A Possible Proof Of The Riemann Hypothesis

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Abstract

The Zeta Function and one of its analytic continuations are defined as follows:

$$\forall s \in \mathbb{C} \mid Re(s) > 1,\ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$\forall s \in \mathbb{C} \setminus \left\{1 + \frac{2\pi i k}{ln(2)} \mid k \in \mathbb{Z}\right\},\ \zeta(s) = \frac{\eta(s)}{\left(1 - 2^{1-s}\right)},\ where\ \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$$

The Riemann Hypothesis states the following, for all the nontrivial zeros:

$$\zeta(s) = 0 \implies Re(s) = \frac{1}{2}$$

It has already been proved that $Re(s) \in]0, 1[$ for all the nontrivial zeros.

Firstly, for $a = Re(s)$ and $b = Im(s)$, we'll prove that:

$$\zeta(s) = 0 \Rightarrow \eta(s) = 0 \Leftrightarrow \sum_{k=1}^{n} \frac{1}{k^{2a}} + 2 \times \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} \frac{(-1)^{k+j-2} \cos(b \ln(k / j))}{(kj)^a} \rightarrow 0\ as\ n \rightarrow +\infty$$

And since $\forall x \in \mathbb{R},\ -1 \leqslant \cos(x) \leqslant 1$, this implies that there exists a map r_n satisfying $-1 \leqslant r_n \leqslant 1$ for all n sufficiently large, and for which:

$$\sum_{k=1}^{n} \frac{1}{k^{2a}} + 2r_n \times \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} \frac{1}{(kj)^a} \rightarrow 0\ as\ n \rightarrow +\infty$$

Secondly, by reformulating it as a problem of quadratic equations, we will figure out that this holds true only if $\forall n \in \mathbb{N} \setminus [\![0, 3]\!],\ r_n \in \left[-\frac{1}{n-1}, -\frac{1}{n-3}\right] \setminus \left\{-\frac{1}{n-2}\right\}$ where $[\![0, 3]\!] = \{0, 1, 2, 3\}$, and therefore, that $r_n \sim -\frac{1}{n}\ as\ n \rightarrow +\infty$

And through various asymptotic equivalences, we will get:

$$\sum_{k=1}^{n} \frac{1}{k^{2a}} - \frac{1}{n} \times \left(\sum_{k=1}^{n} \frac{1}{k^a}\right)^2 \rightarrow 0\ as\ n \rightarrow +\infty$$

Finally, from there, <u>we'll consider $a = Re(s)$ as a map $a_n = Re(s_n)$</u> converging to a real number $a_{+\infty} \in]0, 1[$, rather than considering it as a fixed value (since we're dealing with infinity).
It is for convenience that we denote $\lim\limits_{n \rightarrow +\infty} a_n = a_{+\infty} \in]0, 1[$.

Then we'll approximate these two sums with integrals depending on $a_{+\infty}$,

and we shall distinguish three different cases:

- $a_{+\infty} \in]0, \frac{1}{2}[$
- $a_{+\infty} \in]\frac{1}{2}, 1[$
- $a_{+\infty} = \frac{1}{2}$

And conclude that the only case that is logically consistent is when $a_{+\infty} = \frac{1}{2}$.

1 Simplifying the expression

First of all, for the sake of simplification, let's write $s = a + ib$ where $a = Re(s)$ and $b = Im(s)$,
We can write the Eta function as follows:

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^{-ib}}{n^a} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} e^{-ib\ln(n)}}{n^a}$$

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cos(-b\ln(n))}{n^a} + i \times \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin(-b\ln(n))}{n^a}$$

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cos(b\ln(n))}{n^a} - i \times \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin(b\ln(n))}{n^a}$$

If we assume $\zeta(s) = 0$, then by the expression of its analytic continuation $\zeta(s) = \frac{\eta(s)}{\left(1-2^{1-s}\right)}$, we also have $\eta(s) = 0$ and then $|\eta(s)|^2$ is null too:

$$|\eta(s)|^2 = \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cos(b\ln(n))}{n^a}\right)^2 + \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin(b\ln(n))}{n^a}\right)^2 = 0$$

$$thus\ as\ n \rightarrow +\infty, \left(\sum_{k=1}^{n} \frac{(-1)^{k-1}\cos(b\ln(k))}{k^a}\right)^2 + \left(\sum_{k=1}^{n} \frac{(-1)^{k-1}\sin(b\ln(k))}{k^a}\right)^2 \rightarrow 0$$

$$\Longleftrightarrow \sum_{k=1}^{n}\sum_{j=1}^{n} \frac{(-1)^{k+j-2}\cos(b\ln(k))\cos(b\ln(j))}{(kj)^a} + \frac{(-1)^{k+j-2}\sin(b\ln(k))\sin(b\ln(j))}{(kj)^a} \rightarrow 0\ as\ n \rightarrow +\infty$$

$$\Longleftrightarrow \sum_{k=1}^{n}\sum_{j=1}^{n} (-1)^{k+j-2}\left(\frac{\cos(b\ln(k))\cos(b\ln(j))}{(kj)^a} + \frac{\sin(b\ln(k))\sin(b\ln(j))}{(kj)^a}\right) \rightarrow 0\ as\ n \rightarrow +\infty$$

$$\Longleftrightarrow \sum_{k=1}^{n}\sum_{j=1}^{n} (-1)^{k+j-2}\left(\frac{\cos(b\ln(k) - b\ ln(j))}{(kj)^a}\right) \rightarrow 0\ as\ n \rightarrow +\infty$$

$$\Longleftrightarrow \sum_{k=1}^{n}\sum_{j=1}^{n} \frac{(-1)^{k+j-2}\cos(b\ln(k/j))}{(kj)^a} \rightarrow 0\ as\ n \rightarrow +\infty$$

$$\iff \sum_{k=1}^{n} \frac{(-1)^{2k-2}}{k^{2a}} + \sum_{k=1}^{n} \sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{(-1)^{k+j-2} \cos(b \ln(k/j))}{(kj)^a} \to 0 \ as \ n \to +\infty$$

$$\iff \sum_{k=1}^{n} \frac{1}{k^{2a}} + 2 \times \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} \frac{(-1)^{k+j-2} \cos(b \ln(k/j))}{(kj)^a} \to 0 \ as \ n \to +\infty$$

$$\forall k, j \in [\![1, n]\!], \forall b \in \mathbb{R}, \ -1 \leqslant \cos(b \ln(k/j)) \leqslant 1 \qquad (1)$$

