20.

De usu legitimo formulae summatoriae Maclaurinianae *).

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1.

Series semiconvergentes, quibus Geometrae ante hos centum annos computare docuerunt summas, quae magno vel infinito numero terminorum constant, eo maxime se commendant, quod signis alternantibus procedere soleant; ita ut serie usque ad $n$tum et usque ad $(n+1)$tum terminum computata, alter eius valor maior, alter minor sit valore summae quaesito. Unde cognoscuntur limites, quos excedere non potest error commissus, si in certo termino seriei summatoriae computationem sistis, Frequentur illud observatum, tantum casibus specialibus, ni fallor, demonstratum est. Quod quoties locum habet, tuto ac legitime ad calculandum summae valorem numericum seriae uti licet, quamvis constet post certum terminorum numerum eam fieri divergentem. Hinc operae pretium videtur, paucis demonstrare, quomodo est observatio precaria, ad certam et accuratam regulam revocetur.

Nota est formula

$$1.\quad \psi(x+h) = \psi(x) + \psi'(x)h + \psi''(x)\frac{h^2}{1.2} \dots + \psi^{(x)}(x)\frac{h^n}{\Pi_n} + \int_0^h \frac{(h-t)^n}{\Pi_n}\psi^{(n+1)}(x+t)\,\partial t,$$

in qua positum est

$$\Pi_n = 1.2.3 \dots n, \quad \psi^{(m)}(x) = \frac{\partial^m \psi(x)}{\partial x^m}.$$

Posito $-h$ loco h, simulque $-t$ loco t, formula illa abit in hanc,

$$2.\quad \psi(x-h) = \psi(x) - \psi'(x)h + \psi''(x)\frac{h^2}{1.2} \dots (-1)^n \psi^{(n)}(x)\frac{h^n}{\Pi_n} + (-1)^{n+1}\int_0^h \frac{(h-t)^n}{\Pi_n}\psi^{(n+1)}(x-t)\,\partial t.$$

Sit

$$\psi(x) = \int_a^x f(x)\,\partial x, \quad \psi(x) - \psi(x-h) = \Phi(x),$$

*) C. Maclaurin treatise on fluxions pg. 672. §. 828.

ac supponamus, esse $x-a$ multiplum ipsius h, quod sequentibus semper positivum accipimus, erit

$$3.\quad \varphi(a+h)+\varphi(a+2h)+\ldots+\varphi(x) = \psi(x)-\psi(a) = \psi(x),$$

quam summam generaliter designemus per

$$\Sigma_a^x \varphi(x) = \varphi(a+h)+\varphi(a+2h)+\varphi(a+3h)+\ldots+\varphi(x),$$

excluso valore infimo $\varphi(a)$, incluso extremo $\varphi(x)$. Qua adhibita notatione, est e (3.):

$$4.\quad \sum_a^x \varphi(x) = \psi(x) = \int_a^x f(x)\partial x.$$

Habetur autem e (2):

$$5.\quad \varphi(x) = \psi(x)-\psi(x-h) =$$

$$\psi'(x)h-\psi''(x)\frac{h^2}{2}\ldots(-1)^{n-1}\psi^{(n)}(x)\frac{h^n}{\Pi_n}+(-1)^n\int_0^h \frac{(h-t)^n}{\Pi_n}\psi^{(n+1)}(x-t)\partial t,$$

sive cum sit

$$\psi'(x) = f(x),\quad \text{ac generaliter}\quad \psi^{(m+1)}(x) = f^{(m)}(x),$$

erit, divisione simul per h facta,

$$6.\quad \frac{\varphi(x)}{h} = f(x)-f'(x)\frac{h}{2}+f''(x)\frac{h^2}{2.3}\ldots(-1)^{n-1}f^{(n-1)}(x)\frac{h^{n-1}}{\Pi_{(n-1)}}$$
$$+(-1)^n\int_0^h \frac{(h-t)^n}{h\Pi_n}f^{(n)}(x-t)\partial t.$$

