The Pure Theory of Utility Curves
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THE PURE THEORY OF UTILITY CURVES.

Introduction.

The object of the following investigations is to clear up certain theoretical points which cannot be dealt with precisely by the ordinary diagrams. Part I. deals with the apparatus to be employed. In the demand curves, used in discussing the exchange of aggregate quantities of goods against utility or money or other goods, the ordinate represents the aggregate sacrificed and the abscissa the aggregate acquired. These curves may also be regarded as the locus of points where the straight lines from the origin, representing the various possible ratios of exchange, touch constant utility curves—a constant utility curve being such that all the bargains represented by points on it would yield the same net utility. Hence we can sometimes get a more precise knowledge of what is involved in drawing a demand curve in a particular way by *going behind* the demand curve, as it were, and considering its relation to the constant utility curves. The underlying significance of the apparatus introduced in Part I. depends on a modification of this idea. Instead of considering the net utility of an *exchange*, we consider the resultant utility obtained from the acquisition of two commodities which both contribute to the utility positively. This gives rise to two new kinds of demand curve; one for the case in which the total expenditure on the two commodities varies while their prices remain constant, the other for the case in which the total expenditure is fixed while the price of *one* of the commodities varies. The first of these new demand curves enables problems to be attacked diagrammatically, in which the marginal utility of money need not be assumed constant. In order to interpret the meaning of the various shapes which these curves can assume, we investigate their relation to the constant utility curves, upon which, as in the former case, they ultimately depend.

In Part II. the analytical conditions which govern the shape of the constant utility curves are elucidated. Criteria are given for distinguishing between three types of the resulting demand curves, and precise definitions are introduced for the terms "complementary" and "competitive" as applied to commodities.

In Part III. the two kinds of demand curve defined above are further discussed, particularly the case where the prices of the commodities are constant and the total expenditure on them varies. It is found possible to analyse the case in which increased total expenditure *diminishes* the amount bought of one of the commodities, as well as the standard case in which more of both commodities is bought when there is more to spend. Similarly, when the demand curve is reached, for which one of the commodities varies in price while the total expenditure is fixed, we can analyse the case in which an increased price leads to an *increase* of the amount of the commodity bought (*i.e.*, Giffen's paradox relating to the expenditure of certain classes on wheat). Incidentally, it is found possible to give a more precise definition of the marginal utility of money than that commonly employed.

In Part IV. the problem of more than two variables is attacked; and the question of the maximum product (or utility) derivable from the expenditure of a given sum on n factors (or commodities) is discussed. The general conditions for a true maximum are then investigated. This is, perhaps, rather of mathematical than of strictly economic interest; but it serves to round off what would otherwise be incomplete.

PART I.

§ 1. For the purpose of raising certain economic problems, a modification of Professor Edgeworth's "indifference curves" is here employed.

In Edgeworth's curves the co-ordinates of any point represent two variable quantities, one of which is acquired and the other is sacrificed in return for the former. The acquisition yields utility, the sacrifice disutility. The net utility thus increases with increase of acquisition or with decrease of sacrifice. The properties of Edgeworth's curves are summarised in § 2.

§ 2. As the diagrams have to be limited to *two* dimensions, we cannot represent by any third co-ordinate the variation in the measure of the net utility, arising out of the acquisition and sacrifice taken together. This difficulty is met by constructing curves of constant utility. These are such that the net utility for bargains represented by all the different points on any *one* curve is the same.

Thus if x measures the amount acquired and z the amount sacrificed, the net utility for the point P, whose co-ordinates are x, z, has a value depending on the amounts represented by these co-ordinates. And the locus of all points, yielding the same net utility as P, forms the constant utility curve through P.

The constant utility curve U_0 which passes through the origin O is that of *zero-utility—i.e.*, for all points along this curve the sacrifice of z balances the acquisition of x, so as to yield neither

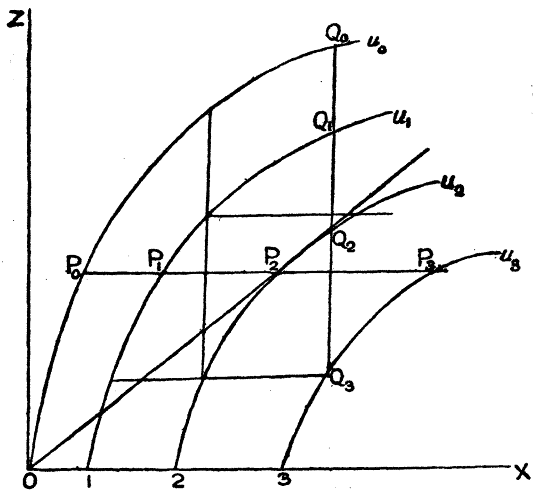


Fig. 1.

more nor less net utility than if no sacrifice and no acquisition had been incurred. For all points along the curve U_1, which passes through the point 1 (on x), the net utility is such as would be yielded by an acquisition, $x=\overline{01}$, if no sacrifice were incurred. Along U_2, which passes through the point 2, the net utility is equal to that yielded by $x=\overline{012}$, if no sacrifice were incurred; and so on. (See Fig. 1.)

In the standard case :—

(*a*) In order to obtain *equal* additional increments of net utility we require continually increasing increments of acquisition or continually decreasing increments of sacrifice.

Thus the lengths $P_0\,P_1$, $P_1\,P_2$, $P_2\,P_3$, &c., increase; also the lengths $Q_0\,Q_1$, $Q_1\,Q_2$, $Q_2\,Q_3$, &c., increase.

(*b*) Between any two utility curves the x distances increase as we move upwards, while the z distances decrease as we move rightwards.

(*c*) The curves are concave to x, convex to z.

(*d*) The curves become flatter as we move rightwards along any horizontal line or upwards along any vertical line.

Suppose, now, that x is offered us in return for z; and that we can acquire x by sacrificing z on certain given terms; that is, by giving z for x at a fixed ratio. The locus of points, for which the ratio of z to x is constant, is a straight line through the origin. Any such straight line may, therefore, be called a Price-Line. When the ratio of exchange is determined for us, we shall obviously maximise our net utility by closing our bargain at the point where the price-line, representing the ratio of exchange, *touches* one of the utility-curves.

If, after we have agreed to acquire x (say) in return for z, we are offered a further amount on more favourable terms, we shall again pass to a position where the line representing the new price, drawn from our former resting-place, touches another utility-curve.

And so on.

The crooked line $OC_1\ C_2\ C_3$ . . . (see Fig. 2) represents the course of a price-line involving changes in the terms on which we can acquire x in return for z.

Now consider the other party to the transaction who sacrifices x in order to acquire z. The general character of *his* utility curves will be the same as that of the first party with x and z interchanged. Hence, if the two parties make a succession of contracts with one another *on alterable terms*, a crooked price-line will represent their transactions, in which the conclusion of each contract is represented by the point where the price-line (representing the terms of the current contract) drawn from the point representing the conclusion of the previous contract *touches* a utility-curve of *one* of the two parties earlier than one of the *other*. Further, as long as the next section lies *between* the two utility curves, further exchanges are made, since they add to the utility of both parties; but, when finally the new section of the price-line touches both the utility curves simultaneously, exchange ceases. (See Fig. 2.)

The construction of the ordinary supply or demand curves is based on the assumption that the whole exchange is transacted on *unaltered* terms. This is a practically legitimate assumption in the case of frequently recurrent exchanges; that is, when there is a *continual* demand and supply for "consumed" commodities. These curves may be called "Offer" curves. There are two ways of constructing such curves, both of which represent precisely the same facts. The more familiar plan is to make the ordinate y stand for "Price," *i.e.*, for the ratio of z to x, where z and x are the total quantities exchanged. The less familiar plan (which is here throughout adopted) is to use as co-ordinates the total quantities z and x themselves. Now, the connection between an "offer" curve, and the series of constant utility curves (for the party making the offer) can be explained as follows: The "offer" curve is the locus of points where the straight lines *from the*

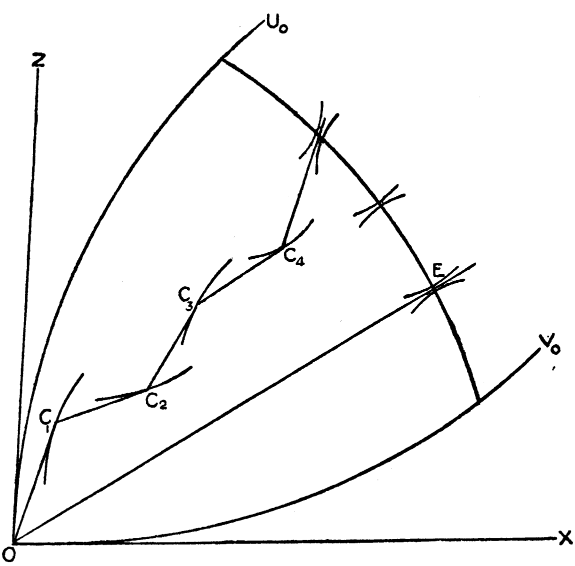


Fig. 2.

origin, representing the various possible ratios of exchange, touch one of the constant utility curves. Thus, where the demand-curve cuts the supply-curve, the utility curves of the two parties have a common tangent *which passes through the origin.* This tangent represents an *unaltered* rate of exchange. In all cases, transactions cease at a point where the two utility curves *touch.* The speciality of the equilibrium, under conditions where the whole exchange is made on unaltered terms, is that this common tangent passes straight through the origin. (See Fig. 2.)