Thus there exists a map r_n satisfying $-1 \leqslant r_n \leqslant 1$ for all n sufficiently large, and for which:

$$\sum_{k=1}^{n} \frac{1}{k^{2a}} + 2r_n \times \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} \frac{1}{(kj)^a} \to 0 \ as \ n \to +\infty$$

And we end up with what curiously resembles a quadratic equation.

2 The "Russian Doll" Quadratic Equations

Now let's assume there is $x_1, ..., x_n \in \mathbb{R}$ with $x_1 = 1$ so that:

$$\sum_{k=1}^{n} x_k^2 + 2r_n \times \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} x_k x_j = 0$$

And let's try and figure out which kind of map r_n is.

But first, let's define $\forall n \in \mathbb{N}^*, \ u_n = \sum_{k=1}^{n} x_k^2$, $v_n = \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} x_k x_j$ and $p_n = \sum_{k=1}^{n} x_k$

Our previous equation becomes:

$$u_n \ + \ 2r_n v_n = x_n^2 + 2r_n p_{n-1} x_n + u_{n-1} + 2r_n v_{n-1} = 0$$

And now let's define $(f_n)_{n \in \mathbb{N} \setminus \{0,1\}}$ and $(g_n)_{n \in \mathbb{N} \setminus \{0,1\}}$ so that $\forall n \in \mathbb{N} \setminus \{0, 1\}$:

$$f_n u_n \ + \ g_n v_n = f_n x_n^2 + g_n p_{n-1} x_n + f_n u_{n-1} + g_n v_{n-1} = 0$$

Let's now express the delta Δ_n of this equation and find the expressions of f_{n-1} and g_{n-1} so that $\Delta_n = f_{n-1} u_{n-1} + g_{n-1} v_{n-1} \geqslant 0$:

$$\Delta_n = (g_n p_{n-1})^2 - 4 f_n (f_n u_{n-1} + g_n v_{n-1}),$$

$$p_{n-1}^2 = \left(\sum_{k=1}^{n-1} x_k \right)^2 = u_{n-1} + 2 v_{n-1},$$

$$thus \ \Delta_n = (g_n p_{n-1})^2 - 4 f_n (f_n u_{n-1} + g_n v_{n-1}) = g_n^2 (u_{n-1} + 2 v_{n-1}) - 4 f_n (f_n u_{n-1} + g_n v_{n-1})$$

$$\Delta_n = \left(g_n^2 - 4 f_n^2 \right) u_{n-1} + \left(2 g_n^2 - 4 f_n g_n \right) v_{n-1}$$

We conclude that $f_{n-1} = g_n^2 - 4f_n^2$ and $g_{n-1} = 2g_n^2 - 4f_n g_n$, and we see Δ_n is in turn a new quadratic equation:

$$\Delta_n = f_{n-1}x_{n-1}^2 + g_{n-1}p_{n-2}x_{n-1} + f_{n-1}u_{n-2} + g_{n-1}v_{n-2}$$

with a new Δ_{n-1} for which we must determine the conditions to ensure $\Delta_{n-1} \geqslant 0,$ and so on until Δ_2 (hence the comparison with a Russian doll).

But also, $\dfrac{g_{n-1}}{f_{n-1}} = \dfrac{2g_n^2 - 4f_n g_n}{g_n^2 - 4f_n^2} = \dfrac{2g_n(g_n - 2f_n)}{(g_n - 2f_n)(g_n + 2f_n)} = \dfrac{2g_n}{(g_n + 2f_n)} = \dfrac{2\frac{g_n}{f_n}}{\frac{g_n}{f_n} + 2}$

We observe that each time we calculate a Δ_{n-k}, we actually apply $h : x \mapsto \dfrac{2x}{x+2}$ to the ratio $\dfrac{g_{n-k}}{f_{n-k}}$ to obtain $\dfrac{g_{n-k-1}}{f_{n-k-1}}$: $\forall k \in [\![1, n-3]\!],\ \dfrac{g_{n-k-1}}{f_{n-k-1}} = \dfrac{2\frac{g_{n-k}}{f_{n-k}}}{\frac{g_{n-k}}{f_{n-k}} + 2}$.

In our precise case, $f_n = 1$ and $g_n = 2r_n$, so $\dfrac{g_n}{f_n} = 2r_n$; our f_{n-1} and g_{n-1} thus become:

$$f_{n-1} = \left(4r_n^2 - 4\right)f_n^2 = 4\left(r_n^2 - 1\right)f_n^2 = 4(r_n - 1)(r_n + 1)f_n^2$$
$$g_{n-1} = \left(2 \times 4r_n^2 - 4 \times 2r_n\right)f_n^2 = 8\left(r_n^2 - r_n\right)f_n^2 = 8r_n(r_n - 1)f_n^2$$

Thus, $\dfrac{g_{n-1}}{f_{n-1}} = \dfrac{8r_n(r_n - 1)f_n^2}{4(r_n - 1)(r_n + 1)f_n^2} = \dfrac{2r_n}{r_n + 1}$.

