ART. VI.—*A Derivation of the Fundamental Relations of Electrodynamics from those of Electrostatics;* by LEIGH PAGE.

MAXWELL'S electrodynamic equations are based upon three experimental laws: (1) the inverse square law for the electric force between two point charges relatively at rest; (2) Ampere's law for the force between current elements, or its equivalent; (3) Faraday's law of current induction. Helmholtz gave a derivation of Faraday's law from Ampere's law by means of the principle of conservation of energy, which, however, has been shown to be erroneous.* Indeed, it has been impossible by any of the methods heretofore used to derive the electrodynamic equations without making use of all three of these experimental laws.

The object of this paper is to show, that if the principle of relativity had been enunciated before the date of Oersted's discovery, the fundamental relations of electrodynamics could have been predicted on theoretical grounds as a direct consequence of the fundamental laws of electrostatics, extended so as to apply to charges relatively in motion as well as to charges relatively at rest. Of course, only that part of the theory derived from the principle of relativity that is independent of any à priori knowledge of the electrodynamic equations, will be made use of. That is to say, we will use only the kinematics of relativity:—to use the dynamics of relativity, which is derived from the electrodynamic equations, would be to reason in a circle.

A material system is defined as an aggregate of material bodies having no relative motion, and showing no linear acceleration or angular velocity as a whole. Suppose now that we have any number of these systems moving in various directions and with various velocities relative to one another. The principle of relativity states that there are no experimental methods, practical or ideal, of distinguishing one such system as being marked out as different from all the others. In other words, if there is an ether, there exist no experimental methods by which we can find out which of these various systems is at rest relative to the ether.

One of the most obvious consequences of this principle is that the velocity of light, as measured in any one system, must be the same as measured in any other system. Otherwise there would be accessible to us an experimental method of locating the luminiferous ether, which is in contradiction to

* Maxwell's "Electricity and Magnetism," 3d edition, vol. ii, p. 192.

the principle of relativity. As a mathematical consequence of the fact that the velocity of light must be the same as observed from different systems, Einstein, in his celebrated paper* in the Annalen der Physik, has derived a set of space time transformations, which, because they were first deduced by Lorentz from entirely different considerations, usually go by his name.

Einstein starts off by a consideration of the meaning that can be attached to time simultaneity at two different points in any one system. Suppose A and B to be two widely separated places in the same system. An observer at A is watching certain phenomena in his immediate neighborhood, while an observer at B is watching certain other phenomena in his (B's) immediate neighborhood. They wish to compare the times of their observations. Obviously they must be provided with synchronous clocks. How are these clocks to be set synchronously? Let A send a light wave toward B when A's clock indicates the time t_A. This light wave reaches B at a time t_B on B's clock, and is returned to A by instantaneous reflection, reaching A at the time t'_A as indicated on A's clock. Since the measured value of the velocity of light is the same in all systems, and the same in all directions in any one system, the clocks at A and B will be synchronous when, and only when, $t_B = \frac{1}{2}(t_A + t'_A)$. Applying this definition of synchronism to two systems in motion relative to one another, Einstein is led to a set of transformations which show that the time at a point P in one system is a function not only of the time at a point Q in the other system, but also of the relative positions of the points P and Q.

When applied to the measurement of distances, these transformations show that a bar which is fixed in the first system with its axis parallel to the direction of relative motion of the two systems, and which has a length l as measured by an observer in the first system, will appear to have a shorter length when measured by an observer in the second system. This apparent shortening is not surprising when we consider the method used in measuring a body which is in motion relative to the observer. Let AB be a bar which has a velocity relative to the observer in the direction AB. In order to measure the length of the bar, the observer must mark the positions of the two ends of the bar at the *same instant*, and then measure the distance between these two marks. If he marks the position of the end B a little earlier than he marks the position of the end A, his measurement will be too short. Hence we see that space measurements as well as time measurements on moving systems, depend on the definition of simultaneity at different points of the same system.

* Annalen der Physik, xvii, 891, 1905.

