# 12.

# De integralibus quibusdam definitis et seriebus infinitis.

(Auctore *E. E. Kummer*, Dr. phil.)

Integralia definita, quae nunc tractare mihi proposui, arctissime conjuncta sunt cum seriebus infinitis, de quibus egi in commentatione hujus diarii de serie hypergeometrica, Tom. XV. pag. 138 sq. quas, ut faciliori modo repraesentari possint, his signis functionalibus designabo:

$$1.\quad 1+\frac{\alpha.x}{\beta.1}+\frac{\alpha(\alpha+1).x^2}{\beta(\beta+1).1.2}+\frac{\alpha(\alpha+1)(\alpha+2).x^3}{\beta(\beta+1)(\beta+2).1.2.3}+\dots=\varphi(\alpha,\beta,x),$$

$$2.\quad 1+\frac{x}{\alpha.1}+\frac{x^2}{\alpha(\alpha+1).1.2}+\frac{x^3}{\alpha(\alpha+1)(\alpha+2).1.2.3}+\dots=\psi(\alpha,x),$$

$$3.\quad 1-\frac{\alpha.\beta}{1.x}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.x^2}-\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.x^3}+\dots=\chi(\alpha,\beta,x).$$

Inde earum serierum transformationes loco citato inventae hoc modo exhiberi possunt:

$$4.\quad \varphi(\alpha,\beta,x)=e^x.\varphi(\beta-\alpha,\beta,-x),$$

$$5.\quad \psi(\alpha,x)=e^{\pm 2\sqrt{x}}\varphi(\alpha-\tfrac{1}{2},2\alpha-1,\pm 4\sqrt{x}),$$

quae formula eadem est ac

$$6.\quad \varphi(\alpha,2\alpha,x)=e^{\frac{x}{2}}\psi\left(\alpha+\frac{1}{2},\frac{x^2}{16}\right)$$

et

$$7.\quad \chi(\alpha,\beta,x)=\frac{x^\alpha\Pi(\beta-\alpha-1)}{\Pi(\beta-1)}\varphi(\alpha,\alpha-\beta+1,x)+\frac{x^\beta\Pi(\alpha-\beta-1)}{\Pi(\alpha-1)}\varphi(\beta,\beta-\alpha+1,x).$$

Quibus praeparatis primum quaestionem instituam de integrali

$$8.\quad y=\int_0^\infty u^{\alpha-1}.e^{-u}.e^{-\frac{x}{u}}du,$$

ex quo sequitur

$$\frac{dy}{dx}=-\int_0^\infty u^{\alpha-2}.e^{-u}.e^{-\frac{x}{u}}du,\qquad \frac{d^2y}{dx^2}=\int_0^\infty u^{\alpha-3}.e^{-u}.e^{-\frac{x}{u}}du,$$

per differentiationem quantitatis $u^{\alpha-1}.e^{-u}.e^{-\frac{u}{x}}$ est:

$$d\left(u^{\alpha-1}.e^{-u}.e^{-\frac{x}{u}}\right)$$

$$=-u^{\alpha-1}.e^{-u}.e^{-\frac{x}{u}}du+(\alpha-1)u^{\alpha-2}.e^{-u}.e^{-\frac{x}{u}}du+x.u^{\alpha-3}.e^{-u}.e^{-\frac{x}{u}}du,$$

et per integrationem intra limites 0 et $\infty$

$$0 = -\int_0^\infty u^{\alpha-1}.e^{-u}.e^{-\frac{x}{u}}du + (\alpha-1)\int_0^\infty u^{\alpha-2}.e^{-u}.e^{-\frac{x}{u}}du + x\int_0^\infty u^{\alpha-3}.e^{-u}.e^{-\frac{x}{u}}du,$$

sive quod idem est

$$9. \quad 0 = y + (\alpha-1)\frac{dy}{dx} - x\frac{d^2y}{dx^2},$$

Aequationis hujus integrale completum per series, quas signo functionali $\psi$ designavimus, facile invenitur

$$10. \quad y = A.\psi(1-\alpha, x) + B.x^\alpha.\psi(1+\alpha, x),$$

ubi $A$ et $B$ sunt constantes arbitrariae. Inde sequitur integralis propositi expressio haec

$$\int_0^\infty u^{\alpha-1}.e^{-u}.e^{-\frac{x}{u}}du = A.\psi(1-\alpha, x) + B.x^\alpha.\psi(1+\alpha, x).$$

Constantis $A$ determinatio facilis est; nam si quantitatem $\alpha$ positivam accipimus, et ponimus $x = 0$, habemus

$$\int_0^\infty u^{\alpha-1}.e^{-u}\,du = A$$

sive

$$A = \Pi(\alpha-1).$$

Ut eodemmodo constans $B$ determinari possit, integrale $y$ per substitutionem $u = \frac{x}{v}$ transformari debet, unde fit

$$\int_0^\infty u^{\alpha-1}.e^{-u}.e^{-\frac{x}{u}}du = x^\alpha\int_0^\infty v^{-\alpha-1}.e^{-v}.e^{-\frac{x}{v}}dv,$$

hac integralis transformatione adhibita aequatio (11.) transit in hanc:

$$\int_0^\infty v^{-\alpha-1}.e^{-v}.e^{-\frac{x}{v}}dv = A.x^{-\alpha}\psi(1-\alpha, x) + B.\psi(1+\alpha, x),$$

inde, si quantitatem $\alpha$ negativam accipimus et ponimus $x = 0$, habemus

$$\int_0^\infty v^{-\alpha-1}e^{-v}\,dv = B$$

sive

$$B = \Pi(-\alpha-1),$$

quibus denique constantium valoribus substitutis est:

$$12. \quad \int_0^\infty u^{\alpha-1}.e^{-u}.e^{-\frac{x}{u}}du = \Pi(\alpha-1)\psi(1-\alpha, x) + \Pi(-\alpha-1)x^\alpha\psi(1+\alpha, x).$$

Ab hac constantium determinatione dubia quaedam removenda sunt, quae inde oriri possint, quod constans altera inventa est posito $\alpha > 0$, alterius vero constantis determinatio hypothesin contrariam poscit. Attamen apparet eas conditiones superfluas fuisse, si in constantibus determinandis non valore $x=0$, sed aliis quibuscunque valoribus positivis usi essemus, neque alios inde constantium valores exstitisse. Praeterea monendum est formulam (12.) non valere nisi $x$ sit quantitas positiva, alioqui integrale illud infinitum evaderet; si vero $x$ est positivum hoc integrale valorem finitum habet, quaecunque sit quantitas $\alpha$, positiva seu negativa.