Now, let's prove by induction that $\forall k \in [\![1, n-2]\!],\ \dfrac{g_{n-k}}{f_{n-k}} = \dfrac{2r_n}{k \times r_n + 1}$:

Let's assume $\exists k \in [\![1, n-3]\!],\ \dfrac{g_{n-k}}{f_{n-k}} = \dfrac{2r_n}{k \times r_n + 1}$,

Then we have:

$$\frac{g_{n-k-1}}{f_{n-k-1}} = h\left(\frac{g_{n-k}}{f_{n-k}}\right) = 2 \times \frac{2r_n}{k \times r_n + 1} \times \frac{1}{\frac{2r_n}{k \times r_n + 1} + 2} = 2 \times 2r_n \times \frac{1}{2r_n + 2(k \times r_n + 1)}$$
$$\Leftrightarrow \frac{g_{n-k-1}}{f_{n-k-1}} = h\left(\frac{g_{n-k}}{f_{n-k}}\right) = \frac{4r_n}{2(r_n + k \times r_n + 1)} = \frac{2r_n}{(k+1) \times r_n + 1}$$

Which proves that $\forall k \in [\![1, n-2]\!],\ \dfrac{g_{n-k}}{f_{n-k}} = \dfrac{2r_n}{k \times r_n + 1}$.

Now, $\forall n \in \mathbb{N} \setminus [\![0, 3]\!],\ \forall k \in [\![1, n-2]\!]$ we can express all the Δ_{n-k}, and above all the following:

$\Delta_3 = f_2 u_2 + g_2 v_2 = f_2 x_2^2 + f_2 x_1^2 + g_2 x_2 x_1 = f_2 x_2^2 + g_2 x_2 + f_2$ (because $x_1 = 1$)

$\Delta_2 = g_2^2 - 4f_2^2 = 4 \times \left(\dfrac{r_n^2}{[(n-2) \times r_n + 1]^2} - 1\right) \times f_2^2$

To determine the positivity of Δ_2 we only focus on the positivity of $\dfrac{r_n^2}{[(n-2)\times r_n+1]^2}-1$,

for we know f_2^2 and 4 are always positive.

$$\frac{r_n^2}{[(n-2)\times r_n+1]^2}-1\geqslant 0 \;\Leftrightarrow\; r_n^2\geqslant[(n-2)\times r_n+1]^2 \Leftrightarrow \left[1-(n-2)^2\right]r_n^2-2(n-2)r_n-1\geqslant 0$$

$$\Delta=4(n-2)^2-4\times(-1)\left[1-(n-2)^2\right]=4\left[(n-2)^2+1-(n-2)^2\right]=4>0$$

So solutions for all of our previous Δ_k exist;

$\forall n\in\mathbb{N}\setminus[\![0,3]\!]$, the quadratic coefficient $\left[1-(n-2)^2\right]$ is strictly negative, so:

$$r_n\in\left[\frac{2(n-2)-\sqrt{4}}{2\left[1-(n-2)^2\right]},\frac{2(n-2)+\sqrt{4}}{2\left[1-(n-2)^2\right]}\right]\setminus\left\{-\frac{1}{n-2}\right\}$$

which means:

$$r_n\in\left[\frac{(n-2)-1}{1-(n-2)^2},\frac{(n-2)+1}{1-(n-2)^2}\right]\setminus\left\{-\frac{1}{n-2}\right\}$$

$$\Leftrightarrow r_n\in\left[\frac{(n-2)-1}{(1-n+2)(1+n-2)},\frac{(n-2)+1}{(1-n+2)(1+n-2)}\right]\setminus\left\{-\frac{1}{n-2}\right\}$$

$$\Leftrightarrow r_n\in\left[-\frac{1}{n-1},-\frac{1}{n-3}\right]\setminus\left\{-\frac{1}{n-2}\right\},\ \forall n\in\mathbb{N}\setminus[\![0,3]\!]$$

(we exclude $-\dfrac{1}{n-2}$ because of the term $\dfrac{r_n^2}{[(n-2)\times r_n+1]^2}$ in Δ_2);

Therefore, as $n\to+\infty,\ r_n\sim-\dfrac{1}{n}$

In conclusion, for the following to be true, as $n\to+\infty$:

$$\sum_{k=1}^{n}x_k^2+2r_n\times\sum_{k=1}^{n-1}\sum_{j=k+1}^{n}x_kx_j\to 0$$

We must have it in the following form:

$$\sum_{k=1}^{n}x_k^2-\frac{2}{n}\times\sum_{k=1}^{n-1}\sum_{j=k+1}^{n}x_kx_j\to 0\ as\ n\to+\infty$$

Now we could simplify this:

$$\sum_{k=1}^{n}x_k^2-\frac{2}{n}\times\sum_{k=1}^{n-1}\sum_{j=k+1}^{n}x_kx_j=\sum_{k=1}^{n}x_k^2-\frac{1}{n}\times\sum_{k=1}^{n}\sum_{\substack{j=1\\ j\neq k}}^{n}x_kx_j$$

$$=\sum_{k=1}^{n}x_k^2-\frac{1}{n}\times\left(\sum_{k=1}^{n}\sum_{j=1}^{n}x_kx_j-\sum_{k=1}^{n}x_k^2\right)=0$$

$$\Leftrightarrow \left(1+\frac{1}{n}\right)\times\sum_{k=1}^{n}x_k^2-\frac{1}{n}\times\sum_{k=1}^{n}\sum_{j=1}^{n}x_kx_j=0$$

And as $n\rightarrow+\infty$ the asymptotic equivalences give us the following:

$$\sum_{k=1}^{n}x_k^2-\frac{1}{n}\times\sum_{k=1}^{n}\sum_{j=1}^{n}x_kx_j\rightarrow 0\ as\ n\rightarrow+\infty$$

$$\Leftrightarrow \sum_{k=1}^{n}x_k^2-\frac{1}{n}\times\left(\sum_{k=1}^{n}x_k\right)^2\rightarrow 0\ as\ n\rightarrow+\infty$$

Now to get back to our problem, if we assume that $\forall k\in[\![1,n]\!],\ x_k=\dfrac{1}{k^a}$, then we get, as $n\rightarrow+\infty$:

$$\sum_{k=1}^{n}\frac{1}{k^{2a}}-\frac{1}{n}\times\left(\sum_{k=1}^{n}\frac{1}{k^a}\right)^2\rightarrow 0$$

