# The Riemann Transform

By

Armando M. Evangelista Jr.

armando781973@gmail.com

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ABSTRACT

In his 1859 paper, Bernhard Riemann used the integral equation $\int_0^\infty f(x)\, x^{-s-1}\, dx$ to develop an explicit formula for estimating the number of prime numbers less than a given quantity. It is the purpose of this present work to explore some of the properties of this equation.

Consider the integral equation given below

$$F(s) = \int_0^\infty f(x)\, x^{-s-1}\, dx \tag{1}$$

Formula (1) is the the integral of $f(x)$ times $x^{-s-1}$ for $x = 0$ to $\infty$ and the resulting function is a function of $s$, say $F(s)$ (or the **transform** of $f(x)$) . It must be assume that $f(x)$ is such that the integral exists (it has finite value).

**Example 1** Apply formula (1) to obtain the transform of $f(x) = e^{-x}$.

***Solution***. Substitute $e^{-x}$ to (1)

$$F(s) = \int_0^\infty e^{-x} x^{-s-1}\, dx = \Gamma(-s) \qquad \Re(s) < 0 \text{ , sine } \Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx, \quad \Re(s) > 0 \text{ ,}$$

where $\Gamma(S)$ is the gamma function and $\Re(s)$ is the real part of the complex quantity $s$.

## Unit Step Function (Heaviside Function)

The **unit step function** or **Heaviside function** $\mu(x - a)$ is 0 for $x < a$, has a jump size 1 at $x = a$ (where it is usually consider as undefined), and is 1 for $x > a$, in a formula:

$$\mu(x-a) = \begin{cases} 0 & \text{if } x < a \\ 1 & \text{if } x > a \end{cases} \qquad a \geq 0.$$

The transform of $\mu(x - a)$ is

$$F(s) = \int_0^\infty x^{-s-1} \mu(x-a) dx = \int_a^\infty x^{-s-1} dx = \left. \frac{-x^{-s}}{s} \right|_a^\infty \quad ;$$

here the integration begins at $x = a$ (>0) because $\mu(x - a)$ is 0 for $x < a$. Hence

$$F(s) = \frac{a^{-s}}{s} \qquad (a > 0 \quad \text{and} \quad s > 0).$$

**Example 2**: The Riemann Zeta Function is given by

$$\zeta(s) \;=\; 1^{-s} + 2^{-s} + 3^{-s} + \ldots \;=\; \sum_{n=1}^{\infty} n^{-s} \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \Re(s) > 1 \quad ,$$

obtain the transform of $\sum_{n=1}^{\infty} \mu(x-n), \quad n = 1,2,3,4,\ldots$ .

$$F(s) = \int_0^{\infty} \{\mu(x-1) + \mu(x-2) + \mu(x-3) + \ldots\} x^{-s-1} dx = \left.\frac{-x^{-s}}{s}\right|_1^{\infty} + \left.\frac{-x^{-s}}{s}\right|_2^{\infty} + \left.\frac{-x^{-s}}{s}\right|_3^{\infty} + \ldots$$

$$= \frac{1}{s}(1+2^{-s}+3^{-s}+4^{-s}+\ldots) = \frac{1}{s}\sum_{n=1}^{\infty}\frac{1}{n^s} = \frac{\zeta(s)}{s}, \qquad \Re(s) > 1 .$$

**Example 3**: Obtain the transform of $\pi(x) \;=\; \sum_{p}^{\infty} \mu(x-p)$ , where $p$ is a prime number, $p$ = 2, 3, 5, 7, 11, ….

$$F(s) \;=\; \int_0^{\infty} \left\{ \sum_{p}^{\infty} \mu(x-p) x^{-s-1} dx \right\} = \int_0^{\infty} \{\mu(x-2) + \mu(x-3) + \mu(x-5) + \mu(x-7) + \ldots\} x^{-s-1} dx$$

$$\pi(s) = \frac{1}{s}(2^{-s} + 3^{-s} + 5^{-s} + 7^{-s} + \ldots) \;=\; \frac{1}{s}\sum_{p}^{\infty} p^{-s} \qquad\qquad \Re(s) > 1.$$

