A Possible Proof Of The Riemann Hypothesis

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Abstract

The Zeta Function and one of its analytic continuations are defined as follows:

$$\forall s \in \mathbb{C} \mid Re(s) > 1,\ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$\forall s \in \mathbb{C} \setminus \left\{1 + \frac{2\pi i k}{ln(2)} \mid k \in \mathbb{Z}\right\},\ \zeta(s) = \frac{\eta(s)}{\left(1 - 2^{1-s}\right)},\ where\ \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$$

The Riemann Hypothesis states the following, for all the nontrivial zeros:

$$\zeta(s) = 0 \implies Re(s) = \frac{1}{2}$$

It has already been proved that $Re(s) \in \,]0,1[$ for all the nontrivial zeros.

Firstly, for $a = Re(s)$ and $b = Im(s)$, we'll prove that:

$$\zeta(s) = 0 \Rightarrow \eta(s) = 0 \Leftrightarrow \sum_{k=1}^{n} \frac{1}{k^{2a}} + 2 \times \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} \frac{(-1)^{k+j-2} \cos(b \ln(k/j))}{(kj)^a} \to 0\ as\ n \to +\infty$$

And since $\forall x \in \mathbb{R},\ -1 \leqslant \cos(x) \leqslant 1$, this implies that there exists a map r_n satisfying $-1 \leqslant r_n \leqslant 1$ for all n sufficiently large, and for which:

$$\sum_{k=1}^{n} \frac{1}{k^{2a}} + 2r_n \times \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} \frac{1}{(kj)^a} \to 0\ as\ n \to +\infty$$

Secondly, by reformulating it as a problem of quadratic equations, we will figure out that this holds true only if $\forall n \in \mathbb{N} \setminus [\![0,3]\!],\ r_n \in \left[-\frac{1}{n-1}, -\frac{1}{n-3}\right] \setminus \left\{-\frac{1}{n-2}\right\}$ where $[\![0,3]\!] = \{0,1,2,3\}$, and therefore, that $r_n \sim -\frac{1}{n}\ as\ n \to +\infty$

And through various asymptotic equivalences, we will get:

$$\sum_{k=1}^{n} \frac{1}{k^{2a}} - \frac{1}{n} \times \left(\sum_{k=1}^{n} \frac{1}{k^a}\right)^2 \to 0\ as\ n \to +\infty$$

Finally, from there, <u>we'll consider $a = Re(s)$ as a map $a_n = Re(s_n)$</u> converging to a real number $a_{+\infty} \in \,]0,1[$, rather than considering it as a fixed value (since we're dealing with infinity).
It is for convenience that we denote $\lim_{n \to +\infty} a_n = a_{+\infty} \in \,]0,1[$.

Then we'll approximate these two sums with integrals depending on $a_{+\infty}$ and asymptotic expansions, and we shall distinguish three different cases:

- $a_{+\infty} \in]0, \frac{1}{2}[$
- $a_{+\infty} \in]\frac{1}{2}, 1[$
- $a_{+\infty} = \frac{1}{2}$

And conclude that the only case that is logically consistent is when $a_{+\infty} = \frac{1}{2}$.

1 Simplifying the expression

First of all, for the sake of simplification, let's write $s = a + ib$ where $a = Re(s)$ and $b = Im(s)$,
We can write the Eta function as follows:

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^{-ib}}{n^a} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} e^{-ib\ln(n)}}{n^a}$$

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cos(-b\ln(n))}{n^a} + i \times \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin(-b\ln(n))}{n^a}$$

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cos(b\ln(n))}{n^a} - i \times \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin(b\ln(n))}{n^a}$$

If we assume $\zeta(s) = 0$, then by the expression of its analytic continuation $\zeta(s) = \dfrac{\eta(s)}{\left(1 - 2^{1-s}\right)}$, we also have $\eta(s) = 0$ and then $|\eta(s)|^2$ is null too:

$$|\eta(s)|^2 = \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cos(b\ln(n))}{n^a}\right)^2 + \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin(b\ln(n))}{n^a}\right)^2 = 0$$

$$thus\ as\ n \to +\infty,\ \left(\sum_{k=1}^{n} \frac{(-1)^{k-1}\cos(b\ln(k))}{k^a}\right)^2 + \left(\sum_{k=1}^{n} \frac{(-1)^{k-1}\sin(b\ln(k))}{k^a}\right)^2 \to 0$$

$$\Longleftrightarrow \sum_{k=1}^{n}\sum_{j=1}^{n} \frac{(-1)^{k+j-2}\cos(b\ln(k))\cos(b\ln(j))}{(kj)^a} + \frac{(-1)^{k+j-2}\sin(b\ln(k))\sin(b\ln(j))}{(kj)^a} \to 0\ as\ n \to +\infty$$

$$\Longleftrightarrow \sum_{k=1}^{n}\sum_{j=1}^{n} (-1)^{k+j-2}\left(\frac{\cos(b\ln(k))\cos(b\ln(j))}{(kj)^a} + \frac{\sin(b\ln(k))\sin(b\ln(j))}{(kj)^a}\right) \to 0\ as\ n \to +\infty$$

$$\Longleftrightarrow \sum_{k=1}^{n}\sum_{j=1}^{n} (-1)^{k+j-2}\left(\frac{\cos(b\ln(k) - b\ ln(j))}{(kj)^a}\right) \to 0\ as\ n \to +\infty$$

$$\Longleftrightarrow \sum_{k=1}^{n}\sum_{j=1}^{n} \frac{(-1)^{k+j-2}\cos(b\ln(k/j))}{(kj)^a} \to 0\ as\ n \to +\infty$$

$$\Longleftrightarrow \sum_{k=1}^{n} \frac{(-1)^{2k-2}}{k^{2a}} + \sum_{k=1}^{n} \sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{(-1)^{k+j-2} \cos(b \ln(k/j))}{(kj)^a} \to 0 \; as \; n \to +\infty$$

$$\Longleftrightarrow \sum_{k=1}^{n} \frac{1}{k^{2a}} + 2 \times \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} \frac{(-1)^{k+j-2} \cos(b \ln(k/j))}{(kj)^a} \to 0 \; as \; n \to +\infty$$

$$\forall k, j \in [\![1, n]\!], \forall b \in \mathbb{R}, \; -1 \leqslant \cos(b \ln(k/j)) \leqslant 1$$

Thus there exists a map r_n satisfying $-1 \leqslant r_n \leqslant 1$ for all n sufficiently large, and for which:

$$\sum_{k=1}^{n} \frac{1}{k^{2a}} + 2r_n \times \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} \frac{1}{(kj)^a} \to 0 \; as \; n \to +\infty$$

And we end up with what curiously resembles a quadratic equation.

2 The "Russian Doll" Quadratic Equations

Now let's assume there is $x_1, \ldots, x_n \in \mathbb{R}$ with $x_1 = 1$ so that:

$$\sum_{k=1}^{n} x_k^2 + 2r_n \times \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} x_k x_j = 0$$

And let's try and figure out which kind of map r_n is.