Let K (o) denote the earth's system at any instant. Then K (v) denotes a system with velocity v relative to the earth.

Let XYZ be a set of orthogonal right-handed axes fixed in the earth's system, and so oriented that K(v) has a velocity v in the positive z direction.

Let X′Y′Z′ be a set of orthogonal right-handed axes fixed in system K(v) and mutually parallel to XYZ.

Unprimed letters denote quantities as measured in the earth's system, and primed letters denote the same quantities as measured in the system K(v).

Then the space time transformations between K(o) and K(v) take the form:

$$t' = \frac{t - \frac{v}{c^2} z}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad t = \frac{t' + \frac{v}{c^2} z'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$x' = x \qquad x = x'$$

$$y' = y \qquad y = y'$$

$$z' = \frac{z - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad z = \frac{z' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where c denotes the velocity of light. and where the time epochs are so chosen that the times at the respective origins of the two systems are zero when these origins coincide.

Let a particle have the velocity V relative to K(o), and V′ relative to K(v). Let V_x, V_y, V_z, be the components of V, and V_x', V_y', V_z' the components of V′. Then the following kinematical transformations follow at once by taking the time derivatives of the space time transformations, with consideration of the relation

$$dt \sqrt{1 - \frac{V^2}{c^2}} = dt' \sqrt{1 - \frac{V'^2}{c^2}}$$

$$\frac{V_x'}{V_x} = \frac{V_y'}{V_y} = \frac{\sqrt{1 - \frac{V'^2}{c^2}}}{\sqrt{1 - \frac{V^2}{c^2}}} = \frac{1 + \frac{vV_z'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{vV_z}{c^2}}$$

$$V_z' = \frac{V_z - v}{1 - \frac{vV_z}{c^2}} \qquad V_z = \frac{V_z' + v}{1 + \frac{vV_z'}{c^2}}$$

$$\frac{\dot{V}_x'}{\left(1-\frac{V'^2}{c^2}\right)^{3/2}}=\frac{\dot{V}_x-\frac{v}{c^2}\left[V_z\dot{V}_x-V_x\dot{V}_z\right]}{\left(1-\frac{V^2}{c^2}\right)^{3/2}\sqrt{1-\frac{v^2}{c^2}}}$$

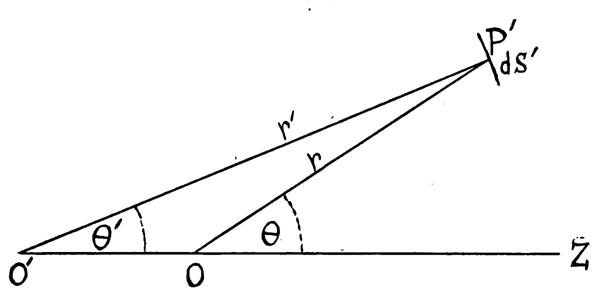
and similarly for $\dot{V}_y$

$$\frac{\dot{V}_z'}{\left(1-\frac{V'^2}{c^2}\right)^{3/2}}=\frac{\dot{V}_z}{\left(1-\frac{V^2}{c_2}\right)^{3/2}}$$

Moving Charges.

We can represent the field due to a charged particle which is at rest relative to the observer by radial lines of force so drawn that equal solid angles contain the same number of lines of force. Then we can define the intensity at any point as having the direction of the line of force at that point and as

FIG. 1.



being proportional, in magnitude, to the density of the lines of force at that point. Now let us extend this definition of intensity to charges which are moving relative to the observer. Consider a charge e at the point O′ (fig. 1) in K (v). Let dS′ be an elementary surface fixed in K (v) at P′, and perpendicular to O′P′. Let O′P′ $= r'$, and the angle between O′P′ and the Z′ axis be θ'. Then E′, the force at P′ as measured in K (v), will be $\frac{e}{r'^2}$. We wish to find the force E due to e, at a point P in K (o), when P coincides with P′. On account of the different definitions of simultaneity in the two systems K (v) and K (o), when P′ and P coincide the charge e as viewed from K (o) will be at some point O not coincident with O′. Let OP $= r$, and let the angle between OP and the Z axis be θ. The space time transformations give the relations