On the other hand, if one party (say, he who is acquiring x by sacrifice of z) has control over the production, then the terminus of exchange is the point where the *supply-curve* (instead of the price-line) *touches* one of his utility curves. This occurs, for example, when a person supplies his own needs by his own labour. (See Fig. 3.)

§ 3. The modification of Edgeworth's diagrams, which is here proposed, is in effect equivalent to turning them upside down. This procedure answers the following purposes :—First, it enables us to deal naturally with cases in which two quantities contribute *positively*—instead of one positively and the other negatively—to the resultant utility. Secondly, it can be applied by analogy to the case in which *any number* of variables contribute to utility or to production. Thirdly, it is specially available for the case in which the monetary resources of the consumer have (for the purposes of the problem in hand) a *fixed limit*.

The immediate results of this transformation are :—

(1) That the utility curves are *convex* to both axes, instead of

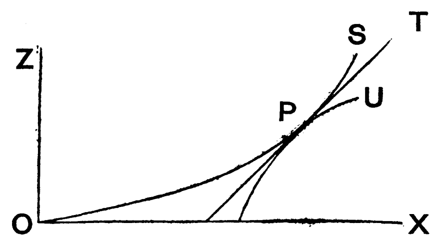


FIG. 3.

being convex to the sacrifice axis and concave to the acquisition axis.

(2) That the values of the two co-ordinates vary along a utility curve *in opposite senses*, instead of in the same sense.

(3) That the price-lines *cut the two axes* instead of starting from the origin.

(4) That the price-lines may be made to vary, not only in direction, but also *in their abscissæ*.

Thus the accompanying diagram shows how three Price-Lines P_1T_1, P_2T_2, P_3T_3 may touch three utility-curves at the respective points P_1, P_2, P_3. (See Fig. 4.)

Thus let ξ, ζ represent prices of x, z; and μ the amount of money expended on them. Then the price-lines are represented by the equations

$$x\xi+z\zeta=\mu,$$

i.e.

$$\frac{x}{\mu/\xi}+\frac{z}{\mu/\zeta}=1.$$

The abscissa OT along $Ox=\mu/\xi$; *i.e.*, the amount of x which could be bought for money μ.

The abscissa Ot along $Oz = \mu/\zeta$; *i.e.*, the amount of z which could be bought for money μ.

Thus any point on tPT represents the amounts of x and of z that might be bought, when the price of x is ξ, that of z is ζ, and the amount spent on the two together is μ.

Provided we assume that the utility curves descend convexly, it is obvious that the resultant utility is a *maximum* where this price-line *touches* a utility-curve.

Again, where two price-lines, such as P_1T_1 and P_2T_2, are parallel, we represent constant values of ξ and ζ with varying

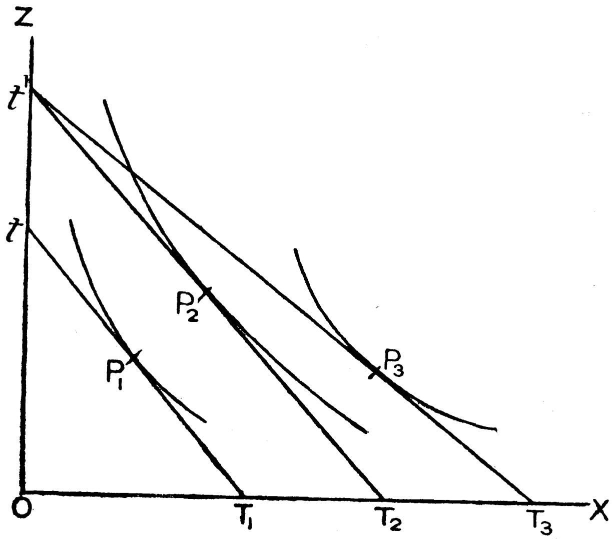


Fig. 4.

values of μ : the ratio of variation being given by the ratio of OT_2 to OT_1.

Where two price-lines, such as P_2T_2 and P_3T_3, cut on the axis of z at the same point t', we represent constant values of ζ and μ with varying values of ξ: the ratio of variation being given (inversely) by the ratio of OT_3 to OT_2.

In the figure given, a rise in money expended such as to shift the point of maximum utility from P_1 to P_2 leads to an increase in the purchase both of x and of z; and a fall in the price of x such as to cause a shift from P_2 to P_3 leads to an increase in the purchase of x and a decrease in that of z.

In the former case the money expended has increased, without changes in the prices; in the latter, the price of x has fallen, while the price of z and the total expenditure have remained unaltered.

This illustration shows how we may construct two kinds of demand-curves : (*a*) that in which the total money expended varies while the prices of the commodities are constant, as from P_1 to P_2; (*b*) that in which the price of *one* commodity (say x) varies, while the total expenditure is constant and the prices of other commodities are constant.

A curve drawn through P_1 and P_2, &c., will represent the former; a curve drawn through P_2 and P_3 the latter. The former kind of demand-curve may be called a varying *expenditure* curve; the second, a varying *price* curve. In both cases all other relevant quantities, potentially variable, are taken as constant.

§ 4. The character of the varying expenditure curve will be considered at length in Part II. From it the characteristics of the varying price curve can be conveniently deduced.

The principal problems to be considered are :—

(1) What precise conditions are involved by our assumption, that the utility curves "descend convexly," which is required if the tangent solution for the price-line is to yield a true maximum. [See § 9.]

(2) Upon what special conditions an increase or a decrease in one or other of the amounts bought depends for changes in the position of the price-line. [See §§ 15, 16, 19, 20.]

Before passing on to these problems, it is worth while to point out that the diagram measures only the quantities x and z. There are no lines in the figure which measure the utility itself. The several utility-curves are arranged in a scale of increasing value as we pass to the right and above; and thus the "distance" (measured arbitrarily) from one curve to another "indicates" (without measuring) the increase in utility. But this impossibility of measurement does not affect any economic problem. Neither does economics need to know the marginal (rate of) utility of a commodity. What is needed is a representation of the *ratio* of one marginal utility to another. In fact, this ratio is precisely represented by the *slope* at any point of the utility-curve.

Thus the sole mathematical datum is summed up in the fact that the ratio of the marginal utility of x to that of z at the point (x, z) is equal to the ratio of the abscissa on z of the tangent to the utility-curve through (x, z) to its abscissa on x.

Moreover, just as we can indicate (without measuring) the total utility (say) at P_2 by the distance from the origin of the curve through P_2, and thus exhibit the fact that at P_2 the utility is intermediate between that at P_1 and that at P_3; so we can *indicate* the different kinds of *Surplus* Utility. A surplus utility, in general, means the excess of the actual utility over what might have been obtained if the individual, under the same objective conditions, had freely chosen to act in some uneconomic way.

What it *usually* means more particularly seems to be as follows : Having a given amount of money to expend on various commodities, and the prices of these commodities being such as they are, the consumer is supposed to maximise his utility by spending his money on certain quantities of each of these commodities. If therefore, at the given prices, he were to spend *all* his money on all *but one of* those commodities in the most useful way, he would procure *less* utility than if he included this one. The difference of utility procured by these two courses of conduct is the (integral) surplus utility actually derived from the commodity singled out. The general problem of maximising utility is applicable just because there *is* this surplus.

In the diagram (see Fig. 4) the surplus utility (say) at P_2, due to the inclusion of x in his purchases, is indicated by the "distance" between the utility-curve through P_2 and the utility-curve drawn through t' (where $x=o$ for the same money expenditure). The convexly descending shape of the utility-curves shows that the utility at t' is less than that at P_2. And the *degree* of this surplus is indicated by some line drawn *from* the curve through t' *to* the curve through P_2. To indicate this differential utility, it would be theoretically most convenient to draw (from the origin) a line through points, on the successive curves, where the tangents are throughout parallel to one another. But the sections—between

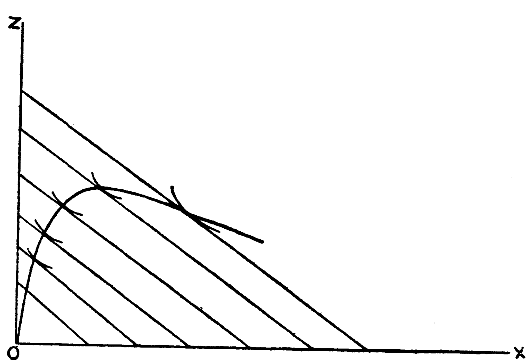


Fig. 5.

any two curves—of the *different* lines (corresponding to the different directions of the systems of parallel tangents) would not necessarily be proportional to one another.