But first, let's define $\forall n \in \mathbb{N}^*, \; u_n = \sum_{k=1}^{n} x_k^2$, $v_n = \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} x_k x_j$ and $p_n = \sum_{k=1}^{n} x_k$

Our previous equation becomes:

$$u_n \; + \; 2r_n v_n = x_n^2 + 2r_n p_{n-1} x_n + u_{n-1} + 2r_n v_{n-1} = 0$$

And now let's define $(f_n)_{n \in \mathbb{N} \setminus \{0,1\}}$ and $(g_n)_{n \in \mathbb{N} \setminus \{0,1\}}$ so that $\forall n \in \mathbb{N} \setminus \{0, 1\}$:

$$f_n u_n \; + \; g_n v_n = f_n x_n^2 + g_n p_{n-1} x_n + f_n u_{n-1} + g_n v_{n-1} = 0$$

Let's now express the delta Δ_n of this equation and find the expressions of f_{n-1} and g_{n-1} so that $\Delta_n = f_{n-1} u_{n-1} + g_{n-1} v_{n-1} \geqslant 0$:

$$\Delta_n = (g_n p_{n-1})^2 - 4 f_n (f_n u_{n-1} + g_n v_{n-1}),$$

$$p_{n-1}^2 = \left(\sum_{k=1}^{n-1} x_k \right)^2 = u_{n-1} + 2 v_{n-1},$$

$$thus \; \Delta_n = (g_n p_{n-1})^2 - 4 f_n (f_n u_{n-1} + g_n v_{n-1}) = g_n^2 (u_{n-1} + 2 v_{n-1}) - 4 f_n (f_n u_{n-1} + g_n v_{n-1})$$

$$\Delta_n = \left(g_n^2 - 4 f_n^2 \right) u_{n-1} + \left(2 g_n^2 - 4 f_n g_n \right) v_{n-1}$$

We conclude that $f_{n-1} = g_n^2 - 4f_n^2$ and $g_{n-1} = 2g_n^2 - 4f_n g_n$, and we see Δ_n is in turn a new quadratic equation:

$$\Delta_n = f_{n-1}x_{n-1}^2 + g_{n-1}p_{n-2}x_{n-1} + f_{n-1}u_{n-2} + g_{n-1}v_{n-2}$$

with a new Δ_{n-1} for which we must determine the conditions to ensure $\Delta_{n-1} \geqslant 0,$ and so on until Δ_2 (hence the comparison with a Russian doll).

But also, $\dfrac{g_{n-1}}{f_{n-1}} = \dfrac{2g_n^2 - 4f_n g_n}{g_n^2 - 4f_n^2} = \dfrac{2g_n(g_n - 2f_n)}{(g_n - 2f_n)(g_n + 2f_n)} = \dfrac{2g_n}{(g_n + 2f_n)} = \dfrac{2\frac{g_n}{f_n}}{\frac{g_n}{f_n} + 2}$

We observe that each time we calculate a Δ_{n-k}, we actually apply $h : x \mapsto \dfrac{2x}{x+2}$ to the ratio $\dfrac{g_{n-k}}{f_{n-k}}$ to obtain $\dfrac{g_{n-k-1}}{f_{n-k-1}}$: $\forall k \in [\![1, n-3]\!]$, $\dfrac{g_{n-k-1}}{f_{n-k-1}} = \dfrac{2\frac{g_{n-k}}{f_{n-k}}}{\frac{g_{n-k}}{f_{n-k}} + 2}$.

In our precise case, $f_n = 1$ and $g_n = 2r_n$, so $\dfrac{g_n}{f_n} = 2r_n$; our f_{n-1} and g_{n-1} thus become:

$$f_{n-1} = \left(4r_n^2 - 4\right)f_n^2 = 4\left(r_n^2 - 1\right)f_n^2 = 4(r_n - 1)(r_n + 1)f_n^2$$
$$g_{n-1} = \left(2 \times 4r_n^2 - 4 \times 2r_n\right)f_n^2 = 8\left(r_n^2 - r_n\right)f_n^2 = 8r_n(r_n - 1)f_n^2$$

Thus, $\dfrac{g_{n-1}}{f_{n-1}} = \dfrac{8r_n(r_n - 1)f_n^2}{4(r_n - 1)(r_n + 1)f_n^2} = \dfrac{2r_n}{r_n + 1}$.

Now, let's prove by induction that $\forall k \in [\![1, n-2]\!]$, $\dfrac{g_{n-k}}{f_{n-k}} = \dfrac{2r_n}{k \times r_n + 1}$:

Let's assume $\exists k \in [\![1, n-3]\!]$, $\dfrac{g_{n-k}}{f_{n-k}} = \dfrac{2r_n}{k \times r_n + 1}$,

Then we have:

$$\frac{g_{n-k-1}}{f_{n-k-1}} = h\left(\frac{g_{n-k}}{f_{n-k}}\right) = 2 \times \frac{2r_n}{k \times r_n + 1} \times \frac{1}{\frac{2r_n}{k \times r_n + 1} + 2} = 2 \times 2r_n \times \frac{1}{2r_n + 2(k \times r_n + 1)}$$
$$\Leftrightarrow \frac{g_{n-k-1}}{f_{n-k-1}} = h\left(\frac{g_{n-k}}{f_{n-k}}\right) = \frac{4r_n}{2(r_n + k \times r_n + 1)} = \frac{2r_n}{(k+1) \times r_n + 1}$$

Which proves that $\forall k \in [\![1, n-2]\!]$, $\dfrac{g_{n-k}}{f_{n-k}} = \dfrac{2r_n}{k \times r_n + 1}$.

Now, $\forall n \in \mathbb{N} \setminus [\![0, 3]\!]$, $\forall k \in [\![1, n-2]\!]$ we can express all the Δ_{n-k}, and above all the following:

$\Delta_3 = f_2u_2 + g_2v_2 = f_2x_2^2 + f_2x_1^2 + g_2x_2x_1 = f_2x_2^2 + g_2x_2 + f_2$ (because $x_1 = 1$)

$\Delta_2 = g_2^2 - 4f_2^2 = 4 \times \left(\dfrac{r_n^2}{[(n-2) \times r_n + 1]^2} - 1\right) \times f_2^2$

To determine the positivity of Δ_2 we only focus on the positivity of $\dfrac{r_n^2}{[(n-2)\times r_n+1]^2}-1$,

for we know f_2^2 and 4 are always positive.

$$\frac{r_n^2}{[(n-2)\times r_n+1]^2}-1 \geqslant 0 \;\Leftrightarrow\; r_n^2 \geqslant [(n-2)\times r_n+1]^2 \Leftrightarrow \left[1-(n-2)^2\right]r_n^2-2(n-2)r_n-1 \geqslant 0$$

$$\Delta = 4(n-2)^2-4\times(-1)\left[1-(n-2)^2\right] = 4\left[(n-2)^2+1-(n-2)^2\right] = 4 > 0$$

So solutions for all of our previous Δ_k exist;