$$r' = \frac{\sqrt{1 - \frac{v^2}{c^2} \sin^2 \theta}}{\sqrt{1 - \frac{v^2}{c^2}}} r$$

$$\sin \theta' = \frac{\sqrt{1 - \frac{v^2}{c^2}}}{\sqrt{1 - \frac{v^2}{c^2} \sin^2 \theta}} \sin \theta$$

$$\cos \theta' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2} \sin^2 \theta}} \cos \theta \quad .$$

The direction of the lines of force, as viewed from K (o), and hence the direction of the intensity, will be OP, and *not* O′P′. Now dS' as viewed from K (o) will not be perpendicular to OP. Let dS be the component of dS', as viewed from K (o), which is perpendicular to OP. Then a short calculation gives

$$dS' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2} \sin^2 \theta}} dS$$

Now the density of the lines of force at P in K (o) is to the density of the lines of force at P′ in K (v), at the instant when P and P′ coincide, as dS' is to dS; that is to say in the ratio $1 : \sqrt{1 - \frac{v^2}{c^2} \sin^2 \theta}$. Hence we have

$$\frac{E}{E'} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2} \sin^2 \theta}}$$

$$\text{But } E' = \frac{e}{r'^2} = \frac{e}{r^2} \frac{\left(1 - \frac{v^2}{c^2}\right)}{\left(1 - \frac{v^2}{c^2} \sin^2 \theta\right)}$$

$$\text{Therefore } E = \frac{e}{r^2} \frac{\left(1 - \frac{v^2}{c^2}\right)}{\left(1 - \frac{v^2}{c^2} \sin^2 \theta\right)^{\frac{3}{2}}} \qquad \text{I}$$

The force E, as already noted, has the direction of the line of force through P, as observed in K (o); that is, the direction OP, where O is the apparent position of the charge to an observer in K (o) at the instant considered.

Thus, by means of the principle of relativity we have been able to derive from the laws of electrostatics, with considerable ease, an expression which Heaviside has derived from the electromagnetic equations only by the use of somewhat complicated mathematical processes.

The relations between the components of E at P and E′ at P′ follow at once from the expressions we have already derived.

$$E_x = \frac{E_x'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$E_y = \frac{E_y'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$E_z = E_z'$$

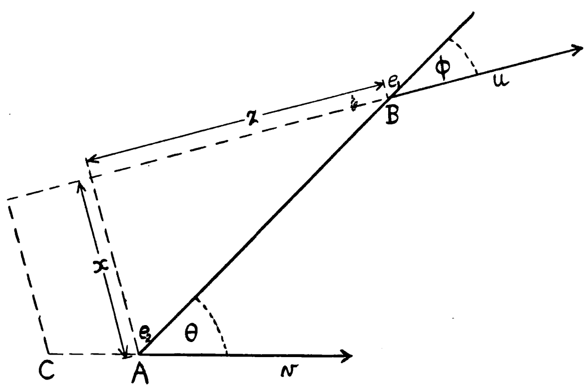
Force between Current Elements.

We can consider a current as made up of a given quantity of positive electricity moving with a given drift velocity along the wire in the direction of the current, and some other given quantity of negative electricity moving with some other given drift velocity in the direction opposite to that of the current. Let u_1 be the velocity of the positive electricity, and u_2 that of the negative electricity. Let λ_1 be the linear density, or the quantity of moving positive electricity per unit length of wire, and λ_2 the quantity of moving negative electricity per unit length of wire. Consider an element of the wire of length ds. Then we can define a current element as $(\lambda_1 u_1 + \lambda_2 u_2)\, ds$. Now this element of wire is as a whole uncharged. So there must be a quantity of positive electricity $(k - \lambda_1)\, ds$, and a quantity of negative electricity $(k - \lambda_2)\, ds$ at rest in the element, k being some constant. As the current is due to that part of the charge in the wire which is in motion, our problem reduces to a consideration of the forces between two charges both of which are moving relative to the observer.