Before entering upon the analytical discussion of Part II., we may examine Figs. 5 and 6, which illustrate the purport of this section. In these figures, small bits of the successive constant utility-curves are drawn where they touch the varying price-lines. In Fig. 5 the demand-curve is exhibited which depends on varying expenditure, with the prices of x and of z constant. In Fig. 6 the demand-curve is exhibited which depends on variation in the price of z alone. In both figures the increasing power of purchase (and, hence, the increase of utility) involves

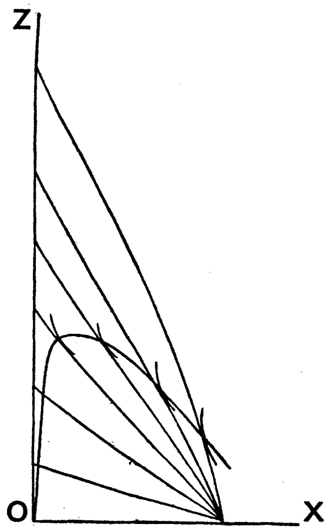


FIG. 6.

at first an increase both of x and of z, and *afterwards* an increase of x with a decrease of z. In this way, the solution of the two chief problems above mentioned is shown to the eye.

PART II.

In this Part, we shall confine ourselves, in order to be able to use diagrammatic representation, to the cases in which the consumer's utility (u) is a function of *two* quantities only, x and z.

We shall assume two characteristics of this function, and shall shew in § 9 that, of these two characteristics, the first proves the curve to be *descending to the right*, the second proves it to be *convex* to the two axes.

§ 5. The first assumption is that any increment of x or of z increases u.

i.e. $\frac{du}{dx}$ and $\frac{du}{dz}$ are both positive. . . . . (1)

Before coming to the second assumption some new symbols must be introduced. The standard case is that in which an increase of any factor causes a decrease in its marginal utility. Hence, as a rule, $\frac{d^2u}{dx^2}$ and $\frac{d^2u}{dz^2}$ are both *negative*.

It would, therefore, be convenient to use the following symbols for measuring the *relative* changes in $\frac{du}{dx}$ or $\frac{du}{dz}$ due to changes in x or z :—

Change in $\frac{du}{dx}$, due to $dx \equiv \tau_{11} = -\frac{d^2u}{dx^2} \div \left(\frac{du}{dx} \cdot \frac{du}{dx}\right)$.

Change in $\frac{du}{dz}$, due to $dz \equiv \tau_{22} = -\frac{d^2u}{dz^2} \div \left(\frac{du}{dz} \cdot \frac{du}{dz}\right)$.

Change in $\frac{du}{dz}$, due to $dx \equiv \tau_{12} = -\frac{d^2u}{dxdz} \div \left(\frac{du}{dz} \cdot \frac{du}{dx}\right)$.

Change in $\frac{du}{dx}$, due to $dz \equiv \tau_{21} = -\frac{d^2u}{dzdx} \div \left(\frac{du}{dx} \cdot \frac{du}{dz}\right)$.

Again, in the standard case, a change in the amount of x would produce a greater relative change in the marginal utility of x than in that of z; and a change in z would produce a greater relative change in the marginal utility of z than in that of x. That is, usually,

$$\tau_{11} > \tau_{12} \text{ and } \tau_{22} > \tau_{21}.$$

However, these standard relations do not hold universally. It may be that a change in z, as well as a change in x, produces a greater relative change in the marginal utility of x than in that of z.

That is, it may be that $\tau_{21} > \tau_{22}$ as well as $\tau_{11} > \tau_{12}$.

This leads up to the second postulate. Whatever relations there may be otherwise, we shall lay down the following restrictive assumption :—

Although a change in z as well as a change in x may produce a greater relative change in the marginal utility of x than in the marginal utility of z, yet such excess as is due to a change in x (measured relatively to the marginal utility of x) will invariably be greater than such excess as is due to a change in z (measured relatively to the marginal utility of z).

This assumption, expressed analytically, is that

$$\tau_{11} - \tau_{12} > \tau_{21} - \tau_{22}$$

i.e.

$$\tau_{11} + \tau_{22} - 2\tau_{12} \textit{ is positive} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

§ 6. In dealing with two variables, it is convenient to make the following substitutions :—

$$V \equiv \frac{du}{dx} \div \frac{du}{dz}\,; \quad W \equiv \frac{du}{dz} \div \frac{du}{dx},$$

so that $V.W = 1$

Then

$$\frac{dV}{dx}\cdot\left(\frac{du}{dz}\right)^2=\frac{du}{dz}\cdot\frac{d^2u}{dx^2}-\frac{du}{dx}\cdot\frac{d^2u}{dxdz},$$

$$\frac{dV}{dz}\cdot\left(\frac{du}{dz}\right)^2=\frac{du}{dz}\cdot\frac{d^2u}{dxdz}-\frac{du}{dx}\cdot\frac{d^2u}{dz^2}$$

Thus $\frac{dV}{dx}$ has the sign of $\tau_{12}-\tau_{11}$.

$\frac{dV}{dz}$ has the sign of $\tau_{22}-\tau_{12}$.

Similarly, $\frac{dW}{dz}$ has the sign of $\tau_{12}-\tau_{22}$,

and $\frac{dW}{dx}$ has the sign of $\tau_{11}-\tau_{12}$.

That is, in the standard case,

$\frac{dV}{dx}$ and $\frac{dW}{dz}$ are *negative.*

By algebraical substitution, it may be shewn that condition (2) may be written

$$V\frac{dV}{dz}-\frac{dV}{dx}>0,$$

or, using the symbols $\frac{dV}{dx}$ and $\frac{dW}{dz}$,

$$\frac{V^2}{W}\cdot\frac{dW}{dz}+\frac{dV}{dx}<0.$$

This last expression shows, symmetrically, that $\frac{dV}{dx}$ and $\frac{dW}{dz}$ cannot both be *positive*, the standard case being that in which they are both *negative.*

§ 7. The course of the demand-curves will fall into three divisions depending essentially upon the signs of $\frac{dV}{dx}$ and $\frac{dW}{dz}$. The standard or *mediate* section is where $\frac{dV}{dx}$ and $\frac{dW}{dz}$ are both *negative*: the section in which z is more urgently needed than x is where $\frac{dW}{dz}$ is positive $\left(\frac{dV}{dx}\text{ being negative}\right)$; and that in which x is more urgently needed than z is where $\frac{dV}{dx}$ is positive $\left(\frac{dW}{dz}\text{ being negative}\right)$.

For in speaking of z as more urgently needed, we imply that the amount of z is comparatively *small*; and, hence, that the marginal utility of z is comparatively *great.* Hence the proportional changes produced in the marginal utility of z (due to changes in the amounts either of x or of z) are comparatively *small*.

That is, τ_{22} is small and τ_{11} is great;

i. e. $$\tau_{11}>\tau_{12} \text{ and } \tau_{22}<\tau_{12},$$

i. e. $$\frac{dV}{dx} \text{ is negative, but } \frac{dW}{dz} \text{ is positive.}$$

§ 8. Two extreme cases are of interest.

In Fig. 7 (a) the curves of utility degenerate into a series of parallel straight lines. Here we may call x and z strictly or absolutely *competitive*: *i.e.*, any given amount of x gives the same utility as a proportional amount of z. These curves have *minimum* curvature.

In Fig. 7 (b) the curves of utility degenerate into a pair of straight lines parallel to the axes terminated at a series of points

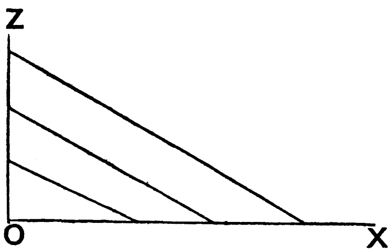


Fig. 7 (a).

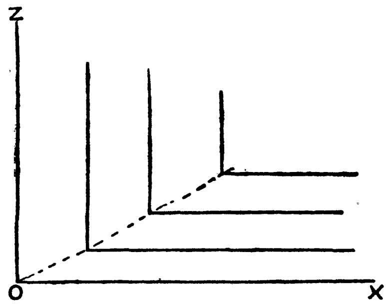


Fig. 7 (b).

along a line through the origin. Here we may call x and z strictly or absolutely *Complementary*; *i.e.*, when x and z are acquired together in a certain fixed proportion, no increment of utility is obtained by increasing one amount unless we also increase the other. These curves have *maximum* curvature (at their principal point).

As at any point in the curves the curvature approximates to one or other of these limiting cases, we may speak of the factors x and z as being roughly competitive or complementary.

A more exact definition of competitive and complementary may be suggested.

Since the general condition (2) is that

$$2\frac{d^2u}{dxdz}\div\left(\frac{du}{dx}\right)\left(\frac{du}{dz}\right)-\frac{d^2u}{dx^2}\div\left(\frac{du}{dx}\right)^2-\frac{d^2u}{dz^2}\div\left(\frac{du}{dz}\right)^2>0,$$

therefore, $\frac{d^2u}{dxdz}$ cannot be less than both $W\frac{d^2u}{dx^2}$ and $V\frac{d^2u}{dz^2}$.