$\forall n \in \mathbb{N} \setminus [\![0,3]\!]$, the quadratic coefficient $\left[1-(n-2)^2\right]$ is strictly negative, so:

$$r_n \in \left[\frac{2(n-2)-\sqrt{4}}{2\left[1-(n-2)^2\right]}, \frac{2(n-2)+\sqrt{4}}{2\left[1-(n-2)^2\right]}\right] \setminus \left\{-\frac{1}{n-2}\right\}$$

which means:

$$r_n \in \left[\frac{(n-2)-1}{1-(n-2)^2}, \frac{(n-2)+1}{1-(n-2)^2}\right] \setminus \left\{-\frac{1}{n-2}\right\}$$

$$\Leftrightarrow r_n \in \left[\frac{(n-2)-1}{(1-n+2)(1+n-2)}, \frac{(n-2)+1}{(1-n+2)(1+n-2)}\right] \setminus \left\{-\frac{1}{n-2}\right\}$$

$$\Leftrightarrow r_n \in \left[-\frac{1}{n-1}, -\frac{1}{n-3}\right] \setminus \left\{-\frac{1}{n-2}\right\},\ \forall n \in \mathbb{N} \setminus [\![0,3]\!]$$

(we exclude $-\dfrac{1}{n-2}$ because of the term $\dfrac{r_n^2}{[(n-2)\times r_n+1]^2}$ in Δ_2);

Therefore, as $n \to +\infty,\ r_n \sim -\dfrac{1}{n}$

In conclusion, for the following to be true, as $n \to +\infty$:

$$\sum_{k=1}^{n} x_k^2 + 2r_n \times \sum_{k=1}^{n-1}\sum_{j=k+1}^{n} x_k x_j \to 0$$

We must have it in the following form:

$$\sum_{k=1}^{n} x_k^2 - \frac{2}{n} \times \sum_{k=1}^{n-1}\sum_{j=k+1}^{n} x_k x_j \to 0 \; as \; n \to +\infty$$

Now we could simplify this:

$$\sum_{k=1}^{n} x_k^2 - \frac{2}{n} \times \sum_{k=1}^{n-1}\sum_{j=k+1}^{n} x_k x_j = \sum_{k=1}^{n} x_k^2 - \frac{1}{n} \times \sum_{k=1}^{n}\sum_{\substack{j=1 \\ j\neq k}}^{n} x_k x_j$$

$$= \sum_{k=1}^{n} x_k^2 - \frac{1}{n} \times \left(\sum_{k=1}^{n}\sum_{j=1}^{n} x_k x_j - \sum_{k=1}^{n} x_k^2\right) = 0$$

$$\Leftrightarrow \left(1+\frac{1}{n}\right) \times \sum_{k=1}^{n} x_k^2 - \frac{1}{n} \times \sum_{k=1}^{n}\sum_{j=1}^{n} x_k x_j = 0$$

And as $n \longrightarrow +\infty$ the asymptotic equivalences give us the following:

$$\sum_{k=1}^{n} x_k^2 - \frac{1}{n} \times \sum_{k=1}^{n}\sum_{j=1}^{n} x_k x_j \rightarrow 0 \; as \; n \rightarrow +\infty$$

$$\Leftrightarrow \sum_{k=1}^{n} x_k^2 - \frac{1}{n} \times \left(\sum_{k=1}^{n} x_k\right)^2 \rightarrow 0 \; as \; n \rightarrow +\infty$$

Now to get back to our problem, if we assume that $\forall k \in [\![1, n]\!],\ x_k = \frac{1}{k^a}$, then we get, as $n \longrightarrow +\infty$:

$$\sum_{k=1}^{n} \frac{1}{k^{2a}} - \frac{1}{n} \times \left(\sum_{k=1}^{n} \frac{1}{k^a}\right)^2 \rightarrow 0$$

Which is therefore - thanks to all we've seen up to now - the new formula on which we'll work from now on, and which is way more easy-to-handle and less obscure than:

$$\sum_{k=1}^{n} \frac{1}{k^{2a}} + 2 \times \sum_{k=1}^{n-1}\sum_{j=k+1}^{n} \frac{(-1)^{k+j-2}\cos(b\ln(k/j))}{(kj)^a} \rightarrow 0 \; as \; n \rightarrow +\infty$$

3 Comparison Of Asymptotic Behaviours

Now, We got this expression from the previous part:

$$\sum_{k=1}^{n} \frac{1}{k^{2a}} - \frac{1}{n} \times \left(\sum_{k=1}^{n} \frac{1}{k^a}\right)^2 \rightarrow 0, \; as \; n \rightarrow +\infty$$

Since we're dealing with infinity, instead of distinguishing the cases for different fixed values for $a \in]0,1[$, **I will speak of a map $(a_n)_{n\in\mathbb{N}^*}$ converging to a real number in $]0,1[$:** $\lim_{n\rightarrow+\infty} a_n \longrightarrow a_{+\infty} \in]0,1[$ with a rate of convergence $\epsilon_n = a_n - a_{+\infty}$.

The sums with their corrections (first-order Taylor expansions) become, as $n \longrightarrow +\infty$:

$$\sum_{k=1}^{n} \frac{1}{k^{2a_{+\infty}}} - 2\epsilon_n \sum_{k=1}^{n} \frac{\ln(k)}{k^{2a_{+\infty}}} - \frac{1}{n} \times \left(\sum_{k=1}^{n} \frac{1}{k^{a_{+\infty}}} - \epsilon_n \sum_{k=1}^{n} \frac{\ln(k)}{k^{a_{+\infty}}}\right)^2 \rightarrow 0$$

The correction terms can be ignored for a fast convergence of a_n;

We'll deal with fast and slow convergences, and also figure out the expansions of the p-th order.

A fast convergence with such sums typically means $\epsilon_n = o\left(\frac{1}{\ln(n)}\right)$.

It has already been well-established in the literature [1, 2] that $a_{+\infty} \in]0,1[$ for all the nontrivial zeros, so $1 - a_{+\infty} > 0$ and then the **squared sum** can be approximated with the **following squared integral as follows if a_n converges fastly to its limit**:

$$\left(\int_1^n \frac{1}{t^a} dt\right)^2 = \frac{\left(n^{1-a} - 1\right)^2}{(1-a)^2} \sim \frac{n^{2-2a}}{(1-a)^2} \; as \; n \to +\infty$$

to obtain the following (I omit the n index of a_n for convenience in these calculations):

$$\forall a \in]0,1[\; and \; as \; n \to +\infty, \; \frac{1}{n} \times \left(\sum_{k=1}^{n} \frac{1}{k^a}\right)^2 \sim \frac{1}{n} \times \frac{n^{2-2a}}{(1-a)^2} = \frac{n^{1-2a}}{(1-a)^2}$$