## Dirac’s Delta Function

Consider the function

$$f_{\tau}(x-a) \;=\; \begin{cases} 1/\tau \;\text{ if }\; a \le x \le a+\tau \\ 0 \qquad\qquad \text{otherwise}. \end{cases}$$

Its integral is

$$I \;=\; \int_0^{\infty} f_{\tau}(x-a) dx \;=\; \int_a^{a+\tau} \frac{1}{\tau} dx \;=\; 1.$$

We let now let $\tau$ becomes smaller and smaller and take the limit as $\tau \to 0$ ($\tau > 0$). This limit is denoted by $\delta(x - a)$, that is,

$$\delta(x-a) \;=\; \lim_{\tau \to 0} f_{\tau}(x-a).$$

and obtain

$$\delta(x-a) = \begin{cases} \infty & \text{if } x = a \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \int_0^\infty \delta(x-a)dx = 1.$$

$\delta(x - a)$ is called the **Dirac delta function** or the **unit impulse function**. For a *continuous* function $f(x)$ one uses the **sifting** property of $\delta(x - a)$,

$$\int_0^\infty f(x)\delta(x-a)dx = f(a).$$

To obtain the transform of $\delta(x - a)$, we write

$$f_\tau(x-a) = \frac{1}{\tau}[\mu(x-a) - \mu(x-(a+\tau))]$$

and take the transform

$$F(s) = \int_0^\infty f_\tau(x-a)x^{-s-1}dx = \frac{1}{\tau s}\left[a^{-s} - (a+\tau)^{-s}\right] = a^{-s}\frac{1 - \left(1+\frac{\tau}{a}\right)^{-s}}{\tau s}, \qquad a > 0 \text{ and } \Re(s) > 0.$$

Take the limit as $\tau \to 0$. By l'Hopital's rule, the quotient on the right has the limit $1/a$. Hence, the right side has the limit $a^{-(s+1)}$. The transform of $\delta(x - a)$ define by this limit is

$$F(s) = \int_0^\infty \delta(x-a)x^{-s-1}dx = a^{-(s+1)} \qquad a > 0.$$

**Example 4** Obtain the transform of $\sum_{n=1}^{\infty} \delta(x-n)$ .

$$F(s) = \int_0^\infty \{\sum_{n=1}^{\infty} \delta(x-n)\}x^{-s-1}dx = \sum_{n=1}^{\infty} n^{-(s+1)} = \zeta(s+1), \quad \Re(s) > 0.$$

**The Riemann Transform**

Many common functions like $\sin x,\ \cos x,\ \ln x,\ etc.,$ when applied to formula (1) don't have finite values. But if the lower limit for (1) starts at $x = 1$, then there are suitable functions such that the integral in (1) exist.

If $f(x)$ is a function defined for all $x \geq 1$, its **Riemann transform** is the integral of $f(x)$ times $x^{-s-1}$ for $x = 1$ to $\infty$. It is a function of $s$, say $F(s)$, and is denoted by $R(f)$ ; thus

**(2)** $$F(s) = R(f) = \int_1^{\infty} f(x)\, x^{-s-1}\, dx\ .$$

The given function $f(x)$ in (2) is called the **inverse transform** of $F(s)$ and is denoted by $R^{-1}(F)$; that is,

$$f(x) = R^{-1}(F).$$

**Example 5** Let $f(x) = 1$. Find $F(s)$.

***Solution***. From (2) we obtain by integration

$$R(f) = R(1) = \int_1^{\infty} x^{-s-1} dx = -\frac{1}{s} x^{-s} \Big|_1^{\infty} = \frac{1}{s} \qquad (s>0).$$

