ON THE EULERIAN NUMBERS AND POWER SUMS

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Abstract. In this short report we discuss a relation between Triangle of Eulerian Numbers and Power sums of the form $\Sigma_k^n k^m$, where n, m are positive integers.

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1. Introduction and Main Results

This manuscript was inspired by the article of Yuyang Zhu, [1], 2018. The author of [1] has done an overview of classical problem of simplifying of power sum

$$\sum_{1\leq k\leq n} k^m,\ (n,m)\geq 0,\ m=\text{const} \tag{1.1}$$

and proposed "A Fast Algorithm to Calculate Power Sum of Natural Numbers". The main result of [1] is based on analysis of certain matrices. In this paper we show the relation between results of [1] and Eulerian Numbers and their role in power sum (1.1). Recall the definition of Eulerian Number $E_{n,k}$

$$E_{n,k} \stackrel{\text{def}}{=} \begin{cases} \sum_{j=0}^{k}(-1)^j\binom{n+1}{j}(k-j+1)^n, & \text{if } 0\leq k\leq n; \\ 0, & \text{otherwise} \end{cases} \tag{1.2}$$

Consider the five initial rows of Triangle of Eulerian numbers

$$\begin{array}{lccccccccccc}
n=0 & & & & & & 1 & & & & & \\
n=1 & & & & & 1 & & 0 & & & & \\
n=2 & & & & 1 & & 1 & & 0 & & & \\
n=3 & & & 1 & & 4 & & 1 & & 0 & & \\
n=4 & & 1 & & 11 & & 11 & & 1 & & 0 & \\
n=5 & 1 & & 26 & & 66 & & 26 & & 1 & & 0
\end{array} \tag{1.3}$$

Figure 1. Triangle of Eulerian Numbers, $E_{n,k},\ n\geq 0,\ 0\leq k\leq n$, [5].

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We denote the terms of above triangle as $E_{n,k},\ n \geq 0,\ 0 \leq k \leq n$, where n is row and k is corresponding term of n-th row. In this manuscript we assume that Eulerian Triangle starts from term $E_{0,0} = 1$ and continues similarly $E_{1,0} = 1,\ E_{1,1} = 0, \dots$. See [2] for a detailed discussion of the Eulerian numbers and many related topics. Now, review some examples form Yuyang's paper [1]. Consider the power sums $\Sigma_{1 \leq k \leq n} k^4$ and $\Sigma_{1 \leq k \leq n} k^{12}$, the following identities hold

$$\sum_{1 \leq k \leq n} k^4 = \binom{n+4}{n-1} + 11\binom{n+3}{n-2} + 11\binom{n+2}{n-3} + \binom{n+1}{n-4}$$

And

$$\begin{aligned}
\sum_{1 \leq k \leq n} k^{12} &= \binom{n+12}{n-1} + 4083\binom{n+11}{n-2} + 478271\binom{n+10}{n-3} \\
&+ 10187685\binom{n+9}{n-4} + 66318474\binom{n+8}{n-5} + 162512286\binom{n+7}{n-6} \\
&+ 162512286\binom{n+6}{n-7} + 66318474\binom{n+5}{n-8} + 10187685\binom{n+4}{n-9} \\
&+ 478271\binom{n+3}{n-10} + 4083\binom{n+2}{n-11} + \binom{n+1}{n-12}
\end{aligned}$$

Now we have to mention that the coefficients in corresponding sums $\Sigma_{1 \leq k \leq n} k^4$ and $\Sigma_{1 \leq k \leq n} k^{12}$ are terms of forth and twelfth rows of Eulerian Triangle (1.3), therefore, these identities can be simplified as follows

$$\sum_{1 \leq k \leq n} k^4 = \sum_{0 \leq k \leq 4} E_{4,k} \binom{n+4-k}{n-1-k} \tag{1.4}$$

And

$$\sum_{1 \leq k \leq n} k^{12} = \sum_{0 \leq k \leq 12} E_{12,k} \binom{n+12-k}{n-1-k} \tag{1.5}$$

Therefore, for every positive integers (n, m) holds

$$\sum_{k=1}^{n} k^m = \sum_{k=0}^{m} E_{m,k} \binom{n+m-k}{n-1-k} = \sum_{k=0}^{m-1} E_{m,k} \binom{n+1+k}{m+1} \tag{1.6}$$

where $E_{m,k}$ are Eulerian numbers.

Proof. Expression (1.6) is direct consequence of Lemma (2.6), Theorem (2.7) and Lemma (2.8) that already proven in [1], pp. 3-4. $\square$

Result of (1.6) is direct consequence of Worpitzky Identity and Symmetry of Binomial coefficients. Recall the power sum, see [6]

$$(*) \quad \sum_{k=1}^{n} k^m = \sum_{k=0}^{m-1} E_{m,k} \binom{n+1+k}{m+1}$$

Now, let compare the binomial coefficients in $(\star)$ and $(*)$, we start to check from the values of corresponding binomial coefficients for $k = 0$ and $k = m-1$ respectively, we have

$$\binom{n+m-(m-1)}{n-1-(m-1)} \equiv \binom{n+1}{m+1} \to \binom{n+1}{n-m} = \binom{n+1}{m+1}$$

Therefore, denote $j = n+1$ and $r = m+1$, symmetry of binomial coefficients holds

$$\binom{j}{r} = \binom{j}{r-j}$$

Now if we substitute the step a instead 1, the identity holds again

$$\binom{n+a}{n-m} = \binom{n+a}{m+a}$$

As per Don Knuth's "Two notes on notation", [3] we don't use upper bound of summation in theorem (1.6) as Eulerian numbers are defined to be 0 when out of range $0 \leq k \leq n$.

2. Conclusion

In this paper we have shown the relation between results of [1] and Eulerian Numbers and their role in power sum (1.1) for every positive integers m, n. Therefore, arXiv:1805.11445 [math.GM] another time proves the Worpitzky Identity.

References

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[6] Eulerian number, From Wikipedia, the free encyclopedia