Si in hac formula loco x ponimus $a+h$, $a+2h$, $a+3h$, ... x, atque summationem instituimus, obtinemus e (4.):

$$7.\quad \sum_a^x \frac{\varphi(x)}{h} = \int_a^x \frac{f(x)}{h}\partial x$$

$$= \sum_a^x \left\{f(x)-f'(x)\frac{h}{2}+f''(x)\frac{h^2}{2.3}\ldots+(-1)^{n-1}f^{(n-1)}(x)\frac{h^{n-1}}{\Pi_{(n-1)}}\right\}$$
$$+(-1)^n\int_0^h \frac{(h-t)^n}{h\Pi_n}\sum_a^x f^{(n)}(x-t)\partial t.$$

2.

Sit iam, evolutione facta,

$$8.\quad \frac{1}{2}\cdot\frac{e^{\frac{h}{2}}+e^{-\frac{h}{2}}}{e^{\frac{h}{2}}-e^{-\frac{h}{2}}} = \frac{1}{2}+\frac{1}{e^h-1} = \frac{1}{h}+a_1h-a_2h^3+a_3h^5-\ldots;$$

multiplicatione facta per

$$e^h-1 = h+\frac{h^2}{\Pi_2}+\frac{h^3}{\Pi_3}+\frac{h^4}{\Pi_4}+\ldots,$$

nanciscimur relationes sequentes, quibus coëfficientes a_m aliae post alias determinantur, et singulae quidem ex antecedentibus binis modis diversis,

$$9. \begin{cases} \frac{1}{\Pi_3} - \frac{1}{2}\cdot\frac{1}{\Pi_2} + \alpha_1 = 0, \\ \frac{1}{\Pi_4} - \frac{1}{2}\cdot\frac{1}{\Pi_3} + \frac{\alpha_1}{\Pi_2} = 0, \\ \frac{1}{\Pi_5} - \frac{1}{2}\cdot\frac{1}{\Pi_4} + \frac{1}{\Pi_3} - \alpha_2 = 0, \\ \frac{1}{\Pi_6} - \frac{1}{2}\cdot\frac{1}{\Pi_5} + \frac{1}{\Pi_4} - \frac{\alpha_2}{\Pi_2} = 0, \\ \dots\dots\dots\dots \\ \frac{1}{\Pi_{(2m+1)}} - \frac{1}{2}\cdot\frac{1}{\Pi_{2m}} + \frac{\alpha_1}{\Pi_{(2m-1)}} - \frac{\alpha_2}{\Pi_{(2m-3)}}\dots(-1)^m\alpha_m = 0, \\ \frac{1}{\Pi_{(2m+2)}} - \frac{1}{2}\cdot\frac{1}{\Pi_{(2m+1)}} + \frac{\alpha_1}{\Pi_{2m}} - \frac{\alpha_2}{\Pi_{(2m-2)}}\dots(-1)^{m+1}\frac{\alpha_m}{\Pi_2} = 0. \end{cases}$$

Harum relationum beneficio fit, ut si in formula (7.) loco $f(x)$ ponimus

$$f(x),\quad \tfrac{1}{2}f'(x)h,\quad \alpha_1 f''(x)h^2,\quad -\alpha_2 f^{\text{IV}}(x)h^4,\quad \dots\quad (-1)^{m+1}\alpha_m f^{(2m)}(x)h^{2m},$$

atque simul loco n ponimus

$$n,\quad n-1,\quad n-2,\quad n-4,\quad \dots\quad n-2m:$$

instituta additione, in altera aequationis parte sub signo summatorio, quod extra signum integrationis invenitur, abeant termini ducti in

$$f'(x)h,\quad f''(x)h^2,\quad f'''(x)h^3,\quad \dots\quad f^{(2m+1)}(x)h^{2m+1}.$$

