $\frac{1}{2}e'$ . If the chosen tiles then contain the whole set  $S$ , their content will lie between  $S$  and  $S + \frac{1}{2}e'$ ; and therefore their sum will differ from  $S$  by less than  $e'$ , so that the condition (iv) will be satisfied.

It remains therefore only to show how to choose the tiles so as to satisfy the conditions (ii) and (iii).

We may evidently suppose that the tiles are so chipped that their points of attachment are their middle points. Let  $q_x$  denote the right-hand half of  $d_x$ . The point of attachment of such a half-tile  $q_x$  is then its left-hand end-point, and the half-tiles may be chipped on the right as much as we please, but not on the left at all.

These half-tiles for values of  $x$  in the set  $S$  fill up a set of non-overlapping intervals, whose internal points are the same as those of the half-tiles, and whose left-hand end-points are either points of  $S$  or limiting points of  $S$ , while the right-hand end-points are certainly not points of  $S$ . Let these intervals be arranged in the usual way\* in order  $(A_1, B_1), (A_2, B_2), \dots$ .

Let  $(a_1, b_1)$  be a closed interval inside  $(A_1, B_1)$ , which, if  $A_1$  is a point of  $S$ , is such that  $a_1$  coincides with  $A_1$ ; for the moment we shall neglect this latter case, but return to it subsequently. With each point  $x$  of the closed interval  $(a_1, b_1)$  we can associate a pair of intervals  $l_x$  and  $r_x$ , where

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\* So that greater intervals precede smaller, and of two equal intervals that comes first which lies more to the left.

$l_x$  is the least interval with  $x$  as right-hand end-point containing all the points of attachment of half-tiles containing  $x$  as internal point, and  $r_x$  is the least interval with  $x$  as left-hand end-point such that  $l_x + r_x$  contains all those half-tiles. Then, if  $x'$  is a point internal to  $l_x$ , there is a half-tile containing both  $x$  and  $x'$ , so that  $x$  is internal to  $r_{x'}$ . We may therefore apply the Heine-Young theorem.

We have thus a finite number of the intervals  $r_x$  covering every point of the closed interval  $(a_1, b_1)$ . Let these be  $r_1, r_2, \dots, r_m$ , and the points of division be  $P_1$  or  $a_1, P_2, P_3, \dots, P_m$ , and  $P_{m+1}$  beyond  $b_1$ .

Then, by the definition of  $r_x$ , every point internal to it may be the right-hand end-point of a half-tile containing its left-hand end-point  $x$ .

There is therefore a half-tile  $q_1$  containing  $P_m$  and  $b_1$ . If  $Q_1$  be the point of attachment of  $q_1$ ,  $Q_1$  falls between two of the points of division, say  $P_i$  and  $P_{i+1}$ , or coincides with one of them, say  $P_{i+1}$ , where  $i$  is less than  $m$ .

We then complete this half-tile by adding on a piece on the left so small as not to reach to  $P_i$ . We can then, since  $P_i P_{i+1}$  is  $r_{P_i}$  and contains  $Q_1$ , find a half-tile containing  $P_i$  and overlapping with the first chosen tile, but so little as not to contain  $Q_1$ . We then treat this tile as we did the first chosen tile, and go on a stage further. After at most  $m$  stages therefore we shall have obtained at most  $m$  tiles, covering over the closed interval  $(a_1, b_1)$ , and such that none of their points of attachment lie in the parts where they overlap. Let  $(a'_1, b'_1)$  be the interval thus tiled over, containing  $(a_1, b_1)$  and contained in  $(A_1, B_1)$ .

Bisecting the intervals  $(A_1, a'_1)$  and  $(b'_1, B_1)$  at  $a_2$  and  $b_2$  respectively, we can in like manner cover over  $(a_2, a_1)$  and  $(b_1, b_2)$ , previously chipping all the tiles whose points of attachment are not internal to the tiles already chosen so as not to contain any of the points of attachment of those chosen tiles.

We thus add on to the tiles first chosen a finite number of tiles so as to cover the larger interval  $(a_2, b_2)$ , and still to have the required property that the point of attachment of any chosen tile should not be covered by any other of the chosen tiles.

Thus we proceed stage by stage to cover intervals  $(a_3, b_2), (a_3, b_3), \dots$ , each lying inside the next following, and approaching  $(A_1, B_1)$  as limiting interval. We have thus defined such a countable set of the given tiles as was required, covering every point of the completely open interval  $(A_1, B_1)$ , and having the required property.