**Example 6** Let $f(x) = x^a$, where $a$ is a constant. Find $F(s)$.

***Solution***. From (2),

$$R(x^a) = \int_1^{\infty} x^a x^{-s-1} dx = -\frac{1}{s-a} x^{-(s-a)} \Big|_1^{\infty} = \frac{1}{s-a} \qquad (s-a>0).$$

**THEOREM 1 Linearity of the Riemann Transform**

*The Riemann transform is a linear operation; that is, for any functions f(x) and g(x) whose transforms exist and any constants a and b the transform of af(x) + bg(x) exists, and*

$$R\{af(x) + bg(x)\} = aF(s) + bG(s).$$

**Example 7** Find the transforms of cosh ($a$ln$x$) and sinh ($a$ln$x$).

***Solution***. Since $\cosh(a\ln x)=\frac{1}{2}(x^a + x^{-a})$ and $\sinh(a\ln x)=\frac{1}{2}(x^a - x^{-a})$, we obtain from Example 6 and Theorem 1,

$$R\{\cosh(a\ln x)\}=\frac{1}{2}(R(x^a) + R(x^{-a})) = \frac{1}{2}\left(\frac{1}{s-a}+\frac{1}{s+a}\right) = \frac{s}{s^2-a^2}$$

$$R\{\sinh(a\ln x)\}=\frac{1}{2}(R(x^a) - R(x^{-a})) = \frac{1}{2}\left(\frac{1}{s-a}-\frac{1}{s+a}\right) = \frac{a}{s^2-a^2} \quad .$$

**Example 8** Let $f(x)=x^{\alpha i}$ , where $i$ is the imaginary operator $(i=\sqrt{-1})$ . Find $F(s)$.

***Solution***. From Example 6

$$R(x^{\alpha i}) = \frac{1}{s-\alpha i} = \frac{1}{s-\alpha i}\,\frac{s+\alpha i}{s+\alpha i} = \frac{s}{s^2+\alpha^2} + i\frac{\alpha}{s^2+\alpha^2}.$$

**Example 9** Cosine and Sine

Derive the formulas

$$R\{\cos(\alpha\ln x)\} = \frac{s}{s^2+\alpha^2} \quad \text{and} \quad R\{\sin(\alpha\ln x)\} = \frac{\alpha}{s^2+\alpha^2} \quad .$$

***Solution***. From Example 8 and Theorem 1

$$x^{\alpha i} = \cos(\alpha\ln x) + i\sin(\alpha\ln x)$$

$$R(x^{\alpha i}) = R(\cos(\alpha\ln x)) + iR(\sin(\alpha\ln x)) \quad \text{, thus}$$

$$R\{\cos(\alpha\ln x)\} = \frac{s}{s^2+\alpha^2} \quad \text{and} \quad R\{\sin(\alpha\ln x)\} = \frac{\alpha}{s^2+\alpha^2} \quad .$$

**THEOREM 2** ***s*-Shifting Theorem**

*If f(x) has the transform F(s) (where s > k for some k), then* $x^a f(x)$ *has the transform F(s – a) (where s – a > k). In formulas,*

$$R\{x^a f(x)\} = F(s-a)$$

*or, if we take the inverse on both sides*

$$x^a f(x) = R^{-1}\{F(s-a)\}.$$

**PROOF** We obtain *F*(*s* – *a*) by replacing *s* with *s* – *a* in the integral in (1), so that

$$F(s-a) = \int_1^\infty x^{-(s-a)-1} f(x)dx = \int_1^\infty x^{-s-1}[x^a f(x)]dx = R\{x^a f(x)\}.$$

**Example 10** From Example 9 and the *s*-Shifting theorem one can obtain the Riemann transform for

$$R\{x^a \cos(\alpha \ln x)\} = \frac{s-a}{(s-a)^2+\alpha^2} \quad \text{and} \quad R\{x^a \sin(\alpha \ln x)\} = \frac{\alpha}{(s-a)^2+\alpha^2} \ .$$