Unde si statuimus

$$n = 2m+2,$$

post factam additionem indicatam evanescit summa integra, quae in altera parte aequationis (7.) extra signum integrationis invenitur, excepto termino primo $\overset{x}{\underset{a}{\Sigma}} f(x)$, atque prodit formula memorabilis:

$$10.\quad \int_a^x \partial x\left[\frac{fx}{h} + \tfrac{1}{2}f'(x) + \alpha_1 f''h - \alpha_2 f^{\text{IV}}(x)h^3\dots(-1)^{m+1}\alpha_m f^{(2m)}(x)h^{2m-1}\right] = \overset{x}{\underset{a}{\Sigma}} f(x) + \int_0^h T_m \overset{x}{\underset{a}{\Sigma}} f^{(2m+2)}(x-t)\,\partial t,$$

posito

$$11.\quad T_m = \frac{(h-t)^{2m+2}}{h\,\Pi_{(2m+2)}} - \frac{1}{2}\frac{(h-t)^{2m+1}}{\Pi_{(2m+1)}} + \alpha_1\frac{(h-t)^{2m}h}{\Pi_{2m}} - \alpha_2\frac{(h-t)^{2m-2}h^3}{\Pi_{(2m-2)}} + \alpha_3\frac{(h-t)^{2m-4}h^5}{\Pi_{(2m-4)}}\dots(-1)^{m+1}\alpha_m\frac{(h-t)^2h^{2m-1}}{\Pi_2}.$$

Seriem ad laevam aequationis (10.) Cl. *Maclaurin* olim ad valorem summae $\overset{x}{\underset{a}{\Sigma}} f(x)$ computandum proposuit. Aequatio nostra insuper errorem assignat commissum, si in certo termino seriem sistis. Qui error cum per integrale definitum exprimatur, plerisque casibus de magnitudine eius indicare licet.

Numeros α_m notum est omnes esse positivos. Facta enim integratione sequitur e (8.):

$$12.\quad \log\left(e^{\frac{h}{2}}-e^{-\frac{h}{2}}\right) = \log h + \tfrac{1}{2}\alpha_1 h^2 - \tfrac{1}{4}\alpha_2 h^4 + \tfrac{1}{6}\alpha_3 h^6 - \ldots$$
$$= \log h + \log\left[1+\frac{1}{\Pi_3}\left(\frac{h}{2}\right)^2+\frac{1}{\Pi_5}\left(\frac{h}{2}\right)^4+\ldots\right],$$

sive, expressione $e^{\frac{h}{2}}-e^{-\frac{h}{2}}$ in factores infinitos resoluta,

$$13.\quad \tfrac{1}{2}\alpha_1 h^2 - \tfrac{1}{4}\alpha_2 h^4 + \tfrac{1}{6}\alpha_3 h^6 - \ldots = \Sigma_0^\infty \log\left(1+\frac{h^2}{4p^2\pi^2}\right),$$

ipsi p tributis valoribus 1, 2, 3 usque ad infinitum. Hinc habetur

$$14.\quad \tfrac{1}{2}\alpha_m = \frac{1}{(2\pi)^{2m}}\Sigma_0^\infty \frac{1}{p^{2m}} = \frac{1}{(2\pi)^{2m}}\left[1+\frac{1}{2^{2m}}+\frac{1}{3^{2m}}+\frac{1}{4^{2m}}+\ldots\right].$$

Unde facile etiam assignas limites, quibus quantitates α_m includuntur. Habetur enim

$$\Sigma_0^\infty \frac{1}{p^{2m+2}} < 1 + \frac{1}{2^{2m}}\left(\Sigma_0^\infty \frac{1}{p^2}-1\right),$$

sive cum sit

$$\Sigma_0^\infty \frac{1}{p^2} = \tfrac{1}{6}\pi^2,$$

erit

$$\Sigma_0^\infty \frac{1}{p^{2m+2}} < 1 + \frac{1}{2^{2m}}\left(\frac{\pi^2}{6}-1\right),$$

unde

$$15.\quad \frac{1}{(2\pi)^{2m}} < \tfrac{1}{2}\alpha_m < \frac{1}{(2\pi)^{2m}}\left[1+\frac{1}{2^{2m-2}}\left(\frac{\pi^2}{6}-1\right)\right].$$

Qui limites facile, quantum placet, arctiores redduntur.