Hence, when $\frac{d^2u}{dxdz}$ lies between $W\frac{d^2u}{dx^2}$ and $V\frac{d^2u}{dz^2}$, we may say that x and z are *competitive.*

But, when $\frac{d^2u}{dxdz}$ is greater than both $W\frac{d^2u}{dx^2}$ and $V\frac{d^2u}{dz^2}$, then we may say that x and z are *complementary.*

In the former case, either $\frac{dV}{dx}$ or $\frac{dW}{dz}$ is *positive*, in the latter case $\frac{dV}{dx}$ and $\frac{dW}{dz}$ are both *negative.*

In the former case changes along the Demand Curves involve an opposite variation in x and z; in the latter the two increase or decrease together.

§ 9. We may now show how the curves constructed on these considerations will behave.

In the constant utility curve, $u=c$, we have

$$\frac{du}{dx}+\left(\frac{dz}{dx}\right)_u\cdot\frac{du}{dz}=0,$$

i.e.,

$$-\left(\frac{dz}{dx}\right)_u=\frac{du}{dx}\div\frac{du}{dz}\equiv V.$$

Thus V measures the inclination to the axis of x of the tangent to the constant-utility curve at any point.

Since, in accordance with our first assumption, $\frac{du}{dx}$ and $\frac{du}{dz}$ are both positive,

$\therefore\left(\frac{dz}{dx}\right)_u$ is *negative*; *i.e.*, the curves fall to the right.

Again,

$$-\left(\frac{d^2z}{dx^2}\right)_u=\frac{dV}{dx}+\left(\frac{dz}{dx}\right)_u\cdot\frac{dV}{dz}=\frac{dV}{dx}-V\frac{dV}{dz}.$$

Since, in accordance with our second assumption, this last expression is *negative*,

$\therefore\left(\frac{d^2z}{dx^2}\right)_u$ is *positive*; *i.e.*, the curve is throughout *convex.*

It follows that any straight line (between the axes) can touch only *one* utility curve, and only at *one* point; and that at this point the utility will be a *maximum.* [See Figs. 4, 5, 6.] The results of this section resolve the first problem of § 4.

§ 10. The condition of convexity may be written :—

$$\frac{du}{dz}\cdot\frac{dV}{dx}<\frac{du}{dx}\cdot\frac{dV}{dz}\quad\ldots\ldots\ldots\,(3)$$

or

$$\frac{du}{dx}\cdot\frac{dW}{dz}<\frac{du}{dz}\cdot\frac{dW}{dx}\quad\ldots\ldots\ldots\,(4)$$

In the *mediate* section, $\frac{dW}{dz}$ and $\frac{dV}{dx}$ are both negative, and thus $\frac{dV}{dz}$ and $\frac{dW}{dx}$ are both positive, so that the above relations are obviously satisfied.

But in the *z-urgent* section, $\frac{dW}{dz}$ $\left(\text{and } \frac{dW}{dx}\right)$ are positive, and thus $\frac{dV}{dz}$ $\left(\text{and } \frac{dV}{dx}\right)$ are negative. Hence in this case, from (3),

$$\frac{du}{dx} \cdot \left(-\frac{dV}{dz}\right) < \frac{du}{dz} \cdot \left(-\frac{dV}{dx}\right),$$

i.e.,

$$\frac{du}{dx} \div \frac{du}{dz} < \frac{dV}{dx} \div \frac{dV}{dz} \quad \dots\dots\dots\dots \quad (5)$$

also

$$\frac{du}{dx} \div \frac{du}{dz} < \frac{dW}{dx} \div \frac{dW}{dz} \quad \dots\dots\dots\dots \quad (6)$$

But in the *x-urgent* section, $\frac{dV}{dx}$ $\left(\text{and } \frac{dV}{dz}\right)$ are positive, and thus $\frac{dW}{dx}$ $\left(\text{and } \frac{dW}{dz}\right)$ are negative. Hence, in this case, from (4),

$$\frac{du}{dz} \cdot \left(-\frac{dW}{dx}\right) < \frac{du}{dx} \cdot \left(-\frac{dW}{dz}\right),$$

i.e.,

$$\frac{dW}{dx} \div \frac{dW}{dz} < \frac{du}{dx} \div \frac{du}{dz} \quad \dots\dots\dots\dots \quad (7)$$

also

$$\frac{dV}{dx} \div \frac{dV}{dz} < \frac{du}{dx} \div \frac{du}{dz} \quad \dots\dots\dots\dots \quad (8)$$

Thus the relations expressed in (7) and (8) for the *x-urgent* section are the reverse of those expressed in (5) and (6) for the *z-urgent* section.

§ 11. The distinctions between the three sections may usefully be indicated by constructing adjacent utility-curves. [See Figs. 8 (*a*), (*b*), (*c*).]

ADJACENT UTILITY CURVES.

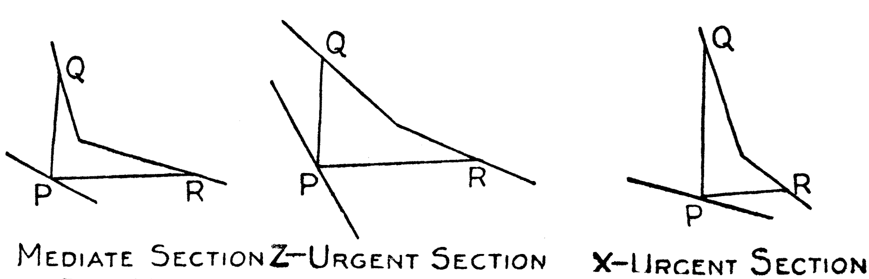


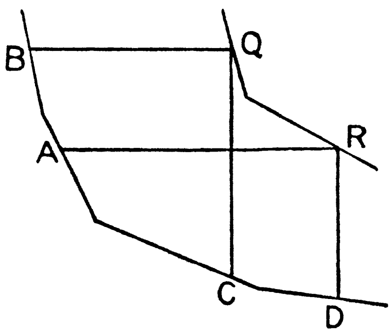
FIG. 8 (*a*). FIG. 8 (*b*). FIG. 8 (*c*).

Let Q and R be two points on a utility curve; and P a point on an adjacent curve of lower utility, where the ordinates at Q and R meet.

Draw tangents to the utility curves at P, Q and R. The slope at P may be related in one of three ways to those at Q and R.

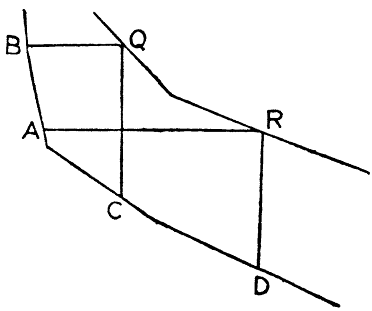
In the *mediate* case, the slope at P is intermediate between those at Q and at R.

i.e. $$\frac{dV}{dx}<0 \quad \text{and} \quad \frac{dW}{dz}<0.$$



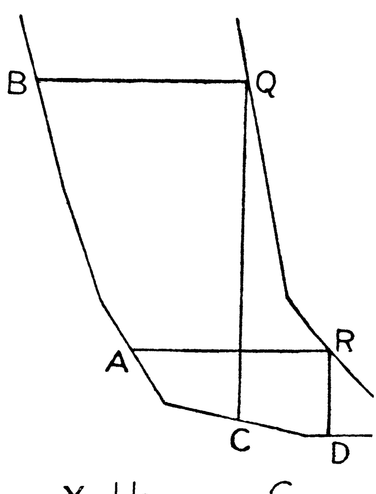
MEDIATE CASE

FIG. 9 (*a*).



Z—URGENT CASE

FIG. 9 (*b*).



X—URGENT CASE

FIG. 9 (*c*).

In the *z-urgent* case, the slope at P is steeper both than that at Q and that at R.

i.e. $$\frac{dV}{dx}<0 \quad \text{but} \quad \frac{dW}{dz}>0.$$

In the *x-urgent* case, the slope at P is flatter both than that at Q and that at R.

i.e. $$\frac{dV}{dx}>0 \text{ while } \frac{dW}{dz}<0.$$

Since the slope at Q is always steeper than that at R, the three figures represent all possible cases.

§ 12. The shapes of the curves may be still more fully shown by Figs. 9 *a*, *b*, *c* for any two utility curves not necessarily adjacent.

We draw horizontal and vertical lines between any two curves.

In the mediate case, $BQ < AR$ and $RD < CQ$,
i.e., the x-distances *decrease* as z increases,
and the z-distances *decrease* as x increases.

In the z-urgent case, $BQ < AR$ but $RD > CQ$,
i.e., the x-distances again *decrease* as z increases;
but the z-distances *increase* as x increases.