And for a slow convergence, the sum of the correction term added in the squared sum:

$$\forall n \in \mathbb{N}^*, \int_1^{n+1} \frac{\ln(t)}{t^{a_{+\infty}}} dt \leqslant \sum_{k=1}^{n} \frac{\ln(k)}{k^{a_{+\infty}}} \leqslant \int_2^{n+1} \frac{\ln(t-1)}{(t-1)^{a_{+\infty}}} dt$$

$$with \int_1^n \frac{\ln(t)}{t^{a_{+\infty}}} dt = \frac{\ln(n) n^{1-a_{+\infty}}}{1 - a_{+\infty}} - \frac{n^{1-a_{+\infty}} - 1}{(1-a)^2}$$

Therefore, since $1 - a_{+\infty} > 0$ we get the following asymptotic equivalence:

$$\sum_{k=1}^{n} \frac{\ln(k)}{k^{a_{+\infty}}} \sim \int_1^n \frac{\ln(t)}{t^{a_{+\infty}}} dt \sim \frac{\ln(n) n^{1-a_{+\infty}}}{1 - a_{+\infty}} - \frac{n^{1-a_{+\infty}}}{(1-a)^2} \; as \; n \to +\infty$$

For any order of expansion p, we figure out the p-th term as follows:

$$\frac{(-\epsilon_n)^p}{p!} \sum_{k=1}^{n} \frac{\ln(k)^p}{k^{a_{+\infty}}} \sim \frac{(-\epsilon_n)^p}{p!} \int_1^n \frac{\ln(t)^p}{t^{a_{+\infty}}} dt \; as \; n \to +\infty$$

$$\int_1^n \frac{\ln(t)^p}{t^{a_{+\infty}}} dt = \frac{\ln(n)^p n^{1-a_{+\infty}}}{(1 - a_{+\infty})} \times \sum_{k=0}^{p} \frac{p!(-1)^k}{(p-k)!(1-a_{+\infty})^k \ln(n)^k} - \frac{p!(-1)^p}{(1-a_{+\infty})^p}$$

$$thus \int_1^n \frac{\ln(t)^p}{t^{a_{+\infty}}} dt \sim \frac{\ln(n)^p n^{1-a_{+\infty}}}{(1 - a_{+\infty})} \; as \; n \to +\infty$$

So the p-th term in the Taylor approximation is written:

$$\frac{n^{1-a_{+\infty}} (-\epsilon_n)^p \ln(n)^p}{(1 - a_{+\infty}) p!}$$

And the total expression involving the asymptotic expansion using the Taylor approximation of order p is then written:

$$\frac{n^{1-a_{+\infty}}}{1 - a_{+\infty}} + \sum_{k=1}^{p} \frac{n^{1-a_{+\infty}} (-\epsilon_n)^k \ln(n)^k}{(1 - a_{+\infty}) k!} = \frac{n^{1-a_{+\infty}}}{1 - a_{+\infty}} \times \left(1 + \sum_{k=1}^{p} \frac{(-\epsilon_n \ln(n))^k}{k!}\right)$$

As to the <u>sum of squares</u>, for a fast convergence:

$$\forall n \in \mathbb{N}^*, \int_1^{n+1} \frac{1}{t^{2a}} dt \leqslant \sum_{k=1}^{n} \frac{1}{k^{2a}} \leqslant 1 + \int_2^{n+1} \frac{1}{(t-1)^{2a}} dt$$

And the sum of the correction term added for a slow convergence:

$$\forall n \in \mathbb{N}^*, \int_1^{n+1} \frac{\ln(t)}{t^{2a_{+\infty}}} dt \leqslant \sum_{k=1}^{n} \frac{\ln(k)}{k^{2a_{+\infty}}} \leqslant \int_2^{n+1} \frac{\ln(t-1)}{(t-1)^{2a_{+\infty}}} dt$$

We get to distinguish $a_{+\infty} \neq \frac{1}{2}$ and $a_{+\infty} = \frac{1}{2}$ for the sum of squares.

If $a_{+\infty} \neq \frac{1}{2}$:
Fast convergence:

$$\frac{(n+1)^{1-2a} - 1}{1-2a} \leqslant \sum_{k=1}^{n} \frac{1}{k^{2a}} \leqslant 1 + \frac{(n+1-1)^{1-2a} - (2-1)^{1-2a}}{1-2a}$$

(I skipped the details of variable substitution on the right side)

We then obtain the following asymptotic equivalences, as $n \longrightarrow +\infty$:

$$\frac{n^{1-2a} - 1}{1-2a} \leqslant \sum_{k=1}^{n} \frac{1}{k^{2a}} \leqslant 1 + \frac{n^{1-2a} - 1}{1-2a}$$

Which means that as $n \longrightarrow +\infty, \exists \lambda \in [0,1], \sum_{k=1}^{n} \frac{1}{k^{2a}} \sim \lambda + \frac{n^{1-2a} - 1}{1-2a}$

Sum of the correction term added for a slow convergence (asymptotic equivalent as $n \longrightarrow +\infty$):

$$\sum_{k=1}^{n} \frac{\ln(k)}{k^{2a_{+\infty}}} \sim \int_1^{n} \frac{\ln(t)}{t^{2a_{+\infty}}} dt = \frac{\ln(n) n^{1-2a_{+\infty}}}{1-2a_{+\infty}} - \frac{n^{1-2a_{+\infty}} - 1}{(1-2a_{+\infty})^2}$$

For any order of expansion p, we figure out the p-th term as follows:

$$\frac{(-2\epsilon_n)^p}{p!} \sum_{k=1}^{n} \frac{\ln(k)^p}{k^{2a_{+\infty}}} \sim \frac{(-2\epsilon_n)^p}{p!} \int_1^{n} \frac{\ln(t)^p}{t^{2a_{+\infty}}} dt \text{ as } n \longrightarrow +\infty$$

$$\int_1^{n} \frac{\ln(t)^p}{t^{2a_{+\infty}}} dt = \frac{\ln(n)^p n^{1-2a_{+\infty}}}{(1-2a_{+\infty})} \times \sum_{k=0}^{p} \frac{p!(-1)^k}{(p-k)!(1-2a_{+\infty})^k \ln(n)^k} - \frac{p!(-1)^p}{(1-2a_{+\infty})^{p+1}}$$

$$\text{thus } \int_1^{n} \frac{\ln(t)^p}{t^{2a_{+\infty}}} dt \sim \frac{\ln(n)^p n^{1-2a_{+\infty}}}{(1-2a_{+\infty})} \text{ as } n \longrightarrow +\infty$$

So the p-th term in the Taylor approximation is written:

$$\frac{n^{1-2a_{+\infty}} (-2\epsilon_n)^p \ln(n)^p}{(1-2a_{+\infty}) p!}$$

And the total expression involving the asymptotic expansion using the Taylor approximation of order p is then written:

$$\lambda + \frac{n^{1-2a_{+\infty}} - 1}{1-2a_{+\infty}} + \sum_{k=1}^{p} \frac{n^{1-2a_{+\infty}} (-2\epsilon_n)^k \ln(n)^k}{(1-2a_{+\infty}) k!}$$

$$= \lambda - \frac{1}{1-2a_{+\infty}} + \frac{n^{1-2a_{+\infty}}}{1-2a_{+\infty}} \times \left(1 + \sum_{k=1}^{p} \frac{(-2\epsilon_n \ln(n))^k}{k!}\right),\ \lambda \in [0,1]$$