Which is therefore - thanks to all we've seen up to now - the new formula on which we'll work from now on, and which is way more easy-to-handle and less obscure than:

$$\sum_{k=1}^{n}\frac{1}{k^{2a}}+2\times\sum_{k=1}^{n-1}\sum_{j=k+1}^{n}\frac{(-1)^{k+j-2}\cos(b\ln(k/j))}{(kj)^a}\rightarrow 0\ as\ n\rightarrow+\infty$$

3 Comparison Of Asymptotic Behaviours

Now, We got this expression from the previous part:

$$\sum_{k=1}^{n}\frac{1}{k^{2a}}-\frac{1}{n}\times\left(\sum_{k=1}^{n}\frac{1}{k^a}\right)^2\rightarrow 0,\ as\ n\rightarrow+\infty$$

Since we're dealing with infinity, instead of distinguishing the cases for different fixed values for $a\in\,]0,1[$, **I will speak of a map $(a_n)_{n\in\mathbb{N}^*}$ converging to a real number in $]0,1[$:** $\lim\limits_{n\rightarrow+\infty}a_n\rightarrow a_{+\infty}\in\,]0,1[$ with a rate of convergence $\epsilon_n=a_n-a_{+\infty}$.

The sums with their corrections (obtained via Taylor expansions) become, as $n\rightarrow+\infty$:

$$\sum_{k=1}^{n}\frac{1}{k^{2a_{+\infty}}}-2\epsilon_n\sum_{k=1}^{n}\frac{\ln(k)}{k^{2a_{+\infty}}}-\frac{1}{n}\times\left(\sum_{k=1}^{n}\frac{1}{k^{a_{+\infty}}}-\epsilon_n\sum_{k=1}^{n}\frac{\ln(k)}{k^{a_{+\infty}}}\right)^2\rightarrow 0$$

The correction terms can be ignored for a fast convergence of a_n;
We'll deal with fast and slow convergences.

It has already been well-established in the literature [1, 2] that $a_{+\infty}\in\,]0,1[$ for all the nontrivial zeros, so $1-a_{+\infty}>0$ and then the **squared sum** can be approximated with the **following squared integral as follows if a_n converges fastly to its limit**:

$$\left(\int_1^n \frac{1}{t^a}dt\right)^2 = \frac{\left(n^{1-a}-1\right)^2}{(1-a)^2} \sim \frac{n^{2-2a}}{(1-a)^2}\ as\ n \to +\infty$$

to obtain the following (I omit the n index of a_n for convenience in these calculations):

$$\forall a \in]0,1[\ and\ as\ n \to +\infty,\ \frac{1}{n} \times \left(\sum_{k=1}^{n} \frac{1}{k^a}\right)^2 \sim \frac{1}{n} \times \frac{n^{2-2a}}{(1-a)^2} = \frac{n^{1-2a}}{(1-a)^2}$$

And for a slow convergence, the sum of the correction term added in the squared sum:

$$\forall n \in \mathbb{N}^*, \int_1^{n+1} \frac{\ln(t)}{t^{a_{+\infty}}}dt \leqslant \sum_{k=1}^{n} \frac{\ln(k)}{k^{a_{+\infty}}} \leqslant 1 + \int_2^{n+1} \frac{\ln(t-1)}{(t-1)^{a_{+\infty}}}dt$$

$$with \int_1^n \frac{\ln(t)}{t^{a_{+\infty}}}dt = \frac{\ln(n)n^{1-a_{+\infty}}}{1-a_{+\infty}} - \frac{n^{1-a_{+\infty}}-1}{(1-a)^2}$$

Therefore, since $1 - a_{+\infty} > 0$ we get the following asymptotic equivalence:

$$\sum_{k=1}^{n} \frac{\ln(k)}{k^{a_{+\infty}}} \sim \int_1^n \frac{\ln(t)}{t^{a_{+\infty}}}dt \sim \frac{\ln(n)n^{1-a_{+\infty}}}{1-a_{+\infty}} - \frac{n^{1-a_{+\infty}}}{(1-a)^2}\ as\ n \to +\infty$$

As to the <u>sum of squares</u>, for a fast convergence:

$$\forall n \in \mathbb{N}^*, \int_1^{n+1} \frac{1}{t^{2a}}dt \leqslant \sum_{k=1}^{n} \frac{1}{k^{2a}} \leqslant 1 + \int_2^{n+1} \frac{1}{(t-1)^{2a}}dt$$

And the sum of the correction term added for a slow convergence:

$$\forall n \in \mathbb{N}^*, \int_1^{n+1} \frac{\ln(t)}{t^{2a_{+\infty}}}dt \leqslant \sum_{k=1}^{n} \frac{\ln(k)}{k^{2a_{+\infty}}} \leqslant 1 + \int_2^{n+1} \frac{\ln(t-1)}{(t-1)^{2a_{+\infty}}}dt$$

So $\exists \mu \in [0,1] \mid \sum_{k=1}^{n} \frac{\ln(k)}{k^{2a_{+\infty}}} \sim \mu + \int_1^n \frac{\ln(t)}{t^{2a_{+\infty}}}dt\ as\ n \to +\infty$

We get to distinguish $a_{+\infty} \neq \frac{1}{2}$ and $a_{+\infty} = \frac{1}{2}$ for the sum of squares.