Ex hac formula (12.) aliud integrale deduci potest, quod per series duas formae $\Phi(\alpha,\beta,x)$ exprimitur. Ponendo $xv$ loco $x$, multiplicando per $e^{-v}.v^{\beta-1}.dv$ et integrando intra limites 0 et $\infty$, est

$$\int_0^\infty\int_0^\infty u^{\alpha-1}.e^{-u}.v^{\beta-1}.e^{-v}.e^{-\frac{xv}{u}}du\,dv = \Pi(\alpha-1)\int_0^\infty v^{\beta-1}.e^{-v}.\psi(1-\alpha,xv)dv$$
$$+\Pi(-\alpha-1)x^\alpha\int_0^\infty v^{\alpha+\beta-1}.e^{-v}\psi(1+\alpha,xv)dv,$$

integrationes secundum variabilem $v$ facile peraguntur; est enim

$$\int_0^\infty v^{\beta-1}e^{-v}\psi(1-\alpha,xv)dv = \Pi(\beta-1)\Phi(\beta,1-\alpha,x),$$
$$\int_0^\infty v^{\alpha+\beta-1}.e^{-v}\psi(1+\alpha,xv)dv = \Pi(\alpha+\beta-1)\Phi(\alpha+\beta,1+\alpha,x),$$
$$\int_0^\infty v^{\beta-1}.e^{-u}.e^{-\frac{xv}{u}}dv = \frac{\Pi(\beta-1)}{\left(1+\frac{x}{u}\right)^\beta}$$

unde

$$\int_0^\infty\int_0^\infty u^{\alpha-1}.e^{-u}.v^{\beta-1}.e^{-v}.e^{-\frac{xv}{u}}du\,dv = \Pi(\beta-1)\int_0^\infty \frac{u^{\alpha-1}e^{-u}du}{\left(1+\frac{x}{u}\right)^\beta},$$

quod integrale ponendo $ux$ loco $u$ mutatur in

$$\Pi(\beta-1)\,x^\alpha\int_0^\infty \frac{u^{\alpha+\beta-1}.e^{-ux}du}{(1+u)^\beta},$$

quibus denique substitutis habemus

$$\Pi(\beta-1)\,x^\alpha\int_0^\infty \frac{u^{\alpha+\beta-1}.e^{-ux}du}{(1+u)^\beta}$$
$$=\Pi(\alpha-1)\Pi(\beta-1)\Phi(\beta,1-\alpha,x)+\Pi(-\alpha-1)\Pi(\alpha+\beta-1)x^\alpha\Phi(\alpha+\beta,1+\alpha,x),$$

quae formula, mutando $\alpha$ in $\alpha-\beta$, in hanc formam commodiorem redigitur

$$13.\quad \frac{x^\alpha}{\Pi(\alpha-1)}\int_0^\infty \frac{u^{\alpha-1}.e^{-ux}.du}{(1+u)^\beta}$$
$$=\frac{\Pi(\alpha-\beta-1)}{\Pi(\alpha-1)}x^\beta.\Phi(\beta,\beta-\alpha+1,x)+\frac{\Pi(\beta-\alpha-1)}{\Pi(\beta-1)}x^\alpha.\Phi(\alpha,\alpha-\beta+1,x).$$

Quia aequationis hujus altera pars, quantitatibus $\alpha$ et $\beta$ inter se permutatis, eadem manet, esse debet

$$14.\quad \frac{x^\alpha}{\Pi(\alpha-1)}\int_0^\infty \frac{u^{\alpha-1}.e^{-ux}.du}{(1+u)^\beta} = \frac{x^\beta}{\Pi(\beta-1)}\int_0^\infty \frac{u^{\beta-1}.e^{-ux}du}{(1+u)^\alpha}.$$

Si ad formulam (13.) transformatio applicatur, quam aequatio (7.) continet, est

$$15.\quad \frac{x^\alpha}{\Pi(\alpha-1)}\int_0^\infty \frac{u^{\alpha-1}.e^{-ux}.du}{(1+u)^\beta} = \chi(\alpha,\beta,x).$$

Quum series $\chi(\alpha,\beta,x)$ ad classem serierum semiconvergentium pertineat, necessarium videtur formulam (15.) demonstratione singulari munire, e qua simul prodeat, per computationem numeri certi terminorum primorum hujus seriei valorem proximum integralis hujus inveniri. Quem ad finem adhibeo aequationem cognitam

$$1-\frac{\beta}{1}z+\frac{\beta(\beta+1)}{1.2}z^2-\dots.(-1)^{k-1}\frac{\beta(\beta+1)\dots.(\beta+k-2)}{1.2\dots.(k-1)}z^{k-1}$$
$$=\frac{1}{(1+z)^\beta}-\frac{(-1)^k\beta(\beta+1)\dots.(\beta+k-1)}{1.2.3\dots.k}z^k\int_0^1\frac{(1-u)^{k-1}du}{(1+zu)^{\beta+k}},$$

ponendo $z=\frac{v}{x}$, multiplicando per $v^{\alpha-1}.e^{-v}.dv$ tum integrando ab $v=0$ usque ad $v=\infty$ et dividendo par $\Pi(\alpha-1)$ fit

$$16.\quad 1-\frac{\alpha.\beta}{1.x}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.x^2}-\dots.(-1)^{k-1}\frac{\alpha(\alpha+1)\dots.(\alpha+k-2)\beta(\beta+1)\dots.(\beta+k-2)}{1.2.3\dots.(k-1).x^{k-1}}$$
$$=\frac{1}{\Pi(\alpha-1)}\int_0^\infty\frac{v^{\alpha-1}.e^{-v}.dv}{\left(1+\frac{v}{x}\right)^\beta}-\frac{(-1)^k\beta(\beta+1)\dots.(\beta+k-1)}{\Pi(\alpha-1)1.2.3\dots.(k-1)x^k}\int_0^1\int_0^\infty\frac{(1-u)^{k-1}.v^{\alpha+k-1}.e^{-v}.dv.du}{\left(1+\frac{uv}{x}\right)^{\beta+k}},$$

hoc integrale duplex cum coefficiente suo errorem indicat, qui committitur si integrale