**Existence and Uniqueness of Riemann Transforms**

A function *f*(*x*) has a Riemann transform if it does not grow too fast, say, if for all *x* ≥ 1 and some constants *M* and *k* it satisfies

**(3)** $$|f(x)| \le Mx^k.$$

**THEOREM 3 Existence Theorem for Riemann Transforms**

*If f(x) is defined and piecewise continuous on every finite interval on x ≥ 1 and satisfies* (3) *for all x ≥ 1 and some constants M and k, then the Riemann transform R(f) exists for all s > k.*

**PROOF** Since $f(x)$ is piecewise continuous, $x^{-s-1}f(x)$ is integrable over any finite interval on the $x$-axis,

$$|R(f)| = \left|\int_1^\infty f(x)x^{-s-1}\right| \le \int_1^\infty |f(x)|x^{-s-1}dx \le \int_1^\infty M x^k x^{-s-1}dx = \frac{M}{s-k}.$$

**Uniqueness.** If the Riemann transform of a given function exists, it is uniquely determined and if two *continuous* functions have the same transform, they are completely identical.

## Transforms of Derivatives and Integrals

**THEOREM 4 Riemann Transform of Derivatives**

*The transforms of the first and second derivatives of f(x) satisfy*

(4) $$R(f') = (s+1)F(s+1) - f(1)$$

(5) $$R(f'') = (s+2)(s+1)F(s+2) - (s+1)f(1) - f'(1)$$

*Formula* (4) *holds if f(x) is continuous for all x ≥ 1 and satisfies* (3) *and f'(x) is piecewise continuous on every finite interval for* $x \ge 1$. *Formula* (5) *holds if f and f' are continuous for all* $x \ge 1$ *and satisfy* (3) *and f'' is piecewise continuous on every finite interval for* $x \ge 1$.

**PROOF** Using integration by parts on formula (4)

$$R(f) = \int_1^\infty f'(x)x^{-s-1}dx = [f(x)x^{-s-1}]|_1^\infty + (s+1)\int_1^\infty f(x)x^{-s-2}dx = -f(1) + (s+1)F(s+1).$$

The proof of (5) now follows by applying integration by parts twice on it, that is

$$R(f'') = \int_1^\infty f''(x)x^{-s-1}dx = [f'(x)x^{-s-1}]|_1^\infty + (s+1)\int_1^\infty f'(x)x^{-s-2}dx$$

$$= -f'(1) + (s+1)\left[f(x)x^{-s-2}|_1^\infty + (s+2)\int_1^\infty f(x)x^{-s-3}dx\right]$$

$$= -f'(1) - (s+1)f(1) + (s+2)(s+1)F(s+2).$$

Repeatedly using integration by parts as in the proof of (5) and using induction, we obtain the following Theorem.

**THEOREM 5 Riemann Transform of the Derivative $f^{(n)}$ of Any Order**

*Let $f, f', \ldots, f^{(n-1)}$ be continuous for all $x \geq 1$ and satisfy* (2). *Furthermore, let $f^{(n)}$ be piecewise continuous on every finite interval for $x \geq 1$. Then the transform of $f^{(n)}$ satisfies*

$$R(f^{(n)}) = (s+n)(s+n-1)\cdots(s+1)F(s+n) - (s+n-1)(s+n-2)\cdots f(1) - (s+n-2)(s+n-3)\cdots f'(1) - \cdots - f^{(n-1)}(1).$$

**Example 11** Let $f(x) = x^2$. Then $f(1) = 1$, $f'(x) = 2x$, $f'(1) = 2$, $f''(x) = 2$. Obtain $R\{f\}$, $R\{f'\}$, and $R\{f''\}$.