In order to make our reasoning as simple as possible, we shall confine ourselves to currents lying in the same plane. There is no difficulty in extending the reasoning to currents which do not lie in the same plane, but in that case the demonstration becomes a little more complicated.

At a given instant in K (o), two charged bodies (fig. 2), one at A and the other at B, have velocities relative to K (o) of v and u cm./sec. respectively. $AB = r$. Choose axes XZ so that z is parallel to u. Let the origin be at B. We wish tö find the force on the charged body at B, due to the other charged body. To find this force we must observe from the

Fig. 2.



system K (u). But according to the time synchronism of K (u), when the one charged body is at B, the other will not be at A. It will be at C, a point whose coördinates are found to be

$$z = -\frac{r\cos\phi}{1-\frac{uv\cos(\theta-\phi)}{c^2}} \qquad x = -\frac{r\sin\phi - r\frac{uv}{c^2}\sin\theta}{1-\frac{uv\cos(\theta-\phi)}{c^2}}$$

These distances, as measured in K (u), are (the primes refer to K (u)),

$$z' = -\frac{r\cos\phi\sqrt{1-\frac{u^2}{c^2}}}{1-\frac{uv\cos(\theta-\phi)}{c^2}} \qquad x' = -\frac{r\sin\phi - r\frac{uv}{c^2}\sin\theta}{1-\frac{uv\cos(\theta-\phi)}{c^2}}$$

and the distance between the charges is

$$r' = \frac{r}{1-\frac{uv\cos(\theta-\phi)}{c^2}}\left\{1-2\frac{uv}{c^2}\sin\phi\sin\theta+\frac{u^2v^2}{c^4}\sin^2\theta-\frac{u^2}{c^2}\cos^2\phi\right\}^{\frac{1}{2}}$$

Applying I and reducing, we get

$$F_x' = \frac{e_1 e_2 \left(1 - \frac{v^2}{c^2}\right)\left(\sin\phi - \frac{uv}{c^2}\sin\theta\right)}{r^2\left(1 - \frac{u^2}{c^2}\right)^{\frac{1}{2}}\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{\frac{3}{2}}}$$

$$F_z' = \frac{e_1 e_2 \left(1 - \frac{v^2}{c^2}\right)\cos\phi}{r^2\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{\frac{3}{2}}}$$

Let F_x and F_z be the forces as measured in K (o) that must be applied to the charge at B in order to produce the same effect as F_x' and F_z'. Then

$$F_x = \frac{e_1 e_2 \left(1 - \frac{v^2}{c^2}\right)\left(\sin\phi - \frac{uv}{c^2}\sin\theta\right)}{r^2\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{\frac{3}{2}}}$$

$$F_z = \frac{e_1 e_2 \left(1 - \frac{v^2}{c^2}\right)\cos\phi}{r^2\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{\frac{3}{2}}}$$

Now replace e_1 by a current element $(\lambda_1 u_1 + \lambda_2 u_2)\, ds$. In this current element there is at rest the positive electricity $(k - \lambda_1)\, ds$, and the negative electricity $(k - \lambda_2)\, ds$. Consider the positive electricity $\lambda_1\, ds$ which is moving, and a portion $\lambda_1\, ds$ of the negative electricity which is at rest. Then the components of the force due to e_2 on the negative electricity $\lambda_1\, ds$ at rest will be

$$F_x = \frac{-\lambda_1 e_2 \left(1 - \frac{v^2}{c^2}\right)\sin\phi\, ds}{r^2\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{\frac{3}{2}}}$$

$$F_z = \frac{-\lambda_1 e_2 \left(1 - \frac{v^2}{c^2}\right)\cos\phi\, ds}{r^2\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{\frac{3}{2}}}$$