$$\frac{1}{\Pi(\alpha-1)}\int_0^\infty\frac{v^{\alpha-1}.e^{-v}.dv}{\left(1+\frac{v}{x}\right)^\beta},\ \text{sive quod idem est,}\ \frac{x^\alpha}{\Pi(\alpha-1)}\int_0^\infty\frac{v^{\alpha-1}.e^{-vx}.dv}{(1+v)^\beta}$$

per seriei illius terminos primos, quorum numerus est $k$, computatur. Si $k$ tam magnum est ut $\beta+k$ sit positivum illa quantitas, quam erroris nomine designavimus, signum mutat simulac $k$ transit in $k+1$, sive, si seriei illius terminorum certus numerus computatur, haec summa aut major est aut minor quam integrale quaesitum, si vero terminus subsequens seriei adjicitur, haec nova summa est minor quam integrale quaesitum, si illa major erat, et major est si illa minor erat. Itaque summae, quas series illa praebet alternatim sunt nimis magnae et nimis parvae, atque elucet valorem proximum inveri, si computatio usque ad terminos minimos seriei semiconvergentis extendatur. Eadem res ex aequatione (16.) hoc modo demonstrari potest. Manifesto pro positivo $\beta+k$ est:

$$\int_0^1\int_0^\infty \frac{(1-u)^{k-1}.\,v^{\alpha+k-1}\,e^{-v}\,dv\,du}{\left(1+\frac{uv}{x}\right)^{\beta+k}} < \int_0^1\int_0^\infty (1-u)^{k-1}.v^{\alpha-k-1}.e^{-v}.\,dv\,du$$

et

$$\int_0^1\int_0^\infty (1-u)^{k-1}.\,e^{-v}.\,v^{\alpha+k-1}\,dv\,du = \frac{\Pi(\alpha+k-1)}{k},$$

ergo error, qui per integrale illud duplex exprimitur, semper minor est quam

$$\frac{\beta(\beta+1)\ldots.(\beta+k-1)\,\Pi(\alpha+k-1)}{1.2.3\ldots.k.\Pi(\alpha-1)\,x^k},$$

qui cum sit terminus primus neglectus, sequitur errorem semper minorem esse quam eum terminum seriei, usque ad quem summatio extendatur.

Posito $\beta=1-\alpha$ aequatio (15.) transit in hanc:

$$\frac{x^\alpha}{\Pi(\alpha-1)}\int_0^\infty (u+u^2)^{\alpha-1}.\,e^{-ux}.\,du$$

$$=\frac{\Pi(2\alpha-2)}{\Pi(\alpha-1)}\,x^{1-\alpha}.\,e^{\frac{x}{2}}.\,\Phi(1-\alpha,2-2\alpha,x)+\frac{\Pi(-2\alpha)}{\Pi(-\alpha)}\,x^\alpha.\,\Phi(\alpha,2\alpha,x),$$

quibus seriebus secundum formulam (6.) transformatis, est

$$\frac{x^\alpha}{\Pi(\alpha-1)}\int_0^\infty (u+u^2)^{\alpha-1}.\,e^{-ux}\,du$$

$$=\frac{\Pi(2\alpha-2)}{\Pi(\alpha-1)}\,x^{1-\alpha}.\,e^{\frac{x}{2}}.\,\psi\left(\tfrac{3}{2}-\alpha,\frac{x^2}{16}\right)+\frac{\Pi(-2\alpha)}{\Pi(-\alpha)}\,x^\alpha.\,e^{\frac{x}{2}}.\,\psi\left(\tfrac{1}{2}+\alpha,\frac{x^2}{16}\right)$$

porro si $x$ mutatur in $4\sqrt{x}$, $\alpha$ in $\alpha+\frac{1}{2}$, per reductiones paucas habemus

$$17.\quad \frac{2^{2\alpha+1}.\sqrt{\pi}.\,x^\alpha.\,e^{-2\sqrt{x}}}{\Pi(\alpha-\frac{1}{2})}\int_0^\infty (u+u^2)^{\alpha-\frac{1}{2}}.\,e^{-4u\sqrt{x}}.\,du$$

$$=\Pi(\alpha-1)\,\psi(1-\alpha,x)+\Pi(-\alpha-1)x^\alpha\,\psi(1+\alpha,x),$$

inde per comparationem cum formula (12.) sequitur

$$\int_0^\infty u^{\alpha-1}.\,e^{-u}.\,e^{-\frac{x}{u}}.\,du = \frac{2^{2\alpha+1}.\sqrt{\pi}.\,x^\alpha.\,e^{-2\sqrt{x}}}{\Pi(\alpha-\frac{1}{2})}\int_0^\infty (u+u^2)^{\alpha-\frac{1}{2}}.\,e^{-4u\sqrt{x}}.\,du,$$

ex hac formula, aut si mavis e formula (12.), posito $\alpha=\frac{1}{2}$, facile deducitur valor persimplex integralis

$$18.\quad \int_0^\infty e^{-u^2}.\,e^{-\frac{x}{u^2}}.\,du = \frac{\sqrt{\pi}}{2}.\,e^{-2\sqrt{x}}.$$

Integralia, quae modo invenimus, applicationes multas habent in analysi, ex. gr. in integranda aequatione Riccatiana, quae per substitutiones faciles in formam aequationis (9.) mutari potest; in iis autem non immorabor, sed de aliis etiam integralibus similibus quaestionem instituam, quorum primum accipio hoc:

$$19.\quad z=\int_0^{\frac{\pi}{2}} \cos v^{\alpha-1}.\cos(\tfrac{1}{2}\,x\,\mathrm{tang}\,v+\beta v)\,dv.$$

Quantitatem $x$ semper positivam accipio, quum ejus signum negativum in quantitatem $\beta$ transferri possit. Per differentiationem quantitatis

$$\cos v^{\alpha-1} . \sin(\tfrac{1}{2} x \operatorname{tang} v + \beta v)$$

est

$$d(\cos v^{\alpha-1} \sin(\tfrac{1}{2} x \operatorname{tang} v + \beta v)) = -(\alpha-1) \cos v^{\alpha-2} . \sin v . \sin(\tfrac{1}{2} x \operatorname{tang} v + \beta v)\, dv$$
$$+ \left(\frac{x}{2\cos v^2} + \beta\right) \cos v^{\alpha-1} \cos(\tfrac{1}{2} x \operatorname{tang} v + \beta v)\, dv,$$