***Solution.*** $R\{f\} = F(s) = \dfrac{1}{s-2}, \quad F(s+1) = \dfrac{1}{s-1}, \quad F(s+2) = \dfrac{1}{s}.$ Hence, by formulas (4) and (5),

$$R(f') = (s+1)\frac{1}{s-1} - 1 = \frac{2}{s-1} \quad \text{and} \quad R(f'') = (s+2)(s+1)\frac{1}{s} - (s+1) - 2 = \frac{2}{s}.$$

**THEOREM 6 Riemann Transform of Integrals**

*Let $F(s)$ denote the transform of a function $f(x)$ which is piecewise continuous for $x \geq 1$ and satisfies formula* (3). *Then, for $s > 0$, $s > k$, and $x > 1$,*

$$R\left\{\int_1^x f(\tau)\,d\tau\right\} = \frac{1}{s}F(s-1), \qquad \text{thus} \qquad \int_1^x f(\tau)\,d\tau = R^{-1}\left\{\frac{1}{s}F(s-1)\right\}. \tag{6}$$

**PROOF** Let the integral in (6) be $g(x)$ then $g'(x) = f(x)$. Since $g(1) = 0$ (the integral from 1 to 1 is zero),

$$R\{f(x)\} = R\{g'(x)\} = (s+1)G(s+1) - g(1) = (s+1)G(s+1) = F(s),$$

replace $s$ by $s - 1$, $([s-1]+1)G([s-1]+1) = F(s-1) = sG(s) = F(s-1)$.

Division by $s$ and interchange of the left and right side gives the first formula in (6), from which the second follows.

**Example 12** Let $f(x) = x$. Obtain the transform of $g(x) = \int_1^x \tau\, d\tau = G(s)$.

***Solution.*** $F(s) = R\{x\} = \dfrac{1}{s-1}$, $F(s-1) = \dfrac{1}{s-2}$, then $G(s) = \dfrac{1}{s(s-2)}$.

## Differentiation and Integration of Transforms

### Differentiation of Transforms

Given a function $f(x)$, the derivative $F'(s) = dF/ds$ of the transform $F(s) = R(f)$ can be obtained by differentiating $F(s)$ under the integral sign with respect to $s$. Thus, if

$$F(s) = \int_1^\infty f(x)\, x^{-s-1}\, dx, \qquad \text{then} \qquad F'(s) = -\int_1^\infty \ln x\; f(x)\, x^{-s-1}\, dx .$$

Consequently, if $R(f) = F(s)$, then

$$R\{\ln x\, f(x)\} = -F'(s) \qquad \text{and} \qquad R^{-1}\{F'(s)\} = -\ln x\, f(x),$$

where the second formula is obtained by applying on both sides of the first formula. In this way, differentiation of a function corresponds to the multiplication of the function by -ln$x$.

**Example 13** Obtain the transform of $\ln x \sin(\alpha \ln x)$ and $\ln x \cos(\alpha \ln x)$ .

***Solution.***

$$R\{\ln x \sin(\alpha \ln x)\} = -\frac{d}{ds}\left\{\frac{\alpha}{s^2+\alpha^2}\right\} = \frac{2\alpha s}{(s^2 + \alpha^2)^2}$$

$$R\{\ln x \cos(\alpha \ln x)\} = -\frac{d}{ds}\left\{\frac{s}{s^2+\alpha^2}\right\} = -\frac{(s^2+\alpha^2)-2s^2}{(s^2 + \alpha^2)^2} = \frac{s^2-\alpha^2}{(s^2+\alpha^2)^2} .$$

**Integration of Transform**

Given a function *f*(*x*), and the limit of *f*(*x*)/ln*x*, as *x* approaches 1 from the right, exists, then for *s* > *k*,

$$R\left\{\frac{f(x)}{\ln}x\right\} = \int_s^\infty F(\sigma)d\sigma \qquad \text{hence} \qquad R^{-1}\left\{\int_s^\infty F(\sigma)d\sigma\right\} = \frac{f(x)}{\ln x}.$$

In this way, integration of the transform of a function *f*(*x*) corresponds to the division of *f*(*x*) by ln*x*. From the definition it follows that

$$\int_s^\infty F(\sigma)d\sigma = \int_s^\infty\left[\int_1^\infty x^{-\sigma-1}f(x)dx\right]d\sigma = \int_1^\infty f(x)\left[\int_s^\infty x^{-\sigma}d\sigma\right]\frac{dx}{x}.$$