3.

Accuratius examinemus expressionem T_m. Qui posito

$$16.\quad \chi_{2m+1}^{(x)} = \frac{x^{2m+2}}{\Pi_{(2m+2)}} + \frac{1}{2}\,\frac{x^{2m+1}}{\Pi_{(2m+2)}} + \alpha_1\frac{x^{2m}}{\Pi_{2m}} - \alpha_2\frac{x^{2m-2}}{\Pi_{(2m-2)}}\ldots(-1)^{m+1}\alpha_m\frac{x^2}{2},$$

fit

$$17.\quad T_m = h^{2m+1}\chi_{2m+1}\left(\frac{t-h}{h}\right).$$

Notum est, et facile e (10.) demonstratur, designante x quemlibet numerum integrum, esse

$$18.\quad \chi_{2m+1}(x) = \Sigma_0^x \frac{x^{2m+1}}{\Pi_{(2m+1)}},$$

siquidem argumenti x incrementum $h=1$ statuimus. Casu vero nostro, quo

$$x = \frac{t-h}{h},$$

atque per integrationem t valores omnes a 0 usque ad h induit, erit x

quantitas fracta negativa, inter 0 et -1 posita. Quo casu non amplius definire licet expressionem $\chi_{2m+1}(x)$ ut summam. Nihilo tamen minus valet aequatio

$$19.\quad \chi_{2m+1}(x+1) = \chi_{2m+1}^{(x)} + \frac{(x+1)^{2m+1}}{\Pi_{(2m+1)}},$$

quicunque sit valor ipsius x. Nam cum aequatio illa, designante x integrum, e (18.) sponte pateat, ideoque pro diversis ipsius x valoribus innumeris valeat, identica illa esse debet. Statuto autem $x = \frac{t-h}{h}$, et multiplicatione per h^{2m+1} facta, fit ea e (17.):

$$20.\quad h^{2m+1}\chi_{2-+1}\left(\frac{t}{h}\right) = T_m + \frac{t^{2m+1}}{\Pi_{(2m+1)}},$$

unde

$$21.\quad T_m =$$

$$\frac{t^{2m+2}}{h\,\Pi_{(2m+2)}} - \frac{1}{2}\frac{t^{2m+1}}{\Pi_{(2m+1)}} + \alpha_1\frac{t^{2m}h}{\Pi_{2m}} - \alpha_2\frac{t^{2m-2}h^2}{\Pi_{(2m-2)}} \dots (-1)^{m+1}\alpha_m\frac{t^2h^{2m-1}}{2}.$$

Qua expressione ipsius T_m collata cum superiore (11.), videmus, ita comparatam esse ipsam T_m, ut posito $h-t$ loco t immutata maneat. Habetur igitur

$$22.\quad T_m = h^{2m+1}\chi_{2m+1}\left(\frac{t-h}{h}\right) = h^{2m+1}\chi_{2m+1}\left(-\frac{t}{h}\right),$$

sive

$$\chi_{2m+1}(x-1) = \chi_{2m+1}(-x).$$

Quae abunde nota sunt. Et constat facile exprimi ipsum T_m per solas dignitates pares ipsius $t-\frac{h}{2}$, quae posito $h-t$ loco t non mutantur. Quam obtinent expressionem per formulam, que sponte patet,

$$23.\quad \underset{a}{\Sigma^x}[f(x), h] = \underset{a}{\Sigma^x}\left\{f\left(x+\frac{h}{2}\right), \frac{h}{2}\right\} - \underset{a}{\Sigma^x}\left\{f\left(x+\frac{h}{2}\right), h\right\};$$