If $a_{+\infty} \neq \frac{1}{2}$:

Fast convergence:

$$\frac{(n+1)^{1-2a}-1}{1-2a} \leqslant \sum_{k=1}^{n} \frac{1}{k^{2a}} \leqslant 1 + \frac{(n+1-1)^{1-2a}-(2-1)^{1-2a}}{1-2a}$$

(I skipped the details of variable substitution on the right side)

We then obtain the following asymptotic equivalences, as $n \to +\infty$:

$$\frac{n^{1-2a}-1}{1-2a} \leqslant \sum_{k=1}^{n} \frac{1}{k^{2a}} \leqslant 1 + \frac{n^{1-2a}-1}{1-2a}$$

Which means that as $n \to +\infty, \exists \lambda \in [0,1], \sum_{k=1}^{n} \frac{1}{k^{2a}} \sim \lambda + \frac{n^{1-2a}-1}{1-2a}$

Sum of the correction term added for a slow convergence (asymptotic equivalent as $n \to +\infty$):

$$\exists \mu \in [0,1] \mid \sum_{k=1}^{n} \frac{\ln(k)}{k^{2a_{+\infty}}} \sim \int_1^n \frac{\ln(t)}{t^{2a_{+\infty}}} dt + \mu = \frac{\ln(n) n^{1-2a_{+\infty}}}{1-2a_{+\infty}} - \frac{n^{1-2a_{+\infty}}-1}{(1-2a_{+\infty})^2} + \mu$$

If $a_{+\infty} = \frac{1}{2}$:

Fast convergence:

$$\sum_{k=1}^{n} \frac{1}{k^{2a}} \sim \ln(n) \text{ } as \text{ } n \to +\infty$$

Sum of the correction term for a slow convergence (asymptotic equivalent as $n \to +\infty$):

$$\sum_{k=1}^{n} \frac{\ln(k)}{k^{2a_{+\infty}}} \sim \int_1^n \frac{\ln(t)}{t} dt \text{ } as \text{ } n \to +\infty$$

$$\int_1^n \frac{\ln(t)}{t} dt = \ln(n)^2 - \int_1^n \frac{\ln(t)}{t}$$

$$\Leftrightarrow \int_1^n \frac{\ln(t)}{t} dt = \frac{\ln(n)^2}{2}$$

We therefore have three different cases:

- $a_{+\infty} \in]0, \frac{1}{2}[$
- $a_{+\infty} \in]\frac{1}{2}, 1[$
- $a_{+\infty} = \frac{1}{2}$

Case $a_{+\infty} \in]0, \frac{1}{2}[$:

If a_n converges fastly enough to its limit, we can take the following for granted:

$1 - 2a_{+\infty} > 0$ so n^{1-2a} grows unboundedly as $n \to +\infty$, so:

$$\sum_{k=1}^{n} \frac{1}{k^{2a}} \sim \frac{n^{1-2a}}{1-2a} \text{ } as \text{ } n \to +\infty$$

Thus our expression:

$$\sum_{k=1}^{n} \frac{1}{k^{2a}} - \frac{n^{1-2a}}{(1-a)^2} \to 0 \; as \; n \to +\infty$$

becomes, as $n \to +\infty$:

$$\frac{n^{1-2a}}{1-2a} - \frac{n^{1-2a}}{(1-a)^2} \to 0 \Leftrightarrow (1-a_{+\infty})^2 = 1-2a_{+\infty} \Leftrightarrow 1-2a_{+\infty} + a_{+\infty}^2 = 1-2a_{+\infty}$$

$\Leftrightarrow a_{+\infty}^2 = 0 \Leftrightarrow a_{+\infty} = 0$, **which contradicts** $a_{+\infty} \in]0, \frac{1}{2}[$.

If the convergence is slow, the expression with the correction terms is as follows:

As $n \to +\infty$:

$$\frac{n^{1-2a_{+\infty}}}{1-2a_{+\infty}} - \frac{2\epsilon_n \ln(n) n^{1-2a_{+\infty}}}{1-2a_{+\infty}} - \frac{n^{1-2a_{+\infty}}}{(1-a_{+\infty})^2} + \frac{2\epsilon_n \ln(n) n^{1-2a_{+\infty}}}{(1-a_{+\infty})^2} - \frac{\epsilon_n^2 \ln(n)^2 n^{1-2a_{+\infty}}}{(1-a_{+\infty})^2} \to 0$$

$$\Leftrightarrow n^{1-2a_{+\infty}} \left(\frac{1}{1-2a_{+\infty}} - \frac{2\epsilon_n \ln(n)}{1-2a_{+\infty}} - \frac{1}{(1-a_{+\infty})^2} + \frac{2\epsilon_n \ln(n)}{(1-a_{+\infty})^2} - \frac{\epsilon_n^2 \ln(n)^2}{(1-a_{+\infty})^2} \right) \to 0$$

$$which \; necessitates:$$

$$\frac{1-2\epsilon_n \ln(n)}{1-2a_{+\infty}} - \frac{1-2\epsilon_n \ln(n) + \epsilon_n^2 \ln(n)^2}{(1-a_{+\infty})^2} \to 0$$

$$\Leftrightarrow \frac{(1-a_{+\infty})^2}{1-2a_{+\infty}} - \frac{1-2\epsilon_n \ln(n) + \epsilon_n^2 \ln(n)^2}{1-2\epsilon_n \ln(n)} \to 0$$

$$\Leftrightarrow \frac{a_{+\infty}^2}{1-2a_{+\infty}} - \frac{\epsilon_n^2 \ln(n)^2}{1-2\epsilon_n \ln(n)} \to 0$$

$$for \; \beta = \epsilon_n \ln(n), \; this \; means:$$

$$a_{+\infty}^2 - 2a_{+\infty}^2 \beta - (1-2a_{+\infty})\beta^2 \to 0 \; as \; n \to +\infty$$

$$\Delta = 4a_{+\infty}^4 + 4a_{+\infty}^2 \left(1-2a_{+\infty}^2\right) = 4a_{+\infty}^2 \left(1-a_{+\infty}^2\right) > 0$$

This equation admits two real solutions, let's not delve into the details but just call them c_1 and c_2, (I'll write c for both to simplify), and just keep in mind that epsilon can then be expressed as:

$$\beta \in \{c_1, c_2\} \Leftrightarrow \epsilon_n \in \left\{ \frac{c_1}{\ln(n)}, \frac{c_2}{\ln(n)} \right\}$$