But the components of the force due to e_2 on the positive electricity $\lambda_1\, ds$ in motion is, as we have just found,

$$F_x = \frac{\lambda_1 e_2 \left(1 - \frac{v^2}{c^2}\right) \left(\sin \phi - \frac{u_1 v}{c^2} \sin \theta\right) ds}{r^2 \left(1 - \frac{v^2}{c^2} \sin^2 \theta\right)^{\frac{3}{2}}}$$

$$F_z = \frac{\lambda_1 e_2 \left(1 - \frac{v^2}{c^2}\right) \cos \phi \, ds}{r^2 \left(1 - \frac{v^2}{c^2} \sin^2 \theta\right)^{\frac{3}{2}}}$$

Combining, we have left the force

$$\frac{-\frac{\lambda_1 e_2 u_1 v}{c^2} \left(1 - \frac{v^2}{c_2}\right) \sin \theta \, ds}{r^2 \left(1 - \frac{v^2}{c^2} \sin^2 \theta\right)^{\frac{3}{2}}}$$

in the X direction.

Proceeding in the same manner, we find the total force on the current element at B due to the moving charge at A is

$$F_x = \frac{\frac{-e_2 v i_1}{c} \sin \theta \, ds}{r^2} \; \frac{\left(1 - \frac{v^2}{c^2}\right)}{\left(1 - \frac{v^2}{c^2} \sin^2 \theta\right)^{\frac{3}{2}}}$$

where i_1 is the current in electromagnetic units. As the drift velocity of the charges constituting a conduction current in a wire is certainly small compared with the velocity of light, we can place the factor

$$\frac{\left(1 - \frac{v^2}{c^2}\right)}{\left(1 - \frac{v^2}{c^2} \sin^2 \theta\right)^{\frac{3}{2}}}$$

equal to unity.

If we replace e_2 by a current element, we will find for the total force exerted by the current element $i_2 \, ds_2$ at A on the current element $i_1 \, ds_1$ at B, as measured on the earth (system K (*o*)), the expression

$$F_x = - \frac{i_1 i_2 \sin \theta \, ds_1 ds_2}{r^2}$$

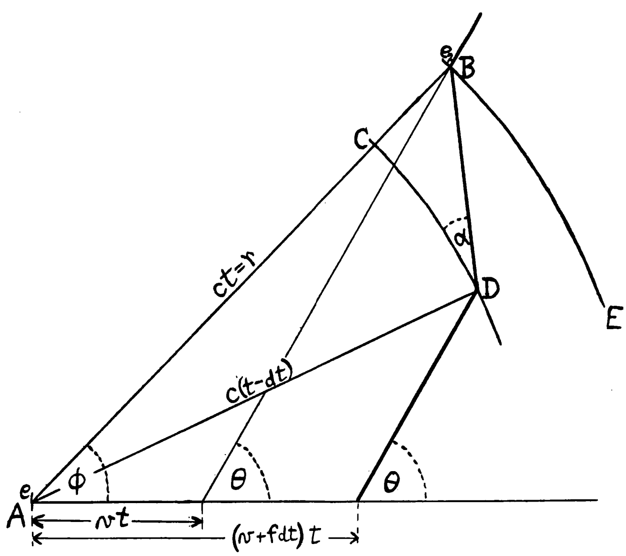
where i_1 and i_2 are measured in electromagnetic units.

This expression gives all the forces between currents, and also the induced current phenomena due to moving a closed circuit through a so-called magnetic field. The induced current effects produced in a secondary circuit by variation of the current in the primary are very simply treated as follows:

Faraday's Law.

Whenever a charged body is accelerated, it is obvious that the lines of force will be kinked. If the charged body is accelerated only for a very short time, these kinks will travel outwards in the form of a pulse. Now this pulse must have the same velocity relative to the system of the field inside the

FIG. 3.



pulse as it has relative to the system of the field outside the pulse. These two systems, however, may be chosen arbitrarily. Therefore the pulse must have the same velocity relative to all systems. The only velocity to satisfy this condition is the velocity of light. Hence the velocity of the pulse must be c.*

*This reasoning may be objected to on the ground that the pulse may expand as it moves outward: i. e., the outside of the pulse may have a greater velocity than the inside. But if this was true under certain conditions, it would be necessary to assume that the reverse was true under certain other conditions. So we would be forced to the most improbable conclusion that the inside of the pulse might outstrip and pass through the outside of the pulse.