Here we have assumed that  $A_1$  is not a point of  $S$ . In the contrary case the argument still holds, the intervals  $(A_1, a_1), (a_2, a_1), \dots$  disappearing altogether. We only have to remark that, in this case the chosen tile

containing  $A_1$  must have  $A_1$  for point of attachment, since there was no half-tile containing points internal to  $(A_1B_1)$  whose point of attachment lay outside  $(A_1, B_1)$ . We may then choose the left-hand end-point of this tile so as not to be internal to any of the intervals  $(A_iB_i)$ , for  $A_1$ , being a point of  $S$  cannot be a right-hand end-point of one of the intervals  $(A_iB_i)$ , so that there is certainly a sequence of points external to the intervals  $(A_iB_i)$ , and having  $A_1$  for limiting point. We may then omit from consideration all those of the intervals  $(A_i, B_i)$  which lie in this tile. Thus, whereas in the former case, when  $A_1$  is not a point of  $S$ , we go on to  $(A_2, B_2)$ , in this latter case we should go on to that  $(A_i, B_i)$  which has the lowest index  $i$  among those not already omitted.

Proceeding thus in order through the intervals  $(A_i, B_i)$  we obtain a countably infinite set of the given tiles covering every point of  $S$ , and having the required property (iii). This proves the theorem since the remaining conditions were already satisfied.

11. Lebesgue\* in his proof of the Heine-Borel theorem has introduced a very valuable principle, which is itself in embryo in Heine's proof. Heine, as we have seen, replaces all the *given* intervals with  $a$  as left-hand end-point by the smallest interval containing them, which is in his case at the same time the largest of these given intervals. Lebesgue replaces all the intervals with  $a$  as left-hand end-point *in which the theorem is true* by the smallest interval containing them. In replacing the actual point  $b$  by a hypothetical point  $X$ , Heine has passed from the consideration of a closed interval  $(a, b)$  about which we cannot yet affirm that the theorem is even approximately true, to a hypothetical interval  $(a, X)$ , open on the right, which is such that in every sufficiently small interval contained in it the oscillation of the function is less than  $6\epsilon$ , that is to say a certain approximation to the theorem is true. Lebesgue's device consists in taking a hypothetical interval  $(a, X)$ , the smallest interval containing all the intervals with  $a$  as left-hand end-point in which the theorem to be proved is itself true. That such an interval, if it does not reach to  $b$ , must be open on the right is evident, since otherwise we could extend it on the right, by adding on one of the given intervals associated with the point  $X$ . It remains therefore only to show that such an interval  $(a, X)$  cannot be open on the right.

This principle of Lebesgue's enables us to prove theorems which would seem to lie beyond the scope of the methods so far used.

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\* H. Lebesgue, *Leçons sur l'Intégration*, 1904, p. 105.

We shall apply this principle first to prove Young's Second Lemma, which includes Young's First Lemma as a special case.

YOUNG'S SECOND LEMMA.\*—*If with each point of a closed interval  $(a, b)$  as left-hand end-point, we are given an interval, or several intervals, forming a finite or infinite set, we can find a finite number of the given intervals, nowhere overlapping, and such that the complementary intervals, or gaps, have lengths which are respectively less than different terms of any chosen monotone descending sequence.*

Let  $r_a$  denote the smallest interval with  $a$  as left-hand end-point containing all the intervals with  $a$  as left-hand end-point in which the theorem is true,  $r_a$  taking the place of  $(a, b)$  in the enunciation. There must be such an interval  $r_a$ , because the least interval containing all the given intervals associated with the point  $a$  is such an interval in which the theorem is true. Then, if  $r_a$  contains  $b$  as internal or end-point, the theorem is true in  $(a, b)$ . If not, let, if possible, a point  $X$ , internal to  $(a, b)$ , be the right-hand end-point of  $r_a$ . Then  $X$  determines at least one interval  $D_X$  of the given set.† If therefore the theorem is true in  $(a, X)$ , it is true in the larger interval got by appending  $D_X$  to  $r_a$ . Since  $r_a$ , however, contains all such intervals with  $a$  as left-hand end-point, this is impossible. Therefore the interval  $(a, X)$ , if it exists, must be open on the right, that is to say, the theorem is not true in  $(a, X)$ .