et integrando intra limites $v = 0$ et $v = \frac{\pi}{2}$

$$20. \quad 0 = -(\alpha-1) \int_0^{\frac{\pi}{2}} \cos v^{\alpha-2} . \sin v . \sin(\tfrac{1}{2} x \operatorname{tang} v + \beta v)\, dv$$
$$+ \frac{x}{2} \int_0^{\frac{\pi}{2}} \cos v^{\alpha-3} . \cos(\tfrac{1}{2} x \operatorname{tang} v + \beta v)\, dv$$
$$+ \beta \int_0^{\frac{\pi}{2}} \cos v^{\alpha-1} . \cos(\tfrac{1}{2} x \operatorname{tang} v + \beta v)\, dv,$$

porro est

$$\frac{dz}{dx} = -\tfrac{1}{2} \int_0^{\frac{\pi}{2}} \cos v^{\alpha-2} . \sin v . \sin(\tfrac{1}{2} x \operatorname{tang} v + \beta v)\, dv,$$
$$\frac{d^2 z}{dx^2} = -\tfrac{1}{4} \int_0^{\frac{\pi}{2}} \cos v^{\alpha-3} . \sin v^2 . \cos(\tfrac{1}{2} x \operatorname{tang} v + \beta v)\, dv,$$

itaque

$$z - 4 \frac{d^2 z}{dx^2} = \int_0^{\frac{\pi}{2}} \cos v^{\alpha-3} \cos(\tfrac{1}{2} x \operatorname{tang} v + \beta v)\, dv,$$

quibus substitutis aequatio (30.) transit in hanc

$$21. \quad 0 = (x + 2\beta) z + 4(\alpha-1) \frac{dz}{dx} - 4x \frac{d^2 z}{dx^2},$$

haec aequatio per substitutionem $z = e^{-\frac{x}{2}} y$ transformatur in hanc

$$0 = \frac{\beta-\alpha+1}{2} y + (\alpha - 1 + x) \frac{dy}{dx} - x \frac{d^2 y}{dx^2},$$

cujus integrale completum est:

$$y = A \varphi\left(\frac{\beta-\alpha+1}{2}, 1-\alpha, x\right) + B x^\alpha \varphi\left(\frac{\beta+\alpha+1}{2}, 1+\alpha, x\right),$$

et quia $z = e^{-\frac{x}{2}} . y$, est

$$22. \quad \int_0^{\frac{\pi}{2}} \cos v^{\alpha-1} . \cos(\tfrac{1}{2} x \operatorname{tang} v + \beta v)\, dv$$
$$= A . \varphi\left(\frac{\beta-\alpha+1}{2}, 1-\alpha, x\right) + B x^\alpha \varphi\left(\frac{\beta+\alpha+1}{2}, 1+\alpha, x\right).$$

Constantis $A$ determinatio facile obtinetur ponendo $x=\infty$ si $\alpha$ est quantitas positiva, alterius vero constantis determinatio artificia peculiaria poscit; utramque simul constantem obtinebimus hac methodo. Aequatio (22.) multiplicetur per $x^{\lambda-1}e^{-\frac{x}{2}}dx$ et integretur intra limites $x=0$ et $x=\infty$, quo facto est

$$23.\quad \int_0^\infty\int_0^{\frac{\pi}{2}}\cos v^{\alpha-1}.x^{\lambda-1}e^{-\frac{x}{2}}\cos(\tfrac{1}{2}x\operatorname{tang}v+\beta v)\,dv\,dx$$
$$=A\int_0^\infty x^{\lambda-1}.e^{-x}\varphi\left(\frac{\beta-\alpha+1}{2},1-\alpha,x\right)dx$$
$$+B\int_0^\infty x^{\lambda+\alpha-1}e^{-x}\varphi\left(\frac{\beta+\alpha+1}{2},1+\alpha,x\right)dx.$$

Omnium horum integralium valores per functiones notas exprimi possunt, est enim

$$\int_0^\infty x^{c-1}.e^{-x}.\varphi(a,b,x)\,dx=\Pi(c-1)\,F(c,a,b,1),$$

ubi $F$ designat notam seriem hypergeometricam, qua per functionem $\Pi$ expressa est

$$\int_0^\infty x^{c-1}.e^{-x}\varphi(a,b,x)\,dx=\frac{\Pi(c-1)\,\Pi(b-1)\,\Pi(b-a-c-1)}{\Pi(b-a-1)\,\Pi(b-c-1)},$$

porro est

$$\int_0^\infty x^{\lambda-1}.e^{-\frac{x}{2}}.\cos(\tfrac{1}{2}x\operatorname{tang}v+\beta v)\,dx=2^\lambda\Pi(\lambda-1)\cos v^\lambda.\cos(\lambda+\beta)v,$$

unde illud integrale duplex transit in hoc

$$2^\lambda\Pi(\lambda-1)\int_0^{\frac{\pi}{2}}\cos v^{\alpha+\lambda-1}.\cos(\lambda+\beta)v.dv,$$

cujus valor per functionem $\Pi$ hoc modo exprimitur

$$\frac{\pi.\Pi(\lambda-1)\,\Pi(\alpha+\lambda-1)}{2^\alpha\Pi\left(\frac{\alpha-\beta-1}{2}\right)\Pi\left(\frac{\alpha+\beta-1}{2}+\lambda\right)},$$

quibus substitutis aequatio (23.) transit in hanc:

$$\frac{\pi.\Pi(\lambda-1)\,\Pi(\alpha+\lambda-1)}{2^\alpha\Pi\left(\frac{\alpha-\beta-1}{2}\right)\Pi\left(\frac{\alpha+\beta-1}{2}+\lambda\right)}$$
$$=A\frac{\Pi(\lambda-1)\,\Pi(-\alpha)\,\Pi\left(-\frac{\alpha+\beta+1}{2}-\lambda\right)}{\Pi\left(-\frac{\alpha+\beta+1}{2}\right)\Pi(-\alpha-\lambda)}+B\frac{\Pi(\alpha+\lambda-1)\,\Pi(\alpha)\,\Pi\left(-\frac{\alpha+\beta+1}{2}-\lambda\right)}{\Pi\left(\frac{\alpha-\beta-1}{2}\right)\Pi(-\lambda)},$$

haec aequatio facile reducitur ad hanc formam commodiorem

$$\frac{\pi.\cos\left(\frac{\alpha+\beta}{2}+\lambda\right)\pi}{2^\alpha\,\Pi\left(\frac{\alpha-\beta-1}{2}\right)} = \frac{A.\Pi(-\alpha)\sin(\alpha+\lambda)\pi}{\Pi\left(-\frac{\alpha+\beta+1}{2}\right)} + \frac{B.\Pi(\alpha)\sin\lambda\pi}{\Pi\left(\frac{\alpha-\beta-1}{2}\right)},$$