Actually, since we only retained the dominant terms of the sums multiplying ϵ_n and ϵ_n^2, we neglected the following ones:

$$n^{1-2a_{+\infty}} \left(2\left(\frac{1}{(1-2a_{+\infty})^2} - \frac{1}{(1-a_{+\infty})^3} \right)\epsilon_n + \left(\frac{2\ln(n)}{(1-a_{+\infty})^3} - \frac{1}{(1-a_{+\infty})^4} \right)\epsilon_n^2 \right)$$

And these **stay hidden** in the parenthesis while $\frac{1-2\epsilon_n \ln(n)}{1-2a_{+\infty}} - \frac{1-2\epsilon_n \ln(n) + \epsilon_n^2 \ln(n)^2}{(1-a_{+\infty})^2}$ **vanishes for**

$\epsilon_n \in \left\{ \frac{c_1}{\ln(n)}, \frac{c_2}{\ln(n)} \right\}$ (I'll write $\epsilon_n = \frac{c}{\ln(n)}$ not to repeat both each time).

As a consequence, **in the end, while this disappears**:

$$n^{1-2a_{+\infty}}\left(\frac{1}{1-2a_{+\infty}}-\frac{2c}{1-2a_{+\infty}}-\frac{1}{(1-a_{+\infty})^2}+\frac{2c}{(1-a_{+\infty})^2}-\frac{c^2}{(1-a_{+\infty})^2}\right)\xrightarrow[n\to+\infty]{}0$$

We are left with this:

$$n^{1-2a_{+\infty}}\left(\frac{2c}{\ln(n)}\left(\frac{c}{(1-a_{+\infty})^3}+\frac{1}{(1-2a_{+\infty})^2}-\frac{1}{(1-a_{+\infty})^3}\right)-\frac{c^2}{\ln(n)^2(1-a_{+\infty})^4}\right)\xrightarrow[n\to+\infty]{}0$$

And this is **impossible** because, if $1-2a_{+\infty}>0$, $\ln(n)=o\left(n^{1-2a_{+\infty}}\right)$ and $\ln(n)^2=o\left(n^{1-2a_{+\infty}}\right)$, so this can never tend to 0, and we have a contradiction here.

(In the following versions, I may make c_1 and c_2 explicit).

For all the Taylor expansions of order $p\in\mathbb{N}\mid p\geqslant 1$, we'll get something of the following form:

$$order\ p\in\mathbb{N}^*:for\ q\in[\![0,2p]\!]\ and\ y_q\in\mathbb{R}:$$

$$n^{1-2a_{+\infty}}\left(y_0+\epsilon_n(\ln(n)y_1+o(\ln(n)))\ +\ldots\ +\epsilon_n^{2p}\left(\ln(n)^{2p}y_{2p}+o\left(\ln(n)^{2p}\right)\right)\right)\xrightarrow[n\to+\infty]{}0,$$

a polynomial where we are able to set the unknow as $\beta=\epsilon_n\ln(n)$, and may find one or more real roots as we did for the expansion of order 1, we shall then have $\epsilon_n=\dfrac{d_p}{\ln(n)},\ with\ d_p\in\mathbb{R}$ again (for simplification, this single d_p denotes any root of the polynomial when we use an expansion of order p if they exist).
This way, in the end, the parenthesis itself may tend to 0 (if roots exist for the polynomial), but because of the "hidden" terms denoted by each $o\left(\ln(n)^q\right)$, we would anyway be left with:

$$n^{1-2a_{+\infty}}\left(d_p\left(\frac{w_{1,1}}{\ln(n)}\right)\ +\ldots\ +d_p^{2p}\left(\frac{w_{2p,1}}{\ln(n)}+\ldots\ +\frac{w_{2p,2p}}{\ln(n)^{2p}}\right)\right)\xrightarrow[n\to+\infty]{}0$$

$$with\ w_{k,j}\in\mathbb{R}\ for\ (k,j)\in[\![1,2p]\!]^2$$

And this can't tend to 0 because $1-2a_{+\infty}>0$ and then $\forall q\in[\![0,2p]\!],\ \ln(n)^q=o\left(n^{1-2a_{+\infty}}\right)$.
A situation similar to that of the expansion of order 1.

Case $a_{+\infty}\in\left]\frac{1}{2},1\right[$:

If a_n converges fastly enough to its limit, we can take the following for granted:

As $n\to+\infty,\ \exists\lambda\in[0,1],\sum_{k=1}^{n}\frac{1}{k^{2a}}\sim\lambda+\frac{n^{1-2a}-1}{1-2a}$

In this case, $1-2a_{+\infty}<0$,

Therefore as $n\to+\infty$, $\sum_{k=1}^{n}\frac{1}{k^{2a}}\to\lambda+\frac{1}{2a-1}$ and then $\sum_{k=1}^{n}\frac{1}{k^{2a}}-\frac{n^{1-2a}}{(1-a)^2}\xrightarrow[n\to+\infty]{}0$ becomes:

$\lambda + \dfrac{1}{2a-1} - \dfrac{n^{1-2a}}{(1-a)^2} \xrightarrow[n \to +\infty]{} 0$, and since $1 - 2a_{+\infty} < 0$ this means $\lambda + \dfrac{1}{2a-1} = 0$

$\Leftrightarrow (2a-1)\lambda + 1 = 0 \Leftrightarrow 2a\lambda = \lambda - 1 \Leftrightarrow a_{+\infty} = \dfrac{\lambda - 1}{2\lambda} \leqslant 0$ because $\lambda - 1 \leqslant 0$ while $2\lambda \geqslant 0$ for $\lambda \in [0, 1]$, **which contradicts** $a_{+\infty} \in]\dfrac{1}{2}, 1[$.