If $a_{+\infty} = \frac{1}{2}$:
Fast convergence:

$$\exists \lambda \in [0,1] \mid \sum_{k=1}^{n} \frac{1}{k^{2a}} = \lambda + \ln(n)\ as\ n \to +\infty$$

Sum of the correction term for a slow convergence (asymptotic equivalent as $n \to +\infty$):

$$\sum_{k=1}^{n} \frac{\ln(k)}{k^{2a_{+\infty}}} \sim \int_1^n \frac{\ln(t)}{t} dt\ as\ n \to +\infty$$

$$\int_1^n \frac{\ln(t)}{t} dt = \ln(n)^2 - \int_1^n \frac{\ln(t)}{t}$$

$$\Leftrightarrow \int_1^n \frac{\ln(t)}{t} dt = \frac{\ln(n)^2}{2}$$

For any order of expansion p, we figure out the p-th term as follows:

$$\frac{(-2\epsilon_n)^p}{p!} \sum_{k=1}^{n} \frac{\ln(k)^p}{k} \sim \frac{(-2\epsilon_n)^p}{p!} \int_1^n \frac{\ln(t)^p}{t} dt\ as\ n \to +\infty$$

$$\int_1^n \frac{\ln(t)^p}{t} dt = \ln(n)^{p+1} - p \times \int_1^n \frac{\ln(t)^p}{t} dt$$

$$\Leftrightarrow \int_1^n \frac{\ln(t)^p}{t} dt = \frac{\ln(n)^{p+1}}{p+1}$$

So the p-th term in the Taylor approximation is written:

$$\frac{(-2\epsilon_n)^p \ln(n)^{p+1}}{(p+1)!} = \ln(n) \times \frac{(-2\epsilon_n)^p \ln(n)^p}{(p+1)!}$$

And the total expression involving the asymptotic expansion using the Taylor approximation of order p is then written:

$$\lambda + \ln(n)\left(1 + \sum_{k=1}^{p} \frac{(-2\epsilon_n \ln(n))^k}{(k+1)!}\right),\ \lambda \in [0,1]$$

Remark:
Instead of expanding the sums first and then approximating each term with integrals, I could have done this right away:

$$\forall a_{+\infty} \in]0,1[\ as\ n \to +\infty:$$

$$\frac{1}{n}\left(\sum_{k=1}^{n} \frac{1}{k^{a_{+\infty}+\epsilon_n}}\right)^2 \sim \frac{1}{n}\left(\int_1^n \frac{1}{t^{a_{+\infty}+\epsilon_n}} dt\right)^2 \sim \frac{n^{1-2(a_{+\infty}+\epsilon_n)}}{(1-(a_{+\infty}+\epsilon_n))^2}$$

$$\sum_{k=1}^{n} \frac{1}{k^{2(a_{+\infty}+\epsilon_n)}} \sim \int_1^n \frac{1}{t^{2(a_{+\infty}+\epsilon_n)}} dt \sim \lambda + \frac{n^{1-2(a_{+\infty}+\epsilon_n)} - 1}{1-2(a_{+\infty}+\epsilon_n)},\ \lambda \in [0,1]$$

This way we would just expand $n^{-2\epsilon_n}$ and the outcome would have been exactly the same as all that follows. Therefore, in upcoming versions I may remove the parts dealing with any finite order of approximation $p \in \mathbb{N}^*$ and directly jump into $p \longrightarrow +\infty$.

We therefore have three different cases:

- $a_{+\infty} \in]0, \frac{1}{2}[$
- $a_{+\infty} \in]\frac{1}{2}, 1[$
- $a_{+\infty} = \frac{1}{2}$

Case $a_{+\infty} \in]0, \frac{1}{2}[$:

If a_n converges fastly enough to its limit, we can take the following for granted:
$1 - 2a_{+\infty} > 0$ so n^{1-2a} grows unboundedly as $n \longrightarrow +\infty$, so:

$$\sum_{k=1}^{n} \frac{1}{k^{2a}} \sim \frac{n^{1-2a}}{1-2a}\ as\ n \longrightarrow +\infty$$

Thus our expression:

$$\sum_{k=1}^{n} \frac{1}{k^{2a}} - \frac{n^{1-2a}}{(1-a)^2} \longrightarrow 0\ as\ n \longrightarrow +\infty$$

becomes, as $n \longrightarrow +\infty$:

$$\lambda - \frac{1}{1-2a} + \frac{n^{1-2a}}{1-2a} - \frac{n^{1-2a}}{(1-a)^2} \longrightarrow 0 \Leftrightarrow \lambda - \frac{1}{1-2a} + n^{1-2a}\left(\frac{1}{1-2a} - \frac{1}{(1-a)^2}\right) \longrightarrow 0$$

Since n^{1-2a} tends to infinity, this requires two things at once:

$$\frac{1}{1-2a} - \frac{1}{(1-a)^2} = 0\ and\ \lambda - \frac{1}{1-2a} = 0$$

Thus:

$$\frac{1}{1-2a} - \frac{1}{(1-a)^2} = 0 \Leftrightarrow (1-a_{+\infty})^2 = 1 - 2a_{+\infty} \Leftrightarrow 1 - 2a_{+\infty} + a_{+\infty}^2 = 1 - 2a_{+\infty}$$
$$\Leftrightarrow a_{+\infty}^2 = 0 \Leftrightarrow a_{+\infty} = 0$$
$$and$$

$$\lambda - \frac{1}{1-2a} = 0 \Leftrightarrow (1-2a)\lambda = 1 \Leftrightarrow 2a_{+\infty}\lambda = \lambda - 1$$
$$\Leftrightarrow a_{+\infty} = \frac{\lambda - 1}{2\lambda} \leqslant 0 \; since \; \lambda \in [0,1]$$

And both of these contradict $a_{+\infty} \in]0, \frac{1}{2}[$.

If the convergence is slow, the expression with the correction terms is as follows, for the asymptotic expansions using the Taylor approximations of infinite order $p \to +\infty$:

We get the following, since $\forall z \in \mathbb{C}, e^z = \sum_{p=0}^{\infty} \frac{(z)^p}{p!}$:

$$\lambda - \frac{1}{1-2a_{+\infty}} + n^{1-2a_{+\infty}}\left(\frac{n^{-2\epsilon_n}}{1-2a_{+\infty}} - \frac{\left(n^{-\epsilon_n}\right)^2}{(1-a_{+\infty})^2}\right) \xrightarrow[n \to +\infty]{} 0$$
$$\Leftrightarrow \lambda - \frac{1}{1-2a_{+\infty}} + n^{1-2(a_{+\infty}+\epsilon_n)}\left(\frac{1}{1-2a_{+\infty}} - \frac{1}{(1-a_{+\infty})^2}\right) \xrightarrow[n \to +\infty]{} 0$$

$n^{1-2(a_{+\infty}+\epsilon_n)}$ tends to infinity, the same reasoning as in the case of a fast convergence apply.