quae, quum pro quolibet valore quantitatis $\lambda$ locum habere debeat, in has duas dilabitur

$$\frac{\pi\cos\frac{\alpha+\beta}{2}\pi}{2^\alpha\,\Pi\left(\frac{\alpha-\beta-1}{2}\right)} = \frac{A.\sin(\alpha\pi)\,\Pi(-\alpha)}{\Pi\left(-\frac{\alpha+\beta+1}{2}\right)},$$

$$-\frac{\pi\sin\frac{\alpha+\beta}{2}\pi}{2^\alpha\,\Pi\left(\frac{\alpha-\beta-1}{2}\right)} = \frac{A.\cos\alpha\pi.\Pi(-\alpha)}{\Pi\left(-\frac{\alpha+\beta+1}{2}\right)} + \frac{B.\Pi(\alpha)}{\Pi\left(\frac{\alpha-\beta-1}{2}\right)},$$

e quibus facile inveniuntur constantium $A$ et $B$ valores

$$A = \frac{\pi.\Pi(\alpha-1)}{2^\alpha\,\Pi\left(\frac{\alpha-\beta-1}{2}\right)\Pi\left(\frac{\alpha+\beta-1}{2}\right)}, \qquad B = -\frac{\pi.\cos\left(\frac{\alpha-\beta}{2}\right)\pi}{2^\alpha.\sin\alpha\pi\,\Pi(\alpha)},$$

quibus denique constantium valoribus in aequatione (22.) substitutis, est

$$24.\quad \int_0^{\frac{\pi}{2}} \cos v^{\alpha-1}.\cos(\tfrac{1}{2}x\,\mathrm{tang}\,v+\beta v)\,dv$$

$$\frac{\pi.\Pi(\alpha-1)e^{-\frac{x}{2}}.\varphi\left(\frac{\beta-\alpha+1}{2},1-\alpha,x\right)}{2^\alpha\,\Pi\left(\frac{\alpha-\beta-1}{2}\right)\Pi\left(\frac{\alpha+\beta-1}{2}\right)} - \frac{\pi.\cos\frac{\alpha-\beta}{2}\pi.x^\alpha.e^{-\frac{x}{2}}\varphi\left(\frac{\beta+\alpha+1}{2},1+\alpha,x\right)}{2^\alpha\sin\alpha\pi\,\Pi(\alpha)}.$$

Hujus formulae casus speciales persimplices sunt:

$$25.\quad \int_0^{\frac{\pi}{2}} \cos v^{\alpha-1}.\cos(x\,\mathrm{tang}\,v-(\alpha+1)v)\,dv = \frac{\pi.x^\alpha.e^{-x}}{\Pi(\alpha)},$$

$$26.\quad \int_0^{\frac{\pi}{2}} \cos v^{\alpha-1}.\cos(x\,\mathrm{tang}\,v+(\alpha+1)v)\,dv = 0,$$

quorum alter obtinetur posito $\beta=-\alpha-1$, alter posito $\beta=\alpha+1$. E conjunctis formulis (25.) et (26.) sequuntur etiam hae

$$27.\quad \int_0^{\frac{\pi}{2}} \cos v^{\alpha-1}.\cos(x\,\mathrm{tang}\,v)\cos(\alpha+1)v.dv = \frac{\pi.x^\alpha.e^{-x}}{2\Pi(\alpha)},$$

$$28.\quad \int_0^{\frac{\pi}{2}} \cos v^{\alpha-1}.\sin(x\,\mathrm{tang}\,v)\sin(\alpha+1)v.dv = \frac{\pi.x^\alpha.e^{-x}}{2\Pi(\alpha)}.$$

Formulae (25.) et (26.) cum formula ab Ill. *Laplace* inventa consentiunt, quam postea alii aliis modis demonstrarunt, cfr. huj. diarii tom. XIII. p. 231, ubi Cl. *Liouville* per mothodum differentiationis ad indices qualescunque invenit

$$\int_{-\infty}^{+\infty} \frac{e^{\alpha\sqrt{-1}}.\, d\alpha}{(x+\alpha\sqrt{-1})^{\mu}} = \frac{2\pi.\, e^{-x}}{\Gamma(\mu)}.$$

Persimplex aliud integrale praebet formula (24.) posito $\beta = \alpha - 1$

$$29. \quad \int_0^{\frac{\pi}{2}} \cos v^{\alpha-1}.\cos(x \operatorname{tang} v + (\alpha-1)v)\, dv = \frac{\pi e^{-x}}{2^{\alpha}}.$$

Series duae, quae in altera parte aequationis (24.) insunt, posito $\beta = 0$, fiunt $\varphi\left(\frac{1-\alpha}{2}, 1-\alpha, x\right)$ et $\varphi\left(\frac{1+\alpha}{2}, 1+\alpha, x\right)$, eaeque per formulam (6.) in series generis $\psi$ transformari possunt. Iis transformationibus peractis, si mutatur $\alpha$ in $2\alpha$, $x$ in $4\sqrt{x}$ prodit formula

$$10. \quad \frac{2\Pi(\alpha-\frac{1}{2})}{\sqrt{\pi}} \int_0^{\frac{\pi}{2}} \cos v^{2\alpha-1}.\cos(2\sqrt{x} \operatorname{tang} v)\, dv$$
$$= \Pi(\alpha-1)\,\psi(1-\alpha, x) + \Pi(-\alpha-1).\, x^{\alpha}.\,\psi(1+\alpha, x),$$

inde per comparationem cum formula (12.) est

$$31. \quad \frac{2\Pi(\alpha-\frac{1}{2})}{\sqrt{\pi}} \int_0^{\frac{\pi}{2}} \cos v^{2\alpha-1}.\cos(2\sqrt{x} \operatorname{tang} v)\, dv = \int_0^{\infty} u^{\alpha-1}.\, e^{-u}.\, e^{-\frac{x}{u}}.\, du.$$