But we see at once that this is impossible. For if  $y$  be a point of  $(a, X)$  at a distance from  $X$  less than  $e_1$ , the theorem is true in  $(a, y)$ . Therefore taking the monotone descending sequence

$$e_2 > e_3 > \dots,$$

each of whose terms is supposed less than  $e_1$ , we can find a finite number of intervals from the given set, non-overlapping and such that the remaining parts of  $(a, y)$  are respectively less than different terms of this sequence. Since  $(y, X)$  is less than  $e_1$ , it follows that these chosen intervals cover over the whole of  $(a, X)$  except a finite number of gaps which are respectively less than different terms of the monotone descending sequence

$$e_1 > e_2 > e_3 \dots$$

Since this is any monotone descending sequence, this proves that we were

\* "On Derivates and their Primitive Functions," 1912, *Proc. London Math. Soc.*, Ser. 2, Vol. 12, pp. 210-212.

† If we wish to be more definite, we can take the smallest interval containing all the intervals  $D_X$ . We can in what follows then append, not  $D_X$ , but say, half of  $D_X$ .

mistaken in supposing that the theorem was not true in  $(a, X)$ . Thus, by a *reductio ad absurdum* it is proved that the theorem is true.

12. As before, using the artifice of associating with each point not belonging to a certain closed set an interval, or set of intervals, contained in the corresponding black interval of the closed set, we have the usual generalisation of the preceding Lemma.

YOUNG'S SECOND LEMMA IN ITS GENERALISED FORM.—*If with each point of a closed set  $S$  as left-hand end-point, we are given an interval, or several intervals forming a finite or infinite set, we can find a finite number of the given intervals, nowhere overlapping, and such that, in the complementary intervals the points of  $S$  can be enclosed in a finite number of extra intervals, not overlapping with one another, or with the chosen intervals, and whose lengths are respectively less than different terms of any chosen monotone descending sequence.*

13. It does not seem that this principle of Lebesgue's will aid us in obtaining a proof of Lebesgue's own Lemma, the enunciation of which is as follows:—

LEBESGUE'S LEMMA.—*If with each point of a closed interval  $(a, b)$  as left-hand end-point we are given an interval, or several intervals, forming a finite or infinite set, we can find a countable set of the intervals, non-overlapping, and containing every point of the closed interval  $(a, b)$  as internal or left-hand end-point.*

Such a set of intervals is said to form a *Lebesgue chain*, stretching from  $a$  to  $b$ .

In fact, if  $(a, X)$  is, as before, the smallest interval containing all the intervals  $(a, y)$  in which the theorem is true, it is perfectly true that if  $y_1$  is a point of  $(a, X)$ , and  $y_2$  another point of  $(a, X)$ , between  $y_1$  and  $X$ , there is a Lebesgue chain stretching from  $a$  to  $y_1$ , and another from  $a$  to  $y_2$ , but it does not follow that these two chains have a common end-point, so that we can replace the first chain by a part of it which is also a part of the second chain. If we could make this assertion, we could find a series of chains, each contained in the next, and having  $X$  for the limiting point of their right-hand end-points, from which we could conclude that there was a Lebesgue chain stretching from  $a$  to  $X$ . As this assertion cannot be made, we cannot draw any such conclusion, and the method seems to fail.

If, however, with Pal,\* we consider only the special case of the Lemma

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\* J. Pal, "Beweis des Lebesgue-Young'schen Satzes," 1912, *Rend. di Pal.*, Vol. xxxiii, pp. 352, 353.

where there is a definite interval connected with each point, and not a set of intervals, we can prove the missing point, and so apply the argument to obtain the required proof without Cantor's numbers.

In fact, from the definition of a Lebesgue chain it follows that, whatever property we may take as a test, there is always a nearest interval of such a chain to  $a$ , having the required property; it is in consequence of this that the intervals of the chain form a well-ordered set. For, if not, we could choose out an infinite succession of intervals of the chain having the property, each lying to the left of the preceding. These would accordingly determine a limiting point  $Y$  to the left of all of them and not to the left of  $a$ . The interval of the chain to which  $Y$  is an internal or left-hand end-point must therefore overlap with some of the intervals of the succession, contrary to the property of the chain that its intervals are non-overlapping. Thus the assumption cannot be true, which proves the statement made at the beginning of this paragraph.