In the x-urgent case, $BQ > AR$ while $RD < CQ$,
i.e., the x-distances here *increase* as z increases;
while the z-distances *decrease* as x increases.

N.B.—If the ratio of the marginal utility of z to that of x varied only with variation of x, so that $W=f(x)$, and $\frac{dW}{dz}=0$; then, the vertical distances would be constant, or the utility curves would be parallel at the same value of x. This is the diagrammatic equivalent of the constancy ascribed to the marginal utility of *money* (z standing here for *money*).

Part III.

Money and Prices.

§ 13. Let μ be the amount of money spent on two factors x and z, whose prices are ξ and ζ respectively. Then

$$x\xi+z\zeta=\mu, \quad i.e. \ \frac{x}{\mu/\xi}+\frac{z}{\mu/\zeta}=1.$$

This straight line may be called the Price-Line.

Its intercept on the axis of x is μ/ξ : its intercept on the axis of z is μ/ζ. The former increases as ξ decreases, the latter as ζ decreases—μ being constant.

The maximum utility is where the price-line touches a utility curve; the inclination of the tangent at any point of the utility curve is given by the ratio of $\frac{du}{dx}$ to $\frac{du}{dz}$; and the inclination of the price-line is given by the ratio of ξ to ζ.

We, therefore, may construct the following three curves of Demand:—

(1) $\frac{du}{dx} \div \frac{du}{dz} = \frac{\xi}{\zeta}$ or $\frac{du}{dz} \div \frac{du}{dx} = \frac{\zeta}{\xi}$.

(2) $\frac{du}{dx} \div \frac{du}{dz} = \frac{\mu/\zeta - z}{x}$.

(3) $\frac{du}{dz} \div \frac{du}{dx} = \frac{\mu/\xi - x}{z}$.

The first curve is to be used, when ξ and ζ are constant, and μ varies. In this case, the price-lines are a series of parallels determined by the ratio ξ to ζ (see Fig. 5).

The second curve is to be used, when μ and ζ are constant, and ξ varies. In this case, the price-lines are a series of lines through the fixed point $x=0$, $z=\mu/\zeta$.

The third curve is to be used, when μ and ξ are constant, and ζ varies. In this case, the price-lines are a series of lines through the fixed point $z=0, x=\mu/\xi$ (see Fig. 6).

The curves (1), (2), (3) may be briefly written in accordance with our former notation:—

(1) $V=\xi/\zeta$ or $W=\zeta/\xi$.
(2) $Vx+z=\mu/\zeta$.
(3) $Wz+x=\mu/\xi$.

We shall begin by considering curve (1), where $V(=\xi/\zeta)$ is constant. This discussion is given in §§ 14, 15, 16, 17. Afterwards we shall deduce the characteristics of curves (2) and (3), and compare the results obtained with those for curve (1). This discussion occupies §§ 18, 19, 20.

§ 14. Whereas in the u-constant curve we have

$$-\left(\frac{dz}{dx}\right)_u = \frac{du}{dx} \div \frac{du}{dz} \equiv V,$$

we have in the V-constant curve

$$-\left(\frac{dz}{dx}\right)_, = \frac{dV}{dx} \div \frac{dV}{dz}.$$

Now, by the condition of convexity,

$$\frac{du}{dz} \cdot \frac{dV}{dx} < \frac{du}{dx} \cdot \frac{dV}{dz}.$$

In the mediate case, where $\frac{dV}{dx}$ is negative and $\frac{dV}{dz}$ is positive, we see that $\left(\frac{dz}{dx}\right)_{\xi,\zeta}$ is positive; *i.e.* along the ξ/ζ-constant curve, z and x increase together.

In the z-urgency case, where $\frac{dV}{dx}$ and $\frac{dV}{dz}$ are both negative, or in the x-urgency case, where $\frac{dV}{dx}$ and $\frac{dV}{dz}$ are both positive, $\left(\frac{dz}{dx}\right)_{\xi,\zeta}$ is negative; *i.e.* in both cases, x and z vary oppositely.

Now, we have previously shown that, where z is urgent, $\frac{dV}{dx} \div \frac{dV}{dz} > \frac{du}{dx} \div \frac{du}{dz}$ *i.e.*, the V-constant curve is *steeper* than the u-constant curve, at any point where z is urgent.

On the other hand, where x is urgent, $\frac{dV}{dx} \div \frac{dV}{dz} < \frac{du}{dx} \div \frac{du}{dz}$; *i.e.*, the V-constant curve is *flatter* than the u-constant curve at any point where x is urgent (see Fig. 5).

§ 15. We will now show how the urgency of z or of x determines the corresponding course of the V-constant curve.

We have
$$\frac{du}{dx}Dx + \frac{du}{dz}Dz \equiv Du,$$

i.e.,
$$\frac{du}{dx} \cdot \frac{Dx}{Du} + \frac{du}{dz} \cdot \frac{Dz}{Du} \equiv 1 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (\alpha)$$

where $\frac{Dx}{Du}$ and $\frac{Dz}{Du}$ are total differential coefficients, derived from any relation in which x and z are determined as functions of u.

Now, along the V-constant curve, we have

$$\frac{dV}{dx}Dx + \frac{dV}{dz}Dz = 0.$$

Thus, this curve combined with $u \equiv f(x, z)$, determines x as a function of u, and z as a function of u, such that

$$\frac{dV}{dx} \cdot \frac{Dx}{Du} + \frac{dV}{dz} \cdot \frac{Dz}{Du} = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (\beta)$$

Equations (α) and (β) determine the values of $\frac{Dx}{Du}$ and $\frac{Dz}{Du}$.

Thus

$$\left(\frac{du}{dx} \cdot \frac{dV}{dz} - \frac{du}{dz} \cdot \frac{dV}{dx}\right)\frac{Dx}{Du} = \frac{dV}{dz} \quad . \quad . \quad . \quad . \quad . \quad (\gamma)$$

$$\left(\frac{du}{dz} \cdot \frac{dV}{dx} - \frac{du}{dx} \cdot \frac{dV}{dz}\right)\frac{Dz}{Du} = \frac{dV}{dx} \quad . \quad . \quad . \quad . \quad . \quad (\delta)$$

Here the coefficient of $\frac{Dx}{Du}$ is positive, and that of $\frac{Dz}{Du}$ is negative. Hence,

$$\frac{Dx}{Du} \text{ has the sign of } \frac{dV}{dz},$$

and

$$\frac{Dz}{Du} \text{ has the sign of } -\frac{dV}{dx}.$$

Now, in the *mediate* case, $\frac{dV}{dz}$ and $-\frac{dV}{dx}$ are both *positive*; hence, here, both x and z increase with increase of utility.

In the *z-urgent* case, $\frac{dV}{dz}$ and $\frac{dV}{dx}$ are both *negative*; hence, here, x decreases and z increases with increase of utility.

In the *x-urgent* case, $\frac{dV}{dz}$ and $\frac{dV}{dx}$ are both *positive*; hence, here, x increases and z decreases with increase of utility. The results of this section solve the second of the principal problems (mentioned in § 4), for the case in which the changes in the price-line are due to changes in amount of money expended.

§ 16. The above results may be shown diagrammatically.

A glance at the curves [see Figs. 9 (*a*), (*b*), (*c*)] of varying convexities, corresponding to the three cases—mediate, z-urgent, x-urgent—will shew the course of the V-constant curve.

In the mediate case, the point in the curve $BACD$ where the tangent is parallel to Q will be to the left of C, and the point where the tangent is parallel to R will be below A.

Hence, the course of the V-curve from $BACD$ to QR will be upwards and rightwards: *i.e.*, an increase of utility will involve an increase both of x and of z.

In the z-urgent case, the point in the curve $BACD$ where the tangent is parallel to Q will be below and to the *right* of C, and the point where the tangent is parallel to R will be (far) below A.

Hence, the course of the V-curve from $BACD$ to QR will be upwards and *leftwards*: *i.e.*, an increase of utility will involve an increase (large) of z and a *decrease* of x.

In the x-urgent case, the point in the curve $BACD$ where the tangent is parallel to Q will be (far) to the left of C, and the point where the tangent is parallel to R will be *above* and to the left of A.

Hence, the course of the V-curve from $BACD$ to QR will be *downwards* and rightwards; *i.e.*, an increase of utility will involve an increase (large) of x and a *decrease* of z.

It is also useful to note that the V-curve rises more *sharply* than the u-curves where they meet, in the z-urgent case, and less *sharply* in the x-urgent case. [See § 14.] In other words, where the u-curves cut a V-curve, the V-curve is more nearly parallel to the axis of Z when z is urgent, and more nearly parallel to the axis of X when x is urgent.

§ 17. We may repeat the explanation that the V-constant curve is to be regarded as the curve of demand for x and z jointly, when the prices of x and z are fixed, and *the amount of money expended on x and z together* is allowed to vary.