Simili modo demonstrari potest nexus duorum integralium, quae in aequationibus (13.) et (24.) continentur; haec enim formula (24.), si ponitur $\alpha-\beta$ loco $\alpha$, $\alpha+\beta-1$ loco $\beta$ et multiplicatur per $\frac{1}{\pi}\Pi(-\beta).2^{\alpha}.e^{\frac{x}{2}}.x^{\beta}$, accipit formam

$$32. \quad \frac{2\Pi(-\beta).e^{\frac{x}{2}}.x^{\beta}}{\pi} \int_0^{\frac{\pi}{2}} (2\cos v)^{\alpha-\beta-1}.\cos(\tfrac{1}{2}x \operatorname{tang} v + (\alpha+\beta-1)v)\, dv$$
$$= \frac{\Pi(\alpha-\beta-1)}{\Pi(\alpha-1)} x^{\beta}\,\varphi(\beta, \beta-\alpha+1, x) + \frac{\Pi(\beta-\alpha-1)}{\Pi(\beta-1)} x^{\alpha}\,\varphi(\alpha, \alpha-\beta+1, x),$$

qua comparata cum formula (13.) cognoscitur esse

$$33. \quad \int_0^{\infty} \frac{u^{\beta-1}.\, e^{-ux}.\, du}{(1+u)^{\alpha}}$$
$$= \frac{2.\, e^{\frac{x}{2}}}{\sin\beta\pi} \int_0^{\frac{\pi}{2}} (2\cos v)^{\alpha-\beta-1}.\cos(\tfrac{1}{2}x \operatorname{tang} v + (\alpha+\beta-1)v)\, dv,$$

praeterea, si aequationis (32.) altera pars per formulam (7.) transformatur, est

$$34. \quad \frac{2\Pi(-\beta)e^{\frac{x}{2}}.x^{\beta}}{\pi} \int_0^{\frac{\pi}{2}} (2\cos v)^{\alpha-\beta-1}.\cos(\tfrac{1}{2}x \operatorname{tang} v + (\alpha+\beta-1)v)\, dv = \chi(\alpha, \beta, x).$$

Generalius etiam integrale simili modo tractabimus

$$y = \int_0^{\frac{\pi}{2}} \sin v^{\alpha-1} . \cos v^{\beta-1} . \cos(x \operatorname{tang} v + \gamma v)\, dv$$

eosque casus eligemus, quibus per series supra citatas exprimi possit. Quantitatem $x$ etiam in hoc integrali semper positivam accipimus, quum ejus signum negativum in quantitatem $\gamma$ transferre liceat. Differentiando formam $\sin v^{\alpha} . \cos v^{\beta} . \cos(x \operatorname{tang} v + \gamma v)$, deinde integrando ab $u = 0$ usque ad $u = \frac{\pi}{2}$, fit

$$\begin{aligned} 0 = \; & \alpha \int_0^{\frac{\pi}{2}} \sin v^{\alpha-1} . \cos v^{\beta+1} . \cos(x \operatorname{tang} v + \gamma v)\, dv \\ & - \beta \int_0^{\frac{\pi}{2}} \sin v^{\alpha+1} . \cos v^{\beta-1} . \cos(x \operatorname{tang} v + \gamma v)\, dv \\ & - x \int_0^{\frac{\pi}{2}} \sin v^{\alpha} . \cos v^{\beta-2} . \sin(x \operatorname{tang} v + \gamma v)\, dv \\ & - \gamma \int_0^{\frac{\pi}{2}} \sin v^{\alpha} . \cos v^{\beta} . \sin(x \operatorname{tang} v + \gamma v)\, dv, \end{aligned}$$

ex hac aequatione, si integralia per $y$ eiusque differentialia exprimuntur, facile deducitur haec aequatio differentialis tertii ordinis:

$$35. \quad 0 = \alpha y + (\gamma + x) \frac{dy}{dx} + (\beta - 2) \frac{d^2 y}{dx^2} - x \frac{d^3 y}{dx^3},$$

nunc si ponitur

$$36. \quad y = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \ldots.$$

facile inveniuntur aequationes conditionales, quae inter coëfficientes hujus seriei locum habere debunt, ut aequationi differentiali haec series satisfaciat:

$$\alpha A_0 + \gamma . 1 . A_1 - 1 . 2 . (2 - \beta) A_2,$$
$$(\alpha + 1) A_1 + \gamma . 2 . A_2 - 2 . 3 . (3 - \beta) A_3,$$

et generaliter

$$37. \quad (\alpha + k) A_k + \gamma . (k + 1) A_{k+1} - (k + 1)(k + 2)(k + 2 - \beta) A_{k+2}.$$

Eodem modo si ponitur

$$38. \quad y = x^{\beta} (B_0 + B_1 x + B_2 x^2 + B_3 x^3 + \ldots.)$$

inveniuntur hae coëfficientium relationes

$$\gamma . \beta . B_0 - \beta(\beta + 1) . 1 . B_1,$$
$$(\alpha + \beta) B_0 + \gamma(\beta + 1) B_1 - (\beta + 1)(\beta + 2) . 2 . B_2,$$

et generaliter

$$39. \quad (\alpha + \beta + k) B_k + \gamma(\beta + k + 1) B_{k+1} - (\beta + k + 1)(\beta + k + 2)(k + 2) B_{k+2},$$

inde patet aequationis (35.) integrale completum esse

40. $y = A_0 + A_1 x + A_2 x^2 + \ldots. + x^\beta (B_0 + B_1 x + B_2 x^2 \ldots.)$,
nam per aequationes (37.) quantitatum $A_0$, $A_1$, $A_2$ etc. duae, et per aequationes (39.) quantitatum $B_0$, $B_1$, $B_2$ etc. una arbitrariae manent, ità ut hoc integrale tres constantes arbitrarias contineat. Itaque, si pro $y$ integrale supra propositum restituitur, est

$$41. \quad \int_0^{\frac{\pi}{2}} \sin v^{\alpha-1} . \cos v^{\beta-1} . \cos(x \operatorname{tang} v + \gamma v)\, dv$$
$$= A_0 + A_1 x + A_2 x^2 + \ldots. + x^\beta (B_0 + B_1 x + B_2 x^2 + \ldots.).$$

E relationibus coëfficientium facile cognoscitur, has series et hoc integrale generale transcendentes altiores esse quam eas de quibus hic agere constituimus; attamen casibus quibusdam specialibus cum illis congruunt. Primum, si accipimus $\gamma = \alpha + \beta$, ex aequationibus (39.) sequitur

$$B_1 = \frac{\alpha + \beta}{1(1+\beta)} B_0,$$
$$B_2 = \frac{(\alpha+\beta)(\alpha+\beta+1)}{1.2.(1+\beta)(2+\beta)} B_0,$$
$$B_3 = \frac{(\alpha+\beta)(\alpha+\beta+1)(\beta+\beta+2)}{1.2.3(1+\beta)(2+\beta)(3+\beta)} B_0$$
etc. etc.