If the convergence is slow, the expression with the correction terms is as follows:

$$\lambda + \frac{1}{2a_{+\infty} - 1} - 2\epsilon_n \left(\frac{1}{(1-2a_{+\infty})^2} + \mu \right) - \frac{1}{n} \left(\frac{n^{1-a_{+\infty}}}{1 - a_{+\infty}} - \frac{\epsilon_n \ln(n) n^{1-a_{+\infty}}}{1-a} \right)^2 \xrightarrow[n \to +\infty]{} 0$$

where $\mu \in [0, 1]$; $\epsilon_n \xrightarrow[n \to +\infty]{} 0$ thus $-2\epsilon_n \left(\dfrac{1}{(1-2a_{+\infty})^2} + \mu \right)$ vanishes anyway:

$$\lambda + \frac{1}{2a_{+\infty} - 1} - \frac{n^{1-2a_{+\infty}}}{(1-a_{+\infty})^2} + \frac{2\epsilon_n \ln(n) n^{1-2a_{+\infty}}}{(1-a_{+\infty})^2} - \frac{\epsilon_n^2 \ln(n)^2 \ n^{1-2a_{+\infty}}}{(1-a_{+\infty})^2} \xrightarrow[n \to +\infty]{} 0$$

and $n^{1-2a_{+\infty}} \xrightarrow[n \to +\infty]{} 0$, then:

$\lambda + \dfrac{1}{2a_{+\infty} - 1} = 0$ which means $a_{+\infty} = \dfrac{\lambda - 1}{2\lambda} \leqslant 0$ **which contradicts** $a_{+\infty} \in]\dfrac{1}{2}, 1[$ once again.

The same thing happens for all the expansions of order $p \in \mathbb{N} \mid p \geqslant 1$:

Same as the first-order approximation, since $n^{1-2a_{+\infty}} \xrightarrow[n \to +\infty]{} 0$, all the powers of ϵ_n will be multiplied by a constant value and since $\epsilon_n \xrightarrow[n \to +\infty]{} 0$, the Taylor expansion of any order would leave us only with $\lambda + \dfrac{1}{2a_{+\infty} - 1} = 0$, and we'll end up with the same $a_{+\infty} = \dfrac{\lambda - 1}{2\lambda} \leqslant 0$ **contradicting** $a_{+\infty} \in]\dfrac{1}{2}, 1[$.

Case $a_{+\infty} = \dfrac{1}{2}$:

If a_n converges fastly enough to its limit, we can take the following for granted:

As $n \to +\infty$, $\displaystyle\sum_{k=1}^{n} \frac{1}{k^{2a_n}} \sim \ln(n)$, so $\displaystyle\sum_{k=1}^{n} \frac{1}{k^{2a_n}} - \frac{n^{1-2a_n}}{(1-a_n)^2} \to 0 \ as \ n \to +\infty$ becomes:

$$\ln(n) - \frac{n^{1-2a_n}}{(1-a_n)^2} \to 0 \ as \ n \to +\infty$$

And now, let's reflect upon the conditions for this statement to hold:

- As said earlier, we deal with a map $(a_n)_{n \in \mathbb{N}^*}$ converging to a real number in $]0, 1[$ as $n \to +\infty$, $\dfrac{1}{2}$ in this case, rather than a fixed value $a = \dfrac{1}{2}$, otherwise it would mean that $\displaystyle\lim_{n \to +\infty} \ln(n) \to 4$ which is absurd,
- $\dfrac{1}{(1-a_n)^2} \to 4 \ as \ n \to +\infty$, so it doesn't affect the asymptotic behaviour of n^{1-2a_n},

- $\ln(n)$ grows unboundedly as $n \to +\infty$, so we must have $1-2a_n > 0$ for all n sufficiently large, for n^{1-2a_n} to grow unboundedly as $n \to +\infty$ as well,
- Had we assumed that $\exists l > 0 \mid \lim_{n \to +\infty} 1-2a_n = l$, we would get $\ln(n) - \frac{n^l}{\left(\frac{l+1}{2}\right)^2} \to 0$ as $n \to +\infty$, which is impossible because $\forall l > 0,\ \ln(n) = o\left(n^l\right)$, therefore this subtraction tends to $-\infty$ and not 0 as $n \to +\infty$,
- So it is necessary that $1-2a_n$ be **strictly positive** <u>for all n sufficiently large</u> while **converging to** 0^+ as $n \to +\infty$, in order to adequately "bend" n^{1-2a_n} for it to match $\ln(n)$, for the subtraction to tend to zero,

You'd think we finished, the problem is, even in this case, the expression of $\epsilon_n = a_n - a_{+\infty}$ becomes something like $\epsilon_n = -\frac{\ln\left(\frac{\ln(n)}{4}\right)}{2\ln(n)}$, which is **too slow a rate of convergence to neglect the correction terms**; which we ironically did here.

So we're left with one case, our last chance:

If the convergence to $\frac{1}{2}$ is slow, the equivalence with the correction terms is as follows:

$$\ln(n) - 2\epsilon_n\left(\frac{\ln(n)^2}{2}\right) - \frac{n^{1-2\frac{1}{2}}\left(1-2\epsilon_n\ln(n)+\epsilon_n^2\ln(n)^2\right)}{\left(1-\frac{1}{2}\right)^2} \to 0,\ as\ n \to +\infty$$

$$\Leftrightarrow \ln(n)(1-\epsilon_n\ln(n)) - 4\left(1-2\epsilon_n\ln(n)+\epsilon_n^2\ln(n)^2\right) \to 0,\ as\ n \to +\infty$$
$$\Leftrightarrow \ln(n) - 4 + \left(8\ln(n) - \ln(n)^2\right)\epsilon_n - 4\epsilon_n^2\ln(n)^2 \to 0,\ as\ n \to +\infty$$

$\epsilon_n = \frac{1}{\ln(n)}$ is an ideal choice:

$$\ln(n) - 4 + \left(8\ln(n) - \ln(n)^2\right)\frac{1}{\ln(n)} - 4\left(\frac{1}{\ln(n)}\right)^2\ln(n)^2$$
$$= \ln(n) - 4 + \frac{8\ln(n)}{\ln(n)} - \frac{\ln(n)^2}{\ln(n)} - 4$$
$$= \ln(n) - 4 + 8 - \ln(n) - 4 = 0$$

We have a good $\epsilon_n = \frac{1}{\ln(n)} \to 0\ as\ n \to +\infty$.