Integration of $x^{-\sigma}$ with respect to *σ* gives $x^{-\sigma}/(-\ln x)$. Hence the integral over *σ* on the right equals $x^{-s}/\ln x$. Therefore,

$$\int_s^\infty F(\sigma)d\sigma = \int_1^\infty x^{-s-1}\frac{f(x)}{\ln x}dx = R\left\{\frac{f(x)}{\ln x}\right\} \qquad (s > k).$$

**Example 14:** Find the inverse transform of $\ln\left(1+\frac{\alpha^2}{s^2}\right) = \ln\left(\frac{s^2+\alpha^2}{s^2}\right)$.

***Solution.*** Denote the given transform by *F*(*s*). Its derivative is

$$F'(s) = \frac{d}{ds}\left[\ln(s^2+\alpha^2) - \ln s^2\right] = \frac{2s}{s^2+\alpha^2} - \frac{2s}{s^2}.$$

Taking the inverse transform, we obtain

$$R^{-1}F'(s) = R^{-1}\left\{\frac{2s}{s^2+\alpha^2} - \frac{2}{s}\right\} = 2\cos(\alpha\ln x) - 2 = -\ln x f(x).$$

Hence the inverse *f*(*x*) of *F*(*s*) is

$$f(x) = \frac{2}{\ln x}(1-\cos(\alpha\ln x)).$$

Alternatively, if we let

$$G(s) = \frac{2s}{s^2+\alpha^2} - \frac{2}{s}, \qquad \text{then} \qquad g(x) = R^{-1}\{G\} = 2[1-\cos(\alpha \ln x)].$$

From this and using the integral of transform we get,

$$R^{-1}\left\{\ln \frac{s^2+\alpha^2}{s^2}\right\} = R^{-1}\left\{\int_s^{\infty} G(s)ds\right\} = -\frac{g(x)}{\ln x} = \frac{2}{\ln x}[1-\cos(\alpha \ln x)].$$

**The Riemann Transform and the Laplace Transform**

The Laplace transform is the integral of $f(y)$ times $e^{-sy}$ from $y = 0$ to $\infty$ where $f(y)$ is defined for all $y \geq 0$. It is denoted by $L\{f\}$,

(7)
$$L\{f\} = \int_0^{\infty} f(y)e^{-sy}dy.$$

The Riemann transform is given below

(8)
$$R\{f\} = \int_1^{\infty} f(x)x^{-s-1}dx.$$

Replace $x = e^y$ ( or $y = \ln x$) in formula (8) and since $x = 1$ to $\infty$, $y = 0$ (ln1) to $\infty$ (ln$\infty$).

$$\int_1^{\infty} f(x)x^{-s-1}dx = \int_0^{\infty} f(e^y)e^{-sy-y}d(e^y) = \int_0^{\infty} f(y)e^{-sy}dy,$$

which is formula (7).

**The Bilateral Laplace Transform**

Formula (7) is usually called the **Unilateral** Laplace transform since the integral is evaluated from 0 to $\infty$. The integral below is known as the Bilateral Laplace transform because the integral is taken from $-\infty$ to $\infty$,

(9)
$$B\{f\} = \int_{-\infty}^{\infty} f(y)e^{-sy}dy.$$

Now, consider the integral equation

(10) $$\int_0^{\infty} f(x)\, x^{-s-1}\, dx\,,$$

Replace $x = e^y$ ( or $y = \ln x$) in formula (4) and since $x = 0$ to $\infty$, $y = -\infty$ to $\infty$, thus

$$\int_0^{\infty} f(x)\, e^{-sx}\, dx \;=\; \int_{-\infty}^{\infty} f(e^y)\, e^{-ys-y}\, d(e^y) \;=\; \int_{-\infty}^{\infty} f(y)\, e^{-sy}\, dy\,,$$

which is (9).