Porro, si $\beta$ est positivum, posito $x = 0$, ex aequat. (41.) sequitur

$$A_0 = \int_0^{\frac{\pi}{2}} \sin v^{\alpha-1} . \cos v^{\beta-1} . \cos(\alpha+\beta) v . dv = \frac{\cos\frac{\alpha\pi}{2} \Pi(\alpha-1)\Pi(\beta-1)}{\Pi(\alpha+\beta-1)},$$

eodem modo si aequatio (41.) differentiatur secundum $x$ et postea ponitur $x = 0$, fit

$$A_1 = -\int_0^{\frac{\pi}{2}} \sin v^{\alpha} . \cos v^{\beta-2} . \sin(\alpha+\beta) v\, dv = -\frac{\cos\frac{\alpha\pi}{2} . \Pi(\alpha)\Pi(\beta-2)}{\Pi(\alpha+\beta-1)}$$

est igitur

$$A_1 = \frac{\alpha}{1(1-\beta)} A_0,$$

inde ex aequationibus (37.) facile sequitur

$$A_2 = \frac{\alpha(\alpha+1) A_0}{1.2(1-\beta)(2-\beta)},$$
$$A_1 = \frac{\alpha(\alpha+1)(\alpha+2) A_0}{1.2.3(1-\beta)(2-\beta)(3-\beta)}$$
etc. etc.

Hoc igitur casu, quo $\gamma = \alpha + \beta$, series duae, per quas integrale nostrum expressimus, ad hoc genus serierum pertinent, quod supra per $\varphi$ designavimus, et formula (41.) transit in hanc:

$$\int_0^{\frac{\pi}{2}} \sin v^{\alpha-1}.\cos v^{\beta-1}.\cos(x \operatorname{tang} v + (\alpha+\beta)v)\,dv$$

$$= \frac{\cos\frac{\alpha\pi}{2}\,\Pi(\alpha-1)\,\Pi(\beta-1)}{\Pi(\alpha+\beta-1)}\,\Phi(\alpha, 1-\beta, x) + B_0 x^\beta \Phi(\alpha+\beta, 1+\beta, x).$$

In determinanda constante $B_0$ methodo eadem utemur ac supra in determinandis constantibus aequationis (22.). Multiplicando per $x^{\lambda-1}.e^{-x}.dx$ et integrando intra limites 0 et $\infty$ fit

$$\Pi(\lambda-1)\int_0^{\frac{\pi}{2}} \sin v^{\alpha-1}.\cos v^{\beta+\lambda-1}.\cos(\alpha+\beta+\lambda)v\,dv$$

$$= \frac{\cos\frac{\alpha\pi}{2}\,\Pi(\alpha-1)\,\Pi(\beta-1)\,\Pi(\lambda-1)}{\Pi(\alpha+\beta-1)}\,F(\lambda, \alpha, 1-\beta, 1)$$
$$+ B_0\,\Pi(\beta+\lambda-1)\,F(\lambda+\beta, \alpha+\beta, 1+\beta, 1),$$

iisque seriebus hypergeometricis cum integrali per functionem $\Pi$ expressis, est

$$\frac{\cos\frac{\alpha\pi}{2}\,\Pi(\lambda-1)\,\Pi(\alpha-1)\,\Pi(\beta+\lambda-1)}{\Pi(\alpha+\beta+\lambda-1)}$$

$$= \frac{\cos\frac{\alpha\pi}{2}\,\Pi(\lambda-1)\,\Pi(\alpha-1)\,\Pi(\beta-1)\,\Pi(-\beta)\,\Pi(-\beta-\alpha-\lambda)}{\Pi(\alpha+\beta-1)\,\Pi(-\alpha-\beta)\,\Pi(-\beta-\lambda)}$$
$$+ B_0\,\frac{\Pi(\beta+\lambda-1)\,\Pi(\beta)\,\Pi(-\beta-\alpha-\lambda)}{\Pi(-\alpha)\,\Pi(-\beta)}$$

post reductiones nonnullas quantitas $\lambda$, quod debet, omnino evanescit, et prodit valor persimpex constantis $B_0$

$$B_0 = \cos\frac{\alpha\pi}{2}\,\Pi(-\beta-1),$$

quo denique substituto habemus

$$42.\quad \int_0^{\frac{\pi}{2}} \sin v^{\alpha-1}.\cos v^{\beta-1}.\cos(x \operatorname{tang} v + (\alpha+\beta)v)$$

$$= \frac{\cos\frac{\alpha\pi}{2}\,\Pi(\alpha-1)\,\Pi(\beta-1)}{\Pi(\alpha+\beta-1)}\,\Phi(\alpha, 1-\beta, x)$$
$$+ x^\beta \cos\frac{\alpha\pi}{2}\,\Pi(-\beta-1)\,\Phi(\alpha+\beta, 1+\beta, x).$$

Formula similis ex hac deducitur mutando $\alpha$ in $\alpha-1$, $\beta$ in $\beta+1$ et differentiando

$$42.\quad \int_0^{\frac{\pi}{2}} \sin v^{\alpha-1}.\cos v^{\beta-1}.\sin(x\,\mathrm{tang}\,v+(\alpha+\beta)v)\,dv$$

$$=\frac{\sin\frac{\alpha\pi}{2}\Pi(\alpha-1)\,\Pi(\beta-1)}{\Pi(\alpha+\beta-1)}\Phi(\alpha,1-\beta,x)$$

$$+x^\beta\sin\frac{\alpha\pi}{2}\Pi(-\beta-1)\,\Phi(\alpha+\beta,1+\beta,x)$$

iisque formulis inter se comparatis, cognoscitur nexus duorum integralium

$$43.\quad \cos\frac{\alpha\pi}{2}\int_0^{\frac{\pi}{2}} \sin v^{\alpha-1}.\cos v^{\beta-1}.\sin(x\,\mathrm{tang}\,v+(\alpha+\beta)v)\,dv$$

$$=\sin\frac{\alpha\pi}{2}\int_0^{\frac{\pi}{2}} \sin v^{\alpha-1}.\cos v^{\beta-1}.\cos(x\,\mathrm{tang}\,v+(\alpha+\beta)v)\,dv,$$

quae formula etiam hoc modo exhiberi potest

$$44.\quad \int_0^{\frac{\pi}{2}} \sin v^{\alpha-1}.\cos v^{\beta-1}.\sin\left(x\,\mathrm{tang}\,v+(\alpha+\beta)v-\frac{\alpha\pi}{2}\right)dv=0.$$

