Perimeter of the Sierpiński Carpet

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1 The Geometric Series

The geometric series is essential for calculating the perimeter of the fractal after n iterations. The equation is as follows:

$$S_c = a\frac{r^c - 1}{r - 1}$$

We will now apply this to the following sequence:

$$x = 4\left(3^n + 3^{n-1}\right) + 4\left(8^1 3^{n-2} + 8^2 3^{n-3} + \cdots + 8^{n-1} 3^0\right)$$

where:

$$\mathbf{a} = 8 \cdot 3^{n-2}, \quad \mathbf{r} = \frac{8}{3}, \quad \mathbf{c} = n - 1$$

Thus, by substituting the values, we obtain:

$$x = 4\left(3^n + 3^{n-1}\right) + 32\left(3^{n-2}\frac{\left(\frac{8}{3}\right)^{n-1} - 1}{\frac{8}{3} - 1}\right)$$

2 Simplification of the Expression

We will now simplify the above expression step by step in detail to it's simplest form.

2.1 Simplifying the First Term

The first term is:

$$4\left(3^n + 3^{n-1}\right)$$

Factor out 3^{n-1} from the two terms inside the parentheses:

$$3^n + 3^{n-1} = 3^{n-1} \cdot (3 + 1) = 3^{n-1} \cdot 4$$

Thus, the first term becomes:

$$4 \cdot 3^{n-1} \cdot 4 = 16 \cdot 3^{n-1}$$

2.2 Simplifying the Second Term

The second term is:

$$32\left(3^{n-2}\frac{\left(\frac{8}{3}\right)^{n-1}-1}{\frac{8}{3}-1}\right)$$

First, simplify the denominator:

$$\frac{8}{3}-1=\frac{8}{3}-\frac{3}{3}=\frac{5}{3}$$

Now substitute this back into the expression:

$$32 \cdot 3^{n-2} \cdot \frac{\left(\frac{8}{3}\right)^{n-1}-1}{\frac{5}{3}}$$

Next, simplify the division by $\frac{5}{3}$ by multiplying by its reciprocal:

$$= 32 \cdot 3^{n-2} \cdot \left(\frac{3}{5}\right) \cdot \left(\left(\frac{8}{3}\right)^{n-1}-1\right)$$

Simplifying further:

$$= \frac{96}{5} \cdot 3^{n-2} \cdot \left(\left(\frac{8}{3}\right)^{n-1}-1\right)$$

Notice that $\left(\frac{8}{3}\right)^{n-1}$ can be expressed as: $\frac{8^{n-1}}{3^{n-1}}$:

$$= \frac{96}{5} \cdot 3^{n-2} \cdot \left(\frac{8^{n-1}}{3^{n-1}}-1\right)$$

Next, simplify the factor 3^{n-2} and the denominator:

$$= \frac{96}{5} \cdot \frac{3^{n-2}}{3^{n-1}} \cdot \left(8^{n-1}-3^{n-1}\right) = \frac{96}{5} \cdot \frac{1}{3} \cdot \left(8^{n-1}-3^{n-1}\right)$$

Simplify the constants:

$$= \frac{96}{15} \cdot \left(8^{n-1}-3^{n-1}\right) = 6.4 \cdot \left(8^{n-1}-3^{n-1}\right)$$

2.3 Combining Both Terms

Now that we have simplified both terms, we can combine them to obtain the expression

$$x = 16 \cdot 3^{n-1} + 6.4 \cdot \left(8^{n-1} - 3^{n-1}\right)$$

We can simplify this expression further by following the steps below, we will begin by expanding the second term:

$$x = 16 \cdot 3^{n-1} + 6.4 \cdot 8^{n-1} - 6.4 \cdot 3^{n-1}$$

Next, group the terms involving 3^{n-1}:

$$x = (16 \cdot 3^{n-1} - 6.4 \cdot 3^{n-1}) + 6.4 \cdot 8^{n-1}$$

Factor out 3^{n-1} from the first group:

$$x = 3^{n-1} \cdot (16 - 6.4) + 6.4 \cdot 8^{n-1}$$

Simplify $16 - 6.4$:

$$16 - 6.4 = 9.6$$

Thus, the expression becomes:

$$x = 9.6 \cdot 3^{n-1} + 6.4 \cdot 8^{n-1}$$

Which can be written as:

$$x = 9.6(3^n) + 6.4(8^n)$$

For simplicity, we assume the exponent $n = n - 1$, as this transformation is relative and can be easily adjusted as needed. It is important to note that this equation calculates the outward growth of the perimeter. For inward growth, the equation can be adjusted by dividing the result by 3^{n+1}. Thus, if we consider inward growth, the equation becomes:

$$x = \frac{9.6(3^n) + 6.4(8^n)}{3^{n+1}}$$

Note that the inward expansion of the perimeter is not inclined to produce integers. except in the special case where $n = -1$, as the geometric representation is a square.