The amount of money (μ) purchases the utility (u), and thus we may speak of the price (say π) of a unit of utility. We may, therefore, write

$$x\xi + z\zeta = \mu \equiv \pi u.$$

Let us introduce the symbol ϵ as a variable depending on the nature of the function $u = f(x, y)$, such that

$$x\frac{du}{dx} + z\frac{du}{dz} \equiv \epsilon u.$$

Then, $\dfrac{\xi}{\dfrac{du}{dx}} = \dfrac{\zeta}{\dfrac{du}{dz}} = \kappa$ (say) along the V-constant curve.

$$\therefore \ \mu \equiv \pi u = \kappa\epsilon u.$$

Now, along the V-constant curve,

$$\frac{du}{dx}Dx + \frac{du}{dz}Dz = Du,$$

and

$$\xi Dx + \zeta Dz = D\mu;$$

$$\therefore \ \frac{D\mu}{Du} = \kappa.$$

i.e., $\dfrac{1}{\kappa}$ measures the marginal utility of money.

And

$$\epsilon = \frac{\mu}{u}\frac{1}{\kappa} = \frac{\mu Du}{uD\mu}.$$

[N.B.—Expressions of the form $\dfrac{zdx}{xdz}$ are of very frequent occurrence in the analytical treatment of economics (and other sciences). This form of expression corresponds to the general notion of elasticity. When x and z are such as to increase or decrease together, and are both positive, so that $\frac{zdx}{xdz}$ is positive, the important variation in its value is according as it is greater, equal, or less than unity. It can easily be shewn that these three cases correspond, respectively, to the three cases according as $\frac{x}{z}$ increases, remains constant, or diminishes when x (or z) increases.]

Thus, ϵ may be regarded as measuring the elasticity of u in terms of money; *i.e.*, the rate at which utility increases proportionally to an increase in money.

This rate gives increasing, constant, or diminishing returns of utility, for money expended, according as ϵ is greater, equal or less than unity.

The result can obviously be extended to any number of factors purchased.

§ 18. *The Demand Curve for Variations in Price of x.*

The equation here is

$$Vx+z=\mu/\zeta.$$

where μ/ζ is constant.

This gives

$$\left(x\frac{dV}{dx}+V\right)+\left(\frac{dz}{dx}\right)_{\mu,\zeta}\left(x\frac{dV}{dz}+1\right)=0.$$

$$\therefore\ -\left(\frac{dz}{dx}\right)_{\mu,\zeta}=\frac{x\frac{dV}{dx}+V}{x\frac{dV}{dz}+1}.$$

$$\therefore\ \left(\frac{dV}{dx}\right)_{\mu,\zeta}\equiv\frac{dV}{dx}+\left(\frac{dz}{dx}\right)_{\mu,\zeta}\cdot\frac{dV}{dz}=\frac{\frac{dV}{dx}-V\frac{dV}{dz}}{x\frac{dV}{dz}+1}.$$

and

$$\left(\frac{dV}{dz}\right)_{\mu,\zeta}\equiv\frac{dV}{dz}+\left(\frac{dx}{dz}\right)_{\mu,\zeta}\cdot\frac{dV}{dx}=\frac{V\frac{dV}{dz}-\frac{dV}{dx}}{x\frac{dV}{dx}+V}.$$

§ 19. We will now show how the urgency of x or of z determines the course of the μ/ζ constant curve.

We have

$$\frac{du}{dx}\cdot\frac{Dx}{Du}+\frac{du}{dz}\cdot\frac{Dz}{Du}=1. \quad \ldots\ldots \quad (a)$$

where $\frac{Dx}{Du}$ and $\frac{Dz}{Du}$ have to satisfy the equation

$$\left(x\frac{dV}{dx}+V\right)\frac{Dx}{Du}+\left(x\frac{dV}{dz}+1\right)\frac{Dz}{Du}=0. \quad . \quad . \quad . \quad . \quad (\beta)$$

Equations (α) and (β) determine the values of $\frac{Dx}{Du}$ and $\frac{Dz}{Du}$.

Thus $$\left(\frac{du}{dx}\cdot\frac{dV}{dz}-\frac{du}{dz}\cdot\frac{dV}{dx}\right)\frac{Dx}{Du}=\frac{dV}{dz}+\frac{1}{x} \quad . \quad . \quad . \quad . \quad (\gamma)$$

$$\left(\frac{du}{dz}\cdot\frac{dV}{dx}-\frac{du}{dx}\cdot\frac{dV}{dz}\right)\frac{Dz}{Du}=\frac{dV}{dx}+\frac{V}{x} \quad . \quad . \quad . \quad . \quad (\delta)$$

Here the coefficient of $\frac{Dx}{Du}$ is positive, and that of $\frac{Dz}{Du}$ is negative.

Thus $\frac{Dx}{Du}$ is positive or negative, according as $\frac{dV}{dz} >$ or $< -\frac{1}{x}$,

and $\frac{Dz}{Du}$ is positive or negative, according as $\frac{dV}{dx} <$ or $> -\frac{V}{x}$.

In this way we learn where the curve moves upwards or downwards, rightwards or leftwards.

It should be noticed that, by aid of the formulæ in § 18, the relations (γ) and (δ) may be shortly written

$$\frac{Du}{Dx}=-x\frac{du}{dz}\cdot\left(\frac{dV}{dx}\right)_{\mu,\zeta} \text{ and } \frac{Du}{Dz}=-x\frac{du}{dz}\cdot\left(\frac{dV}{dz}\right)_{\mu,\zeta}.$$

§ 20. We may compare the results of § 19 with those of § 15; and thus discover the relations between the course of the μ/ζ-constant curve and the ξ/ζ-constant curve.

In the latter, Dx becomes negative when $\frac{dV}{dz}$ is negative, but in the former, Dx becomes negative only when $\frac{dV}{dz} < -\frac{1}{x}$. That is, a higher degree of relative urgency of z is required to lead to a diminution in the amount of x demanded, where merely the price of x falls, than when merely the joint expenditure on x and z increases. Thus, when $\frac{dV}{dz}$ lies between 0 and $-1/x$, the ξ/ζ-constant curve is moving *leftwards*, but the μ/ζ-curve still moves *rightwards*. When $\frac{dV}{dz} < -\frac{1}{x}$, both curves are moving leftwards, and when $\frac{dV}{dz} > 0$, both curves are moving rightwards.

Again, in the ξ/ζ-constant curve, Dz becomes negative only when $\frac{dV}{dx}$ is *positive*, but in the μ/ζ-constant curve, Dz is negative as soon as $\frac{dV}{dx} > -\frac{V}{x}$. That is, a lower degree of relative urgency of x is required to lead to a diminution in the amount of z demanded, when merely the price of x falls, than when merely the joint expenditure on x and z increases. Thus, when $\frac{dV}{dx}$ lies between 0 and $-\frac{V}{x}$, the ξ/ζ-constant curve is moving upwards, but the μ/ζ-constant curve is moving *downwards.* When $\frac{dV}{dx} > 0$, both curves are moving downwards, and when $\frac{dV}{dx} < -\frac{V}{x}$ both curves are moving upwards.

For instance, starting from $x=0$, in the curve $Vx+z=\mu/\zeta$, we have $z=\mu/\zeta$. This is obviously an absolute maximum for z, since V and x are positive. As V diminishes along this curve, correspondingly to the cheapening of x, the curve becomes finally asymptotic to the horizontal line $z=\mu/\zeta$. But it must begin by falling from its initial maximum height $z=\mu/\zeta$. In other words, when x is very small, and z very large, the demand for x must be very urgent relatively to that for z. Hence x begins by increasing at the expense of a rapid fall in z. If, however, the absolute need for x is soon gratified, and the need for z begins to be more felt, then z may reach its first minimum value when x is still small, after which x and z will continue to rise together. When x and z are rising together, $\frac{du}{dx}$ and $\frac{du}{dz}$ may both be presumed to be diminishing. But if, owing to a relative superabundance of x, $\frac{du}{dx}$ rapidly diminishes while $\frac{du}{dz}$ is only slowly diminishing, V would diminish very rapidly if x continued to increase. A *gradual* diminution of V might therefore be effected by decreasing x (and thus increasing $\frac{du}{dx}$).

PART IV.

§ 21. To find the maximum product obtained at a given expense (μ), when the prices of the factors are given.

Let $p \equiv f(a, b, c \ldots)$ (n-factors)

where $$a\alpha + b\beta + c\gamma + \ldots\ldots = \mu,$$

$$Dp = 0 \text{ gives } \frac{df}{da}Da + \frac{df}{db}Db + \frac{df}{dc}Dc + \ldots\ldots = 0,$$

$$D\mu = 0 \text{ gives } \alpha Da + \beta Db + \gamma Dc + \ldots\ldots = 0.$$

This gives

$$\left(\kappa\frac{df}{da}-\alpha\right)Da+\left(\kappa\frac{df}{db}-\beta\right)Db+\ \ldots\ldots=0,$$

for arbitrary increments Da, Db . . . .