Consider two charges e_1 and e_2 at A and B (fig. 3) respectively. Let the charge at B be at rest relative to the observer in K (o), and the charge at A be moving to the right with the velocity v. While e_1 is at A the acceleration f is applied to it in the direction of its velocity v. Let $AB = r = ct$. BE is an arc with A as center and ct as radius, CD an arc with A as center and $c\,(t-dt)$ as radius. If, as before, we define the intensity as proportional to the density of the lines of force at the point considered, the force just to the left of B will be

$$F = \frac{e_1 e_2\left(1 - \frac{v^2}{c^2}\right)}{r^2\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{\frac{3}{2}}} \cdot \frac{1}{\sin\,a}$$

provided v is small compared to c. So the intensity at the same point due to e_1 will be

$$E = \frac{e_1\left(1 - \frac{v^2}{c^2}\right)}{r^2\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{\frac{3}{2}}} \cdot \frac{1}{\sin\,a}$$

If we denote by $E_{/\!/}$ and $E_{/}$ the components of E parallel to and perpendicular to the radius AB,

$$E_{/\!/} = \frac{e_1\left(1 - \frac{v^2}{c^2}\right)}{r^2\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{\frac{3}{2}}} \qquad E_{/} = \frac{e_1\left(1 - \frac{v^2}{c^2}\right)}{r^2\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{\frac{3}{2}}}\cot a$$

So we see that the component of the force parallel to the radius AB is continuous through the pulse.

Now $\cot\,a = \frac{ft\sin\theta}{c}$ if $\frac{v}{c}$ is small.

$$\therefore\ E_{/} = \frac{e_1 f}{rc^2}\,\frac{\left(1 - \frac{v^2}{c^2}\right)}{\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{\frac{3}{2}}}\sin\theta$$

If we replace e_1 by a current element ids

$$E_{/} = \frac{di}{dt}\,\frac{\sin\theta}{r}\,\frac{\left(1 - \frac{v^2}{c^2}\right)}{\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{\frac{3}{2}}}\,ds$$

in electromagnetic units.

If $\frac{v}{c}$ is small, this reduces to

$$E_{,} = \frac{di}{dt} \frac{\sin \theta}{r} ds$$

which is the expression for the induced electromotive force in one wire due to a variation of the current in another.

Conclusions.

Our object was to deduce the fundamental laws of electrodynamics,—the law for the force between currents, and the law governing current induction,—from those of electrostatics. We assumed that part of the theory derived from the principle of relativity which depends only upon the fact that the velocity of light must be the same as measured in different systems, and which depends in no way upon the electrodynamic equations. Then we extended the following conceptions of electrostatics to moving charges:

(1) To an observer at rest relative to a charge, the charge can be replaced by a field of lines of force radiating from the charge in such a way that equal solid angles contain equal numbers of lines of force.

(2) To an observer relative to whom the charge is in motion, as well as to an observer at rest relative to the charge, the electric intensity due to the charge is proportional to the instantaneous density of the lines of the force at the point considered.

By the means of these extensions of electrostatic conceptions to moving charges, we were able to deduce (*a*) the expression for the electric intensity due to a charge moving relative to the observer; (*b*) Ampere's law, or its equivalent; (*c*) Faraday's law, or its equivalent.

Viewed from another standpoint, the fact that we have been able, by means of the principle of relativity, to deduce the fundamental relations of electrodynamics from those of electrostatics, may be considered as some confirmation of the principle of relativity.

I want to express my thanks to Professor H. A. Bumstead for several valuable suggestions, and to Dr. H. M. Dadourian for his help and encouragement.

Sheffield Scientific School, March 7th, 1912.