That is, this requires $a_{+\infty} = 0$ and $a_{+\infty} = \frac{\lambda - 1}{2\lambda} \leqslant 0$ **which contradict** $a_{+\infty} \in]0, \frac{1}{2}[$.

Case $a_{+\infty} \in]\frac{1}{2}, 1[$:

If a_n **converges fastly enough to its limit, we can take the following for granted**:

As $n \to +\infty, \; \exists \lambda \in [0,1], \sum_{k=1}^{n} \frac{1}{k^{2a}} \sim \lambda + \frac{n^{1-2a} - 1}{1-2a}$

In this case, $1 - 2a_{+\infty} < 0$,

Therefore as $n \to +\infty, \sum_{k=1}^{n} \frac{1}{k^{2a}} \to \lambda + \frac{1}{2a-1}$ and then $\sum_{k=1}^{n} \frac{1}{k^{2a}} - \frac{n^{1-2a}}{(1-a)^2} \xrightarrow[n \to +\infty]{} 0$ becomes:

$\lambda + \frac{1}{2a-1} - \frac{n^{1-2a}}{(1-a)^2} \xrightarrow[n \to +\infty]{} 0$, and since $1 - 2a_{+\infty} < 0$ this means $\lambda + \frac{1}{2a-1} = 0$

$\Leftrightarrow (2a-1)\lambda + 1 = 0 \Leftrightarrow 2a\lambda = \lambda - 1 \Leftrightarrow a_{+\infty} = \frac{\lambda - 1}{2\lambda} \leqslant 0$ because $\lambda - 1 \leqslant 0$ while $2\lambda \geqslant 0$ for $\lambda \in [0,1]$, **which contradicts** $a_{+\infty} \in]\frac{1}{2}, 1[$.

If the convergence is slow, the expression with the correction terms is as follows, for the asymptotic expansions using the Taylor approximations of infinite order $p \to +\infty$:

We get the following, since $\forall z \in \mathbb{C}, e^z = \sum_{p=0}^{\infty} \frac{(z)^p}{p!}$:

$$\lambda - \frac{1}{1-2a_{+\infty}} + n^{1-2a_{+\infty}}\left(\frac{n^{-2\epsilon_n}}{1-2a_{+\infty}} - \frac{\left(n^{-\epsilon_n}\right)^2}{(1-a_{+\infty})^2}\right) \xrightarrow[n \to +\infty]{} 0$$

$$with\ \lambda \in [0,1]$$
$$\Leftrightarrow \lambda - \frac{1}{1-2a_{+\infty}} + n^{1-2(a_{+\infty}+\epsilon_n)}\left(\frac{1}{1-2a_{+\infty}} - \frac{1}{(1-a_{+\infty})^2}\right) \xrightarrow[n\to+\infty]{} 0$$

Which implies (again) $\lambda + \frac{1}{2a_{+\infty}-1} = 0$ which implies $a_{+\infty} = \frac{\lambda-1}{2\lambda} \leqslant 0$ **which contradicts** $a_{+\infty} \in]\frac{1}{2}, 1[$ again.

Case $a_{+\infty} = \frac{1}{2}$:

If a_n converges fastly enough to its limit, we can take the following for granted:

As $n \to +\infty$, $\sum_{k=1}^{n} \frac{1}{k^{2a_n}} \sim \lambda + \ln(n),\ \lambda \in [0,1]$, so $\sum_{k=1}^{n} \frac{1}{k^{2a_n}} - \frac{n^{1-2a_n}}{(1-a_n)^2} \to 0\ as\ n \to +\infty$ becomes:

$$\lambda + \ln(n) - \frac{n^{1-2a_n}}{(1-a_n)^2} \to 0\ as\ n \to +\infty$$

And now, let's reflect upon the conditions for this statement to hold:

- As said earlier, we deal with a map $(a_n)_{n\in\mathbb{N}^*}$ converging to a real number in $]0,1[$ as $n \to +\infty$, $\frac{1}{2}$ in this case, rather than a fixed value $a = \frac{1}{2}$, otherwise it would mean that $\lim_{n\to+\infty} \ln(n) \to 4 - \lambda$ which is absurd,
- $\frac{1}{(1-a_n)^2} \to 4\ as\ n \to +\infty$, so it doesn't affect the asymptotic behaviour of n^{1-2a_n},
- $\ln(n)$ grows unboundedly as $n \to +\infty$, so we must have $1 - 2a_n > 0$ for all n sufficiently large, for n^{1-2a_n} to grow unboundedly as $n \to +\infty$ as well,
- Had we assumed that $\exists l > 0 \mid \lim_{n\to+\infty} 1 - 2a_n = l$, we would get $\ln(n) - \frac{n^l}{\left(\frac{l+1}{2}\right)^2} \to 0$ as $n \to +\infty$, which is impossible because $\forall l > 0,\ \ln(n) = o\left(n^l\right)$, therefore this subtraction tends to $-\infty$ and not 0 as $n \to +\infty$,
- So it is necessary that $1 - 2a_n$ be **strictly positive** <u>for all n sufficiently large</u> while **converging to** 0^+ as $n \to +\infty$, in order to adequately "bend" n^{1-2a_n} for it to match $\ln(n)$, for the subtraction to tend to zero,

You'd think we finished, the problem is, even in this case, the expression of $\epsilon_n = a_n - a_{+\infty}$ becomes something like $\epsilon_n = -\frac{\ln\left(\frac{\lambda+\ln(n)}{4}\right)}{2\ln(n)}$, which is **too slow a rate of convergence to neglect the correction terms**; which we ironically did here.

So our last chance is actually dealing with the correction terms:

We need to use the asymptotic expansions using the Taylor approximations to figure out what is required by:

$$\sum_{k=1}^{n} \frac{1}{k^{2a_n}} - \frac{1}{n} \times \left(\sum_{k=1}^{n} \frac{1}{k^{a_n}} \right)^2 \to 0$$

Using the first-order Taylor approximations:

$$\ln(n) - 2\epsilon_n \left(\frac{\ln(n)^2}{2} \right) - \frac{n^{1-2\frac{1}{2}} \left(1 - 2\epsilon_n \ln(n) + \epsilon_n^2 \ln(n)^2\right)}{\left(1 - \frac{1}{2}\right)^2} \to 0,\ as\ n \to +\infty$$

$$\Leftrightarrow \ln(n)(1 - \epsilon_n \ln(n)) - 4\left(1 - 2\epsilon_n \ln(n) + \epsilon_n^2 \ln(n)^2\right) \to 0,\ as\ n \to +\infty$$

$$\Leftrightarrow \ln(n) - 4 + \left(8\ln(n) - \ln(n)^2\right)\epsilon_n - 4\epsilon_n^2 \ln(n)^2 \to 0,\ as\ n \to +\infty$$

$\epsilon_n = \dfrac{1}{\ln(n)}$ is an ideal choice:

$$\ln(n) - 4 + \left(8\ln(n) - \ln(n)^2\right)\frac{1}{\ln(n)} - 4\left(\frac{1}{\ln(n)}\right)^2 \ln(n)^2$$

$$= \ln(n) - 4 + \frac{8\ln(n)}{\ln(n)} - \frac{\ln(n)^2}{\ln(n)} - 4$$

$$= \ln(n) - 4 + 8 - \ln(n) - 4 = 0$$

We have a good $\epsilon_n = \dfrac{1}{\ln(n)} \longrightarrow 0\ as\ n \to +\infty$.