Hence κ is determined by the n-equations,

$$\frac{\alpha}{df/da}=\frac{\beta}{df/db}=\ \ldots\ldots=\kappa.$$

These n-equations, together with

$$a\alpha+b\beta+\ \ldots\ldots=\mu,$$

determine the $(n+1)$ quantities κ, a, b, c. . . .

§ 22. Let f be such that

$$a\frac{dp}{da}+b\frac{dp}{db}+c\frac{dp}{dc}+\ \ldots\ldots=\epsilon p.$$

where, in general, ϵ is a variable function of a, b, c . . . . , and may be called the *elasticity* of production.

Then, from above,

$$\mu=a\alpha+b\beta+c\gamma+\ \ldots\ldots=\epsilon\kappa p$$

$$\therefore\ \kappa=\frac{\mu}{\epsilon p}.$$

Or, if π = cost of unit of p, so that $\mu\equiv\pi p, \kappa=\dfrac{\pi}{\epsilon}$.

Thus $\kappa \gtreqless \pi$, according as $\epsilon \lesseqgtr 1$.

§ 23. Now the price of an agent varies with his own "marginal efficiency" combined with his contribution to the general efficiency. Thus

$$\alpha=\frac{d\mu}{da}=\frac{d(\pi p)}{da}=\pi\frac{dp}{da}+p\frac{d\pi}{da}$$

$$=\frac{\pi}{\kappa}\alpha+p\frac{d\pi}{da}=\epsilon\alpha+p\frac{d\pi}{da},$$

$$\therefore\ \alpha(1-\epsilon)=p\frac{d\pi}{da}.$$

Thus we have two sets of equivalent equations

$$\frac{\alpha}{dp/da}=\frac{\beta}{dp/db}=\ \ldots\ldots=\frac{\pi}{\epsilon}\equiv\frac{\mu}{\epsilon p},$$

$$\frac{\alpha}{d\pi/da}=\frac{\beta}{d\pi/db}=\ \ldots\ldots=\frac{p}{1-\epsilon}\equiv\frac{\mu}{(1-\epsilon)\pi}.$$

§ 24. Eliminating a, b, c . . . . and k, the above give μ as a function of p, say $\mu=\chi(p)$.

Along this curve

$$Dp = \frac{df}{da}Da + \frac{df}{db}Db + \frac{df}{dc}Dc + \ldots .$$

$$D\mu = aDa + \beta Db + \gamma Dc \ldots .$$

$$\therefore \ \frac{D\mu}{Dp} \equiv \kappa = \frac{\mu}{\epsilon p},$$

$$\therefore \qquad \epsilon = \frac{\mu Dp}{pD\mu}.$$

Hence $\epsilon \gtreqless 1$, according as the expense of producing p involves what may be called increasing, constant, or diminishing efficiency of money; or according as

$$a\frac{dp}{da} + b\frac{dp}{db} + c\frac{dp}{dc} + \ldots . \gtreqless p$$

at the values of a, b, c . . . . for which the maximum production is determined.

§ 25. We must consider the general condition that the values obtained for a, b, c . . . . will give a true maximum—rather than a stationary or minimum value—for p. This will depend on the sign of the second differential of p; that is, upon

$$\frac{d^2p}{da^2}Da^2 + 2\frac{d^2p}{dadb}DaDb + \ldots .$$

where, since $a \,.\, Da + \beta \,.\, Db + \ldots . = 0$,

and a, β, &c., are proportional to $\frac{df}{da}$, $\frac{df}{db}$, &c.,

we must have accurately (*i.e.*, not merely to the first approximation) $\frac{dp}{da}Da + \frac{dp}{db}Db + \ldots . = 0$.

We make the following abbreviations:

$$\frac{dp}{da}Da = x_1, \ \frac{dp}{db}Db = x_2, \text{ \&c},$$

$$-\frac{d^2p}{da^2} \div \left(\frac{dp}{da}\right)^2 = \tau_{11}, \quad -\frac{d^2p}{dadb} \div \left(\frac{dp}{da} \cdot \frac{dp}{db}\right) = \tau_{12}, \text{ \&c.}$$

Then the required condition is that

$$\tau_{11}x_1{}^2 + 2\tau_{12}x_1x_2 + \ldots . \text{ should be } \textit{positive},$$

where $$x_1 + x_2 + \ldots . = 0.$$

Substituting $x_1 = -(x_2 + x_3 + \ldots .)$ in the quadratic, we obtain the quadratic in x_2, x_3, &c., of which the typical terms are $x_2^2(\tau_{11} + \tau_{22} - 2\tau_{12}) + 2x_2x_3(\tau_{11} + \tau_{23} - \tau_{12} - \tau_{13})$.

The discriminant so obtained can be shown to be that of which the successive minors are

$$\begin{vmatrix} 0 & 1 \\ -1 & \tau_{11} \end{vmatrix}, \quad \begin{vmatrix} 0 & 1 & 1 \\ -1 & \tau_{11} & \tau_{12} \\ -1 & \tau_{12} & \tau_{22} \end{vmatrix}, \text{ \&c.}$$

all of which must be *positive*. [N.B.—The *first* is identically positive.] Thus, there are $(n-1)$ conditions of sign, where n is the number of independent variables.

These conditions are equivalent to the statement that the "surfaces" $p \equiv f\,(a,\, b,\, c\, \ldots .)$ are in all directions *convex* to the co-ordinate axes. But there are several ways of indicating the economic significance of the result.

§ 26. We will begin by a consideration of the nature of the curve $\mu = \chi\,(p)$, which is analogous to a line of force cutting across the equipotential surfaces $p \equiv f\,(a,\, b,\, c\, \ldots .)$.

It is instructive to determine how much each of the factors $a,\, b,\, c\, \ldots .$ is increased or decreased when the expenditure μ is increased. That is, we must determine the values of the differentials $\frac{Da}{Dp}, \frac{Db}{Dp}$, &c., when μ and therefore p are increased. We shall replace $Da,\, Db\, \ldots .$ in the above by $\frac{Da}{Dp}, \frac{Db}{Dp} \ldots$ and adopt the same abbreviations, where now we have

$$\frac{dp}{da}Da + \frac{dp}{db}Db + \ldots . = Dp,$$

thus

$$x_1 + x_2 + \ldots . = 1,$$

and

$$\frac{\alpha}{dp/da} = \frac{\beta}{dp/db} = \ldots . = \kappa = \frac{D\mu}{Dp}.$$

Then, taking the total differential of the logarithm of these equations, we have

$$\log \kappa + \log \frac{dp}{da} = \log \alpha, \text{ \&c.,}$$

$$\therefore\ -\frac{1}{\kappa}\frac{D\kappa}{Dp} = \left(\frac{d^2p}{da^2} \div \frac{dp}{da}\right)\frac{Da}{Dp} + \left(\frac{d^2p}{da\,db} \div \frac{dp}{da}\right)\frac{Db}{Dp} + \ldots .$$

Writing $\frac{1}{\kappa}\frac{D\kappa}{Dp} = K$, the system of equations for x_1, x_2, *and* K become

$$\begin{aligned} 0 + x_1 + x_2 + x_3 + \ldots\ldots\ldots\ldots &= 1, \\ -K + \tau_{11}x_1 + \tau_{12}x_2 + \tau_{13}x_3 + \ldots . &= 0, \\ -K + \tau_{12}x_1 + \tau_{22}x_2 + \tau_{23}x_3 + \ldots . &= 0, \\ -K + \tau_{13}x_1 + \tau_{23}x_2 + \tau_{33}x_3 + \ldots . &= 0. \end{aligned}$$

The determinant here is that which has been shown to be (with its several minors) necessarily *positive*.

It may be written $|\,0123\,\ldots\,|$, corresponding to the unbordered determinant $|\,123\,\ldots\,|$.

Thus K is given by the equation

$$K \times |\,0123\,\ldots\,| = |\,123\,\ldots\,|.$$

or rather, we must indicate the several values of K by subscripts according to the factors that are to vary. Thus

$$K_1 \times \begin{vmatrix} 0 & 1 \\ -1 & \tau_{11} \end{vmatrix} = \tau_{11}, \text{ i.e., } K_1 = \tau_{11};$$

$$K_{12} \times \begin{vmatrix} 0 & 1 & 1 \\ -1 & \tau_{11} & \tau_{12} \\ -1 & \tau_{12} & \tau_{22} \end{vmatrix} = \begin{vmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{vmatrix}, \text{ \&c.}$$

The following relations can be proved,

$$K_1 - K_{12} = x_2^2 \cdot \frac{|\,012\,|}{|\,01\,|};$$

$$K_{12} - K_{123} = x_3^2 \cdot \frac{|\,0123\,|}{|\,012\,|};$$

$$K_{123} - K_{1234} = x_4^2 \cdot \frac{|\,01234\,|}{|\,0123\,|}.$$

where x_2, x_3, x_4 . . . . are the values of the last terms in the several solutions, for 2, 3, 4 . . . . variables.