**Riemann Transform: General Formulas**

| Formula | Name |
|---|---|
| $F(s) = R\{f(x)\} = \int_1^{\infty} f(x)\, x^{-s-1}\, dx$<br>$f(x) = R^{-1}(F(s))$ | Definition of Transform<br>Inverse Transform |
| $R\{af(x) + bg(x)\} = aR\{f(x)\} + bR\{g(x)\}$ | Linearity |
| $R\{x^a f(x)\} = F(s-a)$<br>$R^{-1}\{F(s-a)\} = x^a f(x)$ | *s*-Shifting Theorem |
| $R(f') = (s+1)F(s+1) - f(1)$<br>$R(f'') = (s+2)(s+1)F(s+2) - (s+1)f(1) - f'(1)$<br>$R\left\{\int_1^x f(\tau)\, d\tau\right\} = \frac{1}{s}F(s-1),$ | Differentiation of Function<br>Integration of Function |
| $R\{\ln x\, f(x)\} = -F'(s)$<br>$R\left\{\frac{f(x)}{\ln x}\right\} = \int_s^{\infty} F(\sigma)\, d\sigma$ | Differentiation of Transform<br>Integration of Transform |

**Table: Some Riemann Transforms**

| | $f(x)$ | $F(s) = R\{f(x)\}$ |
|---|---|---|
| 1 | $1$ | $\frac{1}{s}$ |
| 2 | $x$ | $\frac{1}{s-1}$ |
| 3 | $x^a$ | $\frac{1}{s-a}$ |
| 4 | $x^{\alpha i}$ | $\frac{1}{s-\alpha i}$ |
| 5 | $\cos(\alpha \ln x)$ | $\frac{s}{s^2+\alpha^2}$ |
| 6 | $\sin(\alpha \ln x)$ | $\frac{\alpha}{s^2+\alpha^2}$ |
| 7 | $\cosh(a \ln x)$ | $\frac{s}{s^2-a^2}$ |
| 8 | $\sinh(a \ln x)$ | $\frac{a}{s^2-a^2}$ |
| 9 | $x^b \cos(\alpha \ln x)$ | $\frac{s-b}{(s-b)^2+\alpha^2}$ |
| 10 | $x^b \sin(\alpha \ln x)$ | $\frac{\alpha}{(s-b)^2+\alpha^2}$ |
| 11 | $\delta(x-a)$ | $a^{-(s+1)}$ |
| 12 | $\sum_{n=1}^{\infty} \delta(x-n)$ | $\sum_{n=1}^{\infty} \frac{1}{n^{s+1}} = \zeta(s+1)$ |
| 13 | $\sum_{p}^{\infty} \delta(x-p)$ | $\sum_{p}^{\infty} p^{-(s+1)}$ |
| 14 | $\mu(x-a)$ | $\frac{a^{-s}}{s}$ |
| 15 | $\sum_{n=1}^{\infty} \mu(x-n)$ | $\frac{1}{s}\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{\zeta(s)}{s}$ |
| 16 | $\sum_{p}^{\infty} \mu(x-p)$ | $\frac{1}{s}\sum_{p}^{\infty} p^{-s}$ |
| 17 | $\frac{2}{\ln x}[1-\cos(\alpha \ln x)]$ | $\ln\left(\frac{s^2+\alpha^2}{s^2}\right)$ |

| | | |
|---|---|---|
| 18 | $\frac{1}{\ln x}\sin(\alpha \ln x)$ | $\arctan\frac{\alpha}{s}$ |
| 19 | $\frac{2}{\ln x}[1-\cosh(a \ln x)]$ | $\ln\left(\frac{s^2 - a^2}{s^2}\right)$ |
| 20 | $\frac{1}{\ln x}(x^b - x^a)$ | $\ln\left(\frac{s - a}{s - b}\right)$ |

REFERENCE

Riemann, Bernhard (1859). *On the Number of Prime Numbers less than a Given Quantity.* pp. 5-7.