Now let  $L_1$  denote the part (or whole) of the Lebesgue chain from  $a$  to  $y_1$  got by omitting any of its intervals to the right of that which contains  $y_1$ , and let  $L_2$  denote the second chain, that from  $a$  to  $y_2$ , where  $y_2$  lies between  $y_1$  and  $X$ , as in the argument given above. Then, by what has just been proved, there is a nearest interval of  $L_1$  to  $a$  which is not an interval of  $L_2$ , or else every interval of  $L_1$  is an interval of  $L_2$ , in which latter case  $L_1$  is a part of  $L_2$ , which, as we have seen, will suffice for the purposes of proof. Suppose then that this latter is not the case, and let  $Y$  be the left-hand end-point of the interval of  $L_1$  nearest to  $a$ , not belonging to  $L_2$ . Then all the intervals of  $L_1$  to the left of  $Y$  are intervals of  $L_2$ , and therefore are identical with the part of  $L_2$  to the left of  $Y$ , since the intervals so characterised cover over every point of  $(a, Y)$ , and the intervals of  $L_2$  do not overlap. Hence  $Y$  cannot be an internal point of any intervals of  $L_2$  to the left of  $Y$ , and must therefore be a left-hand end-point of an interval of  $L_2$  not belonging to  $L_1$ . But, by the hypothesis that there is only one of the given intervals associated with the point  $Y$ , and this interval is an interval of  $L_1$ , which is a contradiction. Therefore this assumption is untenable, and  $L_1$  is a part of  $L_2$ .

Thus in the special case when there is only one interval associated with each point, and not a set of intervals, it appears that the principle of Lebesgue will afford us a proof of the Lemma without the use of Cantor's numbers. In the applications this simplified form of the Lemma will often suffice, in fact, it suffices to prove the theorems of Lebesgue on the derivatives of continuous functions, owing to the fact

that the intervals  $(x, x+h)$  associated with a point  $x$ , being characterised by an equality of such a form as the following

$$[f(x+h)-f(x)]/h \leq e,$$

include, by reason of the continuity of  $f(x)$ , a largest such interval, which may accordingly be taken to be the unique interval associated with the point  $x$ . In the more general theorems in the Theory of Functions of a Real Variable of the type of Lebesgue's theorems, however, we have no such unique interval, and we are bound to have recourse to the general form of Lebesgue's Lemma, or to employ other means.

The only proof\* extant of Lebesgue's Lemma is that sketched by him on p. 62 of his *Leçons sur l'Intégration* (1902), and written out at length in "Functions of Bounded Variation," by W. H. Young and Grace Chisholm Young, *Quarterly Journal of Mathematics*, Vol. XLII, p. 78 (1910). It depends on Cantor's numbers of the second class, and involves for the purposes of proof those of still higher classes.

On the other hand, Young's Second Lemma, which has shown itself capable of achieving all that has been effected by Lebesgue's Lemma, and has served to prove fresh results, has here been proved† without the introduction of Cantor's numbers, or even of that of transfinite ordinal types.

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\* In our paper "On the Existence of a Differential Coefficient" (1910), *Proc. London Math. Soc.*, Ser. 2, Vol. 10, pp. 332 *seq.*, an attempt was made to show the virtual identity of Lebesgue's Lemma with Young's First Lemma, and, in point of fact, the reasoning there given deducing the latter from the former was correct; the proof of the remaining half of the statement seems to contain a flaw, so that a rigid proof of Lebesgue's Lemma without Cantor's numbers is still a desideratum. Lebesgue's Lemma, as stated above, though it involves the distinction of right and left, does not contain the idea of order, nor do transfinite numbers enter into its enunciation. On the other hand, we have shown that all the results which have been obtained by means of it including all Lebesgue's results, can be obtained without the use of transfinite numbers. Indeed although the flaw in question is unfortunately repeated in the proof of Young's second Lemma without transfinite numbers in the paper "Derivates and their Primitive Functions" (1912), *Proc. London Math. Soc.*, Ser. 2, Vol. 12, p. 212, the proof given in the present paper may be substituted for it, so that the main argument of the paper applies equally whether the use of transfinite numbers be allowed or not. It may be well to point out that the original flaw, which occurs in the second paragraph of p. 333 of Vol. 10, is due to the fact that at each stage intervals will have in general to be omitted, as being too large for the gaps in which we are operating. This requires the intervals at each point to have zero as the lower bound of their lengths, a condition which is not demanded in the enunciation on p. 332. Moreover, even if we add this demand to the enunciation the interval  $(c, c+h_c)$ , referred to on p. 333, would be certain to drop out at some stage. It is moreover evident that, when there is an unique interval associated with each point, the Lebesgue chain from  $a$  to  $b$  is unique, whereas the intervals obtained as in Young's Lemma may, in general, be chosen in a variety of ways, with end-points not coinciding with those of the chain. This shows the impossibility of deducing Lebesgue's Lemma from Young's.

† See preceding footnote.