Thus, it easily follows that the conditions for a *true* maximum are equivalent to the series of inequations—

$$K_1 > K_{12} > K_{123} > K_{1234}, \text{ \&c.}$$

Now $$K = \frac{1}{\kappa}\frac{D\kappa}{Dp} = \frac{D}{D\mu}\left(1 \div \frac{Dp}{D\mu}\right).$$

Thus K measures the rate at which the inverse of the "marginal efficiency" of money changes. K is not necessarily positive. But the above inequations are equivalent to the economic statement that "any increase of money tends less and less to diminish its 'marginal efficiency' as the variety of factors upon which it is expended increases."

§ 27. Another elementary account can be given of the significance of the various K values.

Returning to the second total differential, we have the quadratic expression, where $\Sigma x = 0$,

$$\tau_{11}x_1^2 + 2\tau_{12}x_1x_2 + \ldots.$$

Let the various values of x be equated to $\bar{x} - \xi$. Then

$$\begin{aligned} \tau_{11}x_1^2 + 2\tau_{12}x_1x_2 + \ldots \equiv\ & \tau_{11}\bar{x}_1^2 + 2\tau_{12}\bar{x}_1\bar{x}_2 + \ldots \\ & + \tau_{11}\xi_1^2 + 2\tau_{12}\xi_1\xi_2 + \ldots \\ & - 2\xi_1(\tau_{11}\bar{x}_1 + \tau_{12}\bar{x}_2 + \ldots) \\ & - 2\xi_2(\tau_{12}\bar{x}_1 + \tau_{22}\bar{x}_2 + \ldots) \\ & - \ldots \end{aligned}$$

Here $\bar{x}_1$, $\bar{x}_2$, &c., may be chosen arbitrarily, when $x_1, x_2 \ldots .$ have any given values.

Let us choose them to satisfy the equations

$$\begin{aligned} \bar{x}_1+\bar{x}_2+ \ldots . &= \Sigma\xi, \\ \tau_{11}\bar{x}_1+\tau_{12}\bar{x}_2+ \ldots . &= K \,.\, \Sigma\xi, \\ \tau_{12}\bar{x}_1+\tau_{22}\bar{x}_2+ \ldots . &= K \,.\, \Sigma\xi. \end{aligned}$$

Then, since $0=\Sigma x=\Sigma\bar{x}-\Sigma\xi$, $\therefore$ $\Sigma\bar{x}=\Sigma\xi$. Thus K must have the value $K_{123} \ldots .$ (obtained before). Multiplying successively by $\bar{x}_1, \bar{x}_2 \ldots .$ and adding, we have

$$\tau_{11}\bar{x}_1{}^2+2\tau_{12}\bar{x}_1\bar{x}_2+ \ldots . = K \,.\, \Sigma\xi \,.\, \Sigma\bar{x} = K \,.\, (\Sigma\xi)^2.$$

Thus

$$\begin{aligned} \tau_{11}x_1{}^2+2\tau_{12}x_1x_2+ \ldots . &= \\ &\tau_{11}\xi_1{}^2+2\tau_{12}\xi_1\xi_2+ \ldots . -K(\xi_1+\xi_2+ \ldots .)^2 \\ &=(\tau_{11}-K)\xi_1{}^2+2(\tau_{12}-K)\xi_1\xi_2+ \ldots . \end{aligned}$$

where ξ_1 $\xi_2 \ldots .$ are unrestricted. Hence another form of the condition for a true maximum is that the series of determinants obtained from $|123 \ldots .|$ by subtracting $K_{123} \ldots .$ from each constituent must be positive. [The last of these is identically zero.]

§ 28. We may give a diagrammatic representation of an important application of this result.

Let us choose (say) r of the quantities ξ_1, ξ_2, &c., so as to satisfy

$$\left.\begin{aligned} \xi_1+ \ldots . +\xi_r&=1 \\ -K_{1 \ldots r}+\tau_{11}\xi_1+ \ldots . +\tau_{1r}\xi_r&=0 \\ -K_{1 \ldots r}+\tau_{12}\xi_1+ \ldots . +\tau_{2r}\xi_r&=0 \end{aligned}\right\} (r+1) \text{ equations,}$$

and let ξ_{r+1}, &c., $=0$.

Then

$$\tau_{11}x_1{}^2+2\tau_{12}x_1x_2+ \ldots . = K_{12 \ldots r}-K_{12 \ldots n}.$$

In the diagram, we represent a tangent "plane" RQ at R to the production "surface."

Then, if P represents any previously obtained maximum position, the movement from P to Q will represent the maximisation obtained by using an extra increment of money on the factors corresponding to ζ, up to ξ_r; whereas the movement from P to R represents that from using the same amount of money on *all* the factors from ξ_1 to ξ_n.

The convexity of the production "surfaces" shows that the value for R is greater than that for Q; and the above equation shows that the increment of utility along the "tangent" plane, (where $\Sigma x = 0$) is measured by the second differential value $\tau_{11}x_1^2 + 2\tau_2 x_1 x_2 + \ldots.$, which must, therefore, be positive.

The difference of utility beween that at R and that at Q is the incrementally measured "*surplus*": *i.e.*, the excess value of

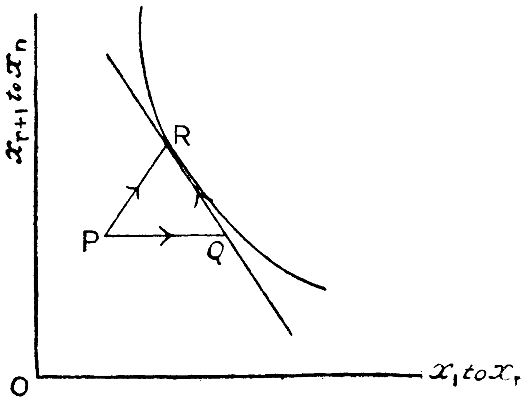


FIG. 10.

expenditure on ($\xi_{r+1} \ldots \xi_n$) over that of the same amount expended on ($\xi_1 \ldots \xi_r$).

On Some Special Forms of the Production Function.

§ 29. *Grouped Factors.*—The factors in production often fall into separate groups. In this case it is legitimate to maximise for one group at a time; and finally maximise for the whole. Or, we may have groups of groups which may be similarly treated.

The grouping of factors, here, means that the product is a function of functions of separate sets of factors.

The simplest case, for illustration, is that in which the product is a function of two functions of two factors each.

Thus, let $p \equiv f\{\psi(a, b), \chi(c, g)\}$,

Then
$$\frac{dp}{da} = \frac{df}{d\psi} \cdot \frac{d\psi}{da}; \quad \frac{dp}{db} = \frac{df}{d\psi} \cdot \frac{d\psi}{db};$$

$$\frac{dp}{dc} = \frac{df}{d\chi} \cdot \frac{d\chi}{dc}; \quad \frac{dp}{dg} = \frac{df}{d\chi} \cdot \frac{d\chi}{dg};$$

$$\therefore \frac{dp}{da} \div \frac{dp}{db} = \frac{d\psi}{da} \div \frac{d\psi}{db}; \quad \frac{dp}{dc} \div \frac{dp}{dg} = \frac{d\chi}{dc} \div \frac{d\chi}{dg};$$

$$\frac{d^2p}{dadc} = \frac{d^2f}{dad\chi} \cdot \frac{d\chi}{dc} = \frac{d^2f}{d\psi d\chi} \cdot \frac{d\psi}{da} \cdot \frac{d\chi}{dc};$$

$$\frac{\dfrac{d^2p}{dadc}}{\dfrac{dp}{da} \dfrac{dp}{dc}} = \frac{\dfrac{d^2f}{d\psi d\chi}}{\dfrac{df}{d\psi} \cdot \dfrac{df}{d\chi}}.$$

This shows that the values of τ which belong to factors in separated groups (such as a and c) are equal to one another, so that we can write

$$\tau_{ac} = \tau_{bc} = \tau_{ag} = \tau_{bg} \equiv \tau_{\psi\chi} \text{ (say).}$$

Thus we may maximise separately for ψ and χ.

For $$\frac{1}{\alpha}\frac{dp}{da} = \frac{1}{\beta}\frac{dp}{db} \text{ becomes } \frac{1}{\alpha}\frac{d\psi}{da} = \frac{1}{\beta}\frac{d\psi}{db},$$

giving values depending only on the form of ψ, independently of f.

A special case of the grouping of factors, which is sometimes assumed, is that in which the function f is a simple summation; *i.e.*,

$$p \equiv \psi(a, b) + \chi(c, g.)$$

In this case we may speak of the separated groups as being independent of one another. This assumption is often permissible when p stands for the *utility* of consumption. Here, of course, the second differentials of p connecting factors in distinct groups vanish; *i.e.*, $\tau_{\psi\chi} = 0$.

In the simple case $p \equiv f\{\psi(a, b), \chi(c, g)\}$ we can represent all the analytical work by diagrams in two dimensions. Thus, taking a, b as axes, we can maximise for ψ (a, b); and represent ψ by x. Then, taking c, g as axes, we can maximise for χ (c, g), and represent χ by z. And finally taking x and z as axes, we can maximise for f (x, z).

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