So if a_n tends to $\dfrac{1}{2}$ slowly, this adequate ϵ_n **exists, and voilà, we get the right result**.

$\dfrac{1}{2}$ is the only limit the map a_n can reach as $n \to +\infty$, if it hopes to satisfy:

$$\sum_{k=1}^{n} \frac{1}{k^{2a_n}} - \frac{1}{n} \times \left(\sum_{k=1}^{n} \frac{1}{k^{a_n}} \right)^2 \to 0\ as\ n \to +\infty$$

And we could ideally write a_n as $a_n = \dfrac{1}{2} + \dfrac{1}{\ln(n)}$, for a first-order Taylor expansion. Alas, it won't satisfy the Taylor approximations of all orders, which leads us to the last part.

What about the asymptotic expansions using Taylor approximations of any order $p \in \mathbb{N}^*$, ideally very large?

$\forall p \in \mathbb{N}^*$, The p-th order Taylor approximations give us the following:

$$\lambda + \ln(n)\left(1 + \sum_{k=1}^{p} \frac{(-2\epsilon_n \ln(n))^k}{(k+1)!}\right) - \frac{1}{n}\left(\frac{n^{1-\frac{1}{2}}}{1-\frac{1}{2}}\right)^2 \times \left(1 + \sum_{k=1}^{p} \frac{(-\epsilon_n \ln(n))^k}{k!}\right)^2 \xrightarrow[n \to +\infty]{} 0$$

$$\Leftrightarrow \lambda + \ln(n)\left(1+\sum_{k=1}^{p}\frac{(-2\epsilon_n\ln(n))^k}{(k+1)!}\right) - 4\times\left(1+\sum_{k=1}^{p}\frac{(-\epsilon_n\ln(n))^k}{k!}\right)^2 \xrightarrow[n\to+\infty]{} 0$$

Let's try and find $x_p, y_p \in \mathbb{C}$ so $\epsilon_n = \dfrac{x_p\ln(n)+y_p}{\ln(n)^2}$; the equation becomes:

$$\lambda + \ln(n)\left(1+\sum_{k=1}^{p}\frac{\left(-2\left(x_p+\frac{y_p}{\ln(n)}\right)\right)^k}{(k+1)!}\right) - 4\times\left(1+\sum_{k=1}^{p}\frac{(-1)^k\left(x_p+\frac{y_p}{\ln(n)}\right)^k}{k!}\right)^2 \xrightarrow[n\to+\infty]{} 0$$

$$\forall k\in\mathbb{N},\ \left(x_p+\frac{y_p}{\ln(n)}\right)^k = \sum_{j=0}^{k}\binom{k}{j}x_p^{k-j}\times\left(\frac{y_p}{\ln(n)}\right)^j = x_p^k + \frac{kx_p^{k-1}y_p}{\ln(n)} + o\left(\frac{1}{\ln(n)}\right)$$

Therefore we get for all $p\in\mathbb{N}^*$:

$$\lambda + \ln(n)\left(1+\sum_{k=1}^{p}\frac{(-2)^k\left(x_p^k+kx_p^{k-1}\times\frac{y_p}{\ln(n)}\right)}{(k+1)!}\right) - 4\times\left(1+\sum_{k=1}^{p}\frac{(-1)^k\left(x_p^k+kx_p^{k-1}\times\frac{y_p}{\ln(n)}\right)}{k!}\right)^2 \xrightarrow[n\to+\infty]{} 0$$

$$\Leftrightarrow \lambda + \ln(n)\left(1+\sum_{k=1}^{p}\frac{(-2x_p)^k}{(k+1)!}\right) + \sum_{k=1}^{p}\frac{(-2)^k kx_p^{k-1}y_p}{(k+1)!} - 4\times\left(1+\sum_{k=1}^{p}\frac{(-x_p)^k}{k!}\right)^2 \xrightarrow[n\to+\infty]{} 0$$

$$\Leftrightarrow \lambda + \ln(n)\left(1+\sum_{k=1}^{p}\frac{(-2x_p)^k}{(k+1)!}\right) + y_p\times\sum_{k=1}^{p}\frac{(-2)^k kx_p^{k-1}}{(k+1)!} - 4\times\left(1+\sum_{k=1}^{p}\frac{(-x_p)^k}{k!}\right)^2 \xrightarrow[n\to+\infty]{} 0$$

And for all $p\in\mathbb{N}^*$ the polynomial $P_p(x) = 1+\sum_{k=1}^{p}\frac{(-2x)^k}{(k+1)!}$ admits p roots (real or complex), we just have to pick one of them to "nullify" $\ln(n)\left(1+\sum_{k=1}^{p}\frac{(-2x_p)^k}{(k+1)!}\right)$, then there will only remain:

$$y_p\times\sum_{k=1}^{p}\frac{(-2)^k kx_p^{k-1}}{(k+1)!} - 4\times\left(1+\sum_{k=1}^{p}\frac{(-x_p)^k}{k!}\right)^2 \xrightarrow[n\to+\infty]{} 0$$

We also notice that $\sum_{k=1}^{p}\frac{(-2)^k kx^{k-1}}{(k+1)!} = P_p{}'(x)$.

Now, let's see why the polynomial $P_p(x)$ only has simple roots, meaning that none of them are roots of $P_p{}'(x)$.
We're already sure that 0 is not a root of $P_p(x)$ because $P_p(0)=1$.
Let $x_p\in\mathbb{C}^*$ be a root of $P_p(x)$, let's assume it's also a root of $P_p{}'(x)$:

$$1+\sum_{k=1}^{p}\frac{(-2x_p)^k}{(k+1)!} = 0 \Leftrightarrow \sum_{k=1}^{p}\frac{(-2x_p)^k}{(k+1)!} = -1$$

On the other hand, we also have:

$$x_pP_p'(x_p) = \sum_{k=1}^{p}\frac{(-2)^k k x_p^k}{(k+1)!} = \sum_{k=1}^{p}\frac{(-2)^k(k+1-1)x_p^k}{(k+1)!} = \sum_{k=1}^{p}\frac{(-2)^k(k+1)x_p^k}{(k+1)!} - \sum_{k=1}^{p}\frac{(-2x_p)^k}{(k+1)!}$$

$$= \sum_{k=1}^{p}\frac{(-2x_p)^k}{k!} - \sum_{k=1}^{p}\frac{(-2x_p)^k}{(k+1)!} = \sum_{k=1}^{p}\frac{(-2x_p)^k}{k!} - (P_p(x_p)-1)$$

$$\sum_{k=1}^{p}\frac{(-2x_p)^k}{(k+1)!} = -1,\ \textit{this means}: \sum_{k=1}^{p}\frac{(-2x_p)^k}{k!} = -1\ \textit{as well}$$

And this is impossible, because the terms of the sum on the left are those of the sum on the right scaled down by a factor $\frac{1}{k+1}$, so we can't have both of them match -1 (I shall develop this in upcoming versions).