So if a_n tends to $\frac{1}{2}$ slowly, this adequate ϵ_n **exists, and voilà, we get the right result**.

$\frac{1}{2}$ is the only limit the map a_n can reach as $n \to +\infty$, if it hopes to satisfy:

$$\sum_{k=1}^{n} \frac{1}{k^{2a_n}} - \frac{1}{n} \times \left(\sum_{k=1}^{n} \frac{1}{k^{a_n}} \right)^2 \to 0 \; as \; n \to +\infty$$

And we could ideally write a_n as $a_n = \frac{1}{2} + \frac{1}{\ln(n)}$, for a first-order Taylor expansion.

What about the Taylor expansions of any order $p \to +\infty$?

We are going determine x_p and y_p to figure out our rate of convergence $\epsilon_n = \frac{x_p \ln(n) + y_p}{\ln(n)^2}$ for which the total sums tends to 0 as $n \to +\infty$ for an infinite order of expansion.

The Taylor expansion of the totals sums is as follows (equivalences as $n \to +\infty$):

$$\ln(n) + \sum_{k=1}^{p} \frac{(-2)^k \epsilon_n^k \ln(n)^{k+1}}{(k+1)!} - \frac{1}{n} \times \left(\frac{\sqrt{n}}{1-\frac{1}{2}} + \sum_{k=1}^{p} \frac{(-1)^k \epsilon_n^k \sqrt{n} \ln(n)^k}{\left(1-\frac{1}{2}\right) k!} \right)^2$$

$$=$$

$$\ln(n) + \sum_{k=1}^{p} \frac{(-2)^k \epsilon_n^k \ln(n)^{k+1}}{(k+1)!} - 4 - 8 \times \sum_{k=1}^{p} \frac{(-\epsilon_n)^k \ln(n)^k}{k!} - 4 \times \sum_{k=1}^{p} \sum_{j=1}^{p} \frac{(-\epsilon_n)^{k+j} \ln(n)^{k+j}}{k! \times j!} \qquad (2)$$

now let's search x_p and y_p for $\epsilon_n = \frac{x_p \ln(n) + y_p}{\ln(n)^2}$ to make the total sum tend to zero,
I spare the details, we have something in the lines of:

$$\ln(n) \left(1 + \sum_{k=1}^{p} \frac{(-2)^k x_p^k}{(k+1)!} \right) - y_p - 4 \; - 8 \times \sum_{k=1}^{p} \frac{(-1)^k x_p^k}{k!} - 4 \times \sum_{k=1}^{p} \sum_{j=1}^{p} \frac{(-1)^{k+j} x_p^{k+j}}{k! \times j!} \xrightarrow[n \to +\infty]{} 0$$

You may ask "how did y_p get out just like this?", it's because at $k = 1$, we had:

$$-\frac{2 \ln(n)^2}{2} \times \frac{(x_p \ln(n) + y_p)}{\ln(n)^2} = -x_p \ln(n) - y_p$$

And that's the only k at which it happens.

Now we must find x_p so that $\ln(n) \left(1 + \sum_{k=1}^{p} \frac{(-2)^k x_p^k}{(k+1)!} \right) \xrightarrow[p \to +\infty]{} 0$; we'll do as follows:

$$1 + \sum_{k=1}^{p} \frac{(-2)^k x_p^k}{(k+1)!} = 0$$

$$\Leftrightarrow 1 - \frac{1}{2x} \sum_{k=1}^{p} \frac{(-2x_p)^{k+1}}{(k+1)!} = 1 - \frac{1}{2x} \left(\sum_{k=2}^{p+1} \frac{(-2x_p)^k}{k!} \right) = 1 - \frac{1}{2x} \left(\sum_{k=0}^{p+1} \frac{(-2x_p)^k}{k!} - 1 + 2x_p \right) = 0 \qquad (3)$$

$$So\ as\ p \to +\infty:\ 1 - \frac{1}{2x}\left(e^{-2x_p} - 1 + 2x_p\right) = 0$$
$$\Leftrightarrow e^{-2x_p} - 1 + 2x_p = 2x_p \Leftrightarrow e^{-2x_p} = 1 \Leftrightarrow x_p = 0$$

As a consequence, we're only left with $-y_p - 4 = 0 \Leftrightarrow y_p = 4$, and then, as $p \to +\infty$, $\epsilon_n = \dfrac{4}{\ln(n)^2}$

Therefore, for an infinite order of expansion, we can ideally write:

$$a_n = \frac{1}{2} + \frac{4}{\ln(n)^2}$$
$$so\ a_n \xrightarrow[n \to +\infty]{} \frac{1}{2}$$

Conclusion:

For any nontrivial zero $s \in \mathbb{C} \setminus \left\{1 + \dfrac{2\pi ik}{ln(2)} \mid k \in \mathbb{Z}\right\}$, $\lim\limits_{n \to +\infty} \sum\limits_{k=1}^{n} \dfrac{(-1)^{k-1}}{k^s} = 0$

implies that $Re(s)$ be a map of n, $a_n = Re(s_n)$, the limit of which **necessarily** is: $\lim\limits_{n \to +\infty} a_n = \dfrac{1}{2}$.

Therefore, since $\zeta(s) = 0 \ \Rightarrow \eta(s) = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n^s} = 0$:

For any nontrivial zero $s \in \mathbb{C} \setminus \left\{1 + \dfrac{2\pi ik}{ln(2)} \mid k \in \mathbb{Z}\right\}$, $\zeta(s) = 0 \implies Re(s) = \dfrac{1}{2}$.

This proves the Riemann Hypothesis.

[1] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function* (1986).
[2] H. Iwaniec and E. Kowalski, *Analytic Number Theory* (2004)