Notatu dignus est formulae (42.) casus specialis, quo $\alpha=0$

$$45.\quad \int_0^{\frac{\pi}{2}} \frac{\cos v^{\beta-1}.\sin(x\,\mathrm{tang}\,v+\beta v)}{\sin v}\,dv=\frac{\pi}{2},$$

cujus casum specialiorem, valori $x=0$ respondentem cl. *Liouville* invenit hoc diario tom. XIII. pag. 232. Praeterea e comparatis formulis (42.) et (13.) sine ulla difficultate cognoscitur nexus hujus integralis cum illis quae supra tractavimus

$$46.\quad \frac{\cos\frac{\alpha\pi}{2}\Pi(\alpha-1)}{\Pi(\alpha+\beta-1)}x^\beta\int_0^\infty \frac{u^{\alpha+\beta-1}.e^{-ux}\,du}{(1+u)^\alpha}$$

$$=\int_0^{\frac{\pi}{2}} \sin v^{\alpha-1}.\cos v^{\beta-1}.\cos(x\,\mathrm{tang}\,v+(\alpha+\beta)v)\,dv.$$

Alius casus, quo series formulae (41.) in series per characterem $\Phi$ designatas redeunt, est $\gamma=-\alpha-\beta$, hoc enim casu facile eodem modo ac supra invenitur formulam (41.) transire in hanc:

$$\int_0^{\frac{\pi}{2}} \sin v^{\alpha-1}\cos v^{\beta-1}.\cos(x\,\mathrm{tang}\,v-(\alpha+\beta)v)\,dv$$

$$=\frac{\cos\frac{\alpha\pi}{2}\Pi(\alpha-1)\,\Pi(\beta-1)}{\Pi(\alpha+\beta-1)}\Phi(\alpha,1-\beta,-x)+B_0x^\beta\,\Phi(\alpha+\beta,1+\beta,-x),$$

sed hoc casu constans $B_0$ alium valorem accipit, quem invenimus multiplicando per $x^{\alpha+\beta}.e^{-x}\,dx$ et integrando intra limites $x=0$ et $x=\infty$, iis integrationibus peractis fit:

$$\Pi(\alpha+\beta-1)\int_0^{\frac{\pi}{2}} \sin v^{\alpha-1}.\cos v^{\alpha+2\beta-1}.\,dv$$
$$= \cos\frac{\alpha\pi}{2}\,\Pi(\alpha-2)\,\Pi(\beta-1)\,F(\alpha+\beta,\alpha,1-\beta,-1)$$
$$+B_0\,\Pi(\alpha+2\beta-1)\,F(\alpha+2\beta,\alpha+\beta,1+\beta,-1),$$

etiam hae series hypergeometricae, quarum elementum quartum est $=-1$, per functionem $\Pi$ exprimi possunt secundum formulam

$$F(\alpha,\beta,\alpha-\beta+1,-1) = \frac{2^{-\alpha}\sqrt{\pi}\,\Pi(\alpha-\beta)}{\Pi\left(\frac{\alpha}{2}-\beta\right)\Pi\left(\frac{\alpha-1}{2}\right)},$$

quam demonstravi in commentatione de serie hypergeometrica h. diar. tom. XV. pag. 135. Inde si integrale illud et series hypergeometrica per functionem $\Pi$ exprimuntur, post faciles quasdam reductiones prodit:

$$B_0 = \cos\left(\frac{\alpha}{2}+\beta\right)\pi\,\Pi(-\beta-1),$$

eoque constantis valore substituta est:

$$47.\quad \int_0^{\frac{\pi}{2}} \sin v^{\alpha-1}.\cos v^{\beta-1}.\cos(x\,\mathrm{tang}\,v-(\alpha+\beta)v)\,dv$$
$$= \frac{\cos\frac{\alpha\pi}{2}\,\Pi(\alpha-1)\,\Pi(\beta-1)}{\Pi(\alpha+\beta-1)}\,\varphi(\alpha,1-\beta,-x)$$
$$+x^{\beta}\cos\left(\frac{\alpha}{2}+\beta\right)\pi\,\Pi(-\beta-1)\,\varphi(\alpha+\beta,1+\beta,-x).$$

Formula similis ex hac facile deducitur mutando $\alpha$ in $\alpha-1$, $\beta$ in $\beta+1$ et differentiando secundum variabilem $x$

$$48.\quad \int_0^{\frac{\pi}{2}} \sin v^{\alpha-1}.\cos v^{\beta-1}.\sin(x\,\mathrm{tang}\,v-(\alpha+\beta)v)$$
$$= -\frac{\sin\frac{\alpha\beta}{2}\,\Pi(\alpha-1)\,\Pi(\beta-1)}{\Pi(\alpha+\beta-1)}\,\varphi(\alpha,1-\beta,-x)$$
$$-x^{\beta}.\sin\left(\frac{\alpha}{2}+\beta\right)\pi\,\Pi(-\beta-1)\,\varphi(\alpha+\beta,1+\beta,-x).$$

Hae formulae (47.) et (48.) duobus modis facile ita conjungi possunt, ut has formas simpliciores obtineant:

$$49.\quad \int_0^{\frac{\pi}{2}} \sin v^{\alpha-1}.\cos v^{\beta-1}.\sin\left(x\,\mathrm{tang}\,v-(\alpha+\beta)v+\left(\frac{\alpha}{2}+\beta\right)\pi\right)dv$$
$$= \frac{\pi\,\Pi(\alpha-1)\,\varphi(\alpha,1-\beta,-x)}{\Pi(-\beta)\,\Pi(\alpha+\beta-1)},$$

50. $$\int_0^{\frac{\pi}{2}} \sin v^{\alpha-1} . \cos v^{\beta-1} . \sin\left(x \operatorname{tang} v - (\alpha+\beta) v + \frac{\alpha\pi}{2}\right) dv$$
$$= \frac{\pi x^\beta}{\Pi(\beta)} \varphi(\alpha+\beta, 1+\beta, -x).$$

In omnibus integralibus que hic tractata sunt, uti jam supra monuimus, $x$ semper esse debet quantitas positiva, si vero $x$ acciperetur negativum, omnes summae inventae falsae essent; in eo praecipue notatu dignum est integrale aequationis (50.), quod pro positivo $x$ seriei illi aequale est, sed pro negativo $x$ evanescit, cfr. aequat. (44.).

d. Lignicii, mense aprili a. 1837.