In conclusion, **no root of** $P_p(x)$ **is a root of** $P_p'(x)$.

Therefore, we just need to pick any root for the value of x_p, and we're sure y_p exists as well, and we can express it as $y_p = \frac{4\times\left(1+\sum_{k=1}^{p}\frac{(-x_p)^k}{k!}\right)^2-\lambda}{\sum_{k=1}^{p}\frac{(-2)^k k x_p^{k-1}}{(k+1)!}}$.

In conclusion, for all orders of expansion $p \in \mathbb{N}^*$, there exist $x_p, y_p \in \mathbb{C}$ so that we get $\epsilon_n = \frac{x_p\ln(n)+y_p}{\ln(n)^2}$ ensuring the limit:

$$\sum_{k=1}^{n}\frac{1}{k^{2\left(\frac{1}{2}+\epsilon_n\right)}} - \frac{1}{n}\times\left(\sum_{k=1}^{n}\frac{1}{k^{\frac{1}{2}+\epsilon_n}}\right)^2 \xrightarrow[n\to+\infty]{} 0$$

required by $\zeta(s)=0 \Rightarrow \eta(s)=0$.

And we would actually write $a_n = a_{+\infty} + Re(\epsilon_n) = a_{+\infty} + Re\left(\frac{x_p\ln(n)+y_p}{\ln(n)^2}\right)$.

And now, for an infinite order of approximation $p\to+\infty$:
In the formula:

$$\lambda + \ln(n)\left(1+\sum_{k=1}^{p}\frac{(-2x_p)^k}{(k+1)!}\right) + y_p\times\sum_{k=1}^{p}\frac{(-2)^k k x_p^{k-1}}{(k+1)!} - 4\times\left(1+\sum_{k=1}^{p}\frac{(-x_p)^k}{k!}\right)^2 \xrightarrow[n\to+\infty]{} 0$$

Let's have a closer look at $P_p(x) = 1+\sum_{k=1}^{p}\frac{(-2x)^k}{(k+1)!}$ and $P_p'(x) = \sum_{k=1}^{p}\frac{(-2)^k k x^{k-1}}{(k+1)!}$:

$$1+\sum_{k=1}^{p}\frac{(-2x_p)^k}{(k+1)!}=\sum_{k=0}^{p}\frac{(-2x_p)^k}{(k+1)!}=\frac{1}{-2x_p}\sum_{k=0}^{p}\frac{(-2x_p)^{k+1}}{(k+1)!}=\frac{1}{-2x_p}\sum_{k=1}^{p+1}\frac{(-2x_p)^k}{k!}$$
$$=\frac{1}{-2x_p}\left(\sum_{k=0}^{p+1}\frac{(-2x_p)^k}{k!}-1\right)\xrightarrow[p\to+\infty]{}\frac{1}{-2x_p}\left(e^{-2x_p}-1\right)$$

If we want $\frac{1}{-2x_p}\left(e^{-2x_p}-1\right)=0$, we can't chose $x_p=0$ (we shall see why soon), but we can very well chose $x_p=k\pi i,\ k\in\mathbb{Z}^*$,
Now let's remember:

$$x_pP_p'(x_p)=\sum_{k=1}^{p}\frac{(-2x_p)^k}{k!}-P_p(x_p)+1$$

As $p\to+\infty$, for $x_p=k\pi i,\ k\in\mathbb{Z}^*$, we then have:

$$k\pi i\times P_p'(x_p)=e^{-2k\pi i}-1+1=1\Leftrightarrow P_p'(x_p)=-\frac{i}{k\pi}$$

So our formula:

$$\lambda+\ln(n)\left(1+\sum_{k=1}^{p}\frac{(-2x_p)^k}{(k+1)!}\right)+y_p\times\sum_{k=1}^{p}\frac{(-2)^kkx_p^{k-1}}{(k+1)!}-4\times\left(1+\sum_{k=1}^{p}\frac{(-x_p)^k}{k!}\right)^2\xrightarrow[n\to+\infty]{}0$$

becomes:

$$\lambda-y_p\times\frac{i}{k\pi}-4\times\left(e^{-k\pi i}\right)^2\xrightarrow[n\to+\infty]{}0$$
$$\Leftrightarrow\lambda-y_p\times\frac{i}{k\pi}-4\xrightarrow[n\to+\infty]{}0$$

Thus our y_p exists and its expression is:

$$y_p=-\frac{k\pi(4-\lambda)}{i}=ik\pi(4-\lambda)$$

As a result, this expression of ϵ_n is valid for all $k\in\mathbb{Z}^*$:

$$\epsilon_n=ik\pi\times\frac{\ln(n)+4-\lambda}{\ln(n)^2},\ \lambda\in[0,1],\ k\in\mathbb{Z}^*$$

So here you see why we can't have $k=0$, for it would mean $\epsilon_n=0$, and this would mean:

$$\ln(n)\xrightarrow[n\to+\infty]{}4-\lambda$$

which is absurd. Hence $k\in\mathbb{Z}^*$.

Also, we notice that it is a purely imaginary value, but do we care so long as it makes everything work? I don't think so. Plus the real part is a plain $\frac{1}{2}$ so it's all good.

In conclusion, the real part of our nontrivial zero is actually a fixed value $\frac{1}{2}$, but it is the imaginary part that includes ϵ_n.

That's an unexpected twist. But it works, just in an unexpected way.

Conclusion:

For any nontrivial zero $s \in \mathbb{C} \setminus \left\{1 + \frac{2\pi ik}{ln(2)} \mid k \in \mathbb{Z}\right\}$, $\lim_{n \to +\infty} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^s} = 0$

requires that the only value the real part $Re(s)$ can have as $n \to +\infty$ be $Re(s) = \frac{1}{2}$, and one of the expressions of the number s depending on n to ensure this limit is the following:

$$s_n = \frac{1}{2} + i\left(b + k\pi \times \frac{\ln(n) + 4 - \lambda}{\ln(n)^2}\right), \; k \in \mathbb{Z}^*, \lambda \in [0,1], \; b \in \mathbb{R}$$

Therefore, since $\zeta(s) = 0 \Rightarrow \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = 0$:

For any nontrivial zero $s \in \mathbb{C} \setminus \left\{1 + \frac{2\pi ik}{ln(2)} \mid k \in \mathbb{Z}\right\}$, $\zeta(s) = 0 \implies Re(s) = \frac{1}{2}$.

This proves the Riemann Hypothesis.

[1] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function* (1986).
[2] H. Iwaniec and E. Kowalski, *Analytic Number Theory* (2004)