ubi per signum $\Sigma[f(x), h]$ intelligo, argumenti x accipiendum esse h incrementum. De qua formula, posito

$$f(x) = \frac{x^{2m+1}}{\Pi^{(2m+1)}},\quad a = 0,\quad x = \frac{t-h}{h},$$

obtines:

$$24.\quad T_m = \frac{\left(t-\frac{h}{2}\right)^{2m+2}}{h\,\Pi_{(2m+2)}} - \frac{\alpha_1}{2}\frac{\left(t-\frac{h}{2}\right)^{2m}h}{\Pi_{2m}} + \frac{\alpha_2}{8}\frac{\left(t-\frac{h}{2}\right)^{2m-2}h^2}{\Pi_{(2m-2)}} \dots$$

$$\dots (-1)^m\left(1-\frac{1}{2^{2m-1}}\right)\alpha\,\frac{\left(t-\frac{h}{2}\right)^2 h^{2m-1}}{\Pi_2} + h^{2m+1}\,\text{Const.}$$

Addo, cum T_m posito $h-t$ loco t non mutetur, theorema nostrum (10.)

etiam ita exhiberi posse:

25. $$\int_a^x \partial x \left\{\frac{f}{h}x + \frac{1}{2}f'(x) + \alpha_1 f''(x)h - \alpha_2 f^{\text{IV}}(x)h^3 \dots (-1)^{m+1}\alpha_m f^{(2m)}(x)h^{2m-1}\right\}$$
$$= \Sigma_a^x f(x) + \int_0^h T_m \Sigma_a^x f^{(2m+2)}(x-h+t)\,\partial t$$
$$= \Sigma_a^x f(x) + \int_0^{\frac{h}{2}} T_m \Sigma_a^x [f^{(2m+2)}(x-t) + f^{(2m+2)}(x-h+t)]\,\partial t.$$

4.

In theoremate nostro (10.) seu (25.) cum valores ipsius t tantum inter 0 et h positi considerentur, iam demonstrabimus, in quo cardo rei nostrae vertitur, pro omnibus illis valoribus ipsius t ipsum T_m signum non mutare. Quam ita adornare licet demonstrationem.

Habetur

26. $$\frac{1}{2}\left\{\frac{1-e^{xz}}{1-e^z} - \frac{1-e^{-xz}}{1-e^{-z}}\right\} = z\chi_1(x-1) + z^3\chi_3(x-1) + z^5\chi_5(x-1) + \dots.$$

Quae, designante x integrum, sponte patet evolutio e (18.), cum sit

$$\frac{1-e^{xz}}{1-e^z} = \Sigma_0^x e^{z(x-1)},$$

ipsius x incremento $=1$ posito. Unde cum aequatio (26.) pro innumeris ipsius x valoribus valeat, pro natura functionum $\chi(x)$, quae sunt rationales, integrae, finitae, eadem pro quolibet ipsius x valore valet. Sit iam

$$x' = 1 - x,$$

erit

27. $$\frac{1-e^{xz}}{1-e^z} - \frac{1-e^{-xz}}{1-e^{-z}} = \frac{1-e^{xz}}{1-e^z} + \frac{e^z - e^{x'z}}{1-e^z} = \frac{(1-e^{xz})(1-e^{x'z})}{1-e^z}$$
$$= -\frac{\left(e^{\frac{x}{2}z} - e^{-\frac{x}{2}z}\right)\left(e^{\frac{x'}{2}z} - e^{-\frac{x'}{2}z}\right)}{e^{\frac{z}{2}} - e^{-\frac{z}{2}}}.$$

Unde fit e (26.), si expressionem hanc in factores infinitos resolvis:

28. $$-zxx'\Pi \frac{\left(1+\frac{x^2z^2}{4p^2\pi^2}\right)\left(1+\frac{x'^2z^2}{4p^2\pi^2}\right)}{\left(1+\frac{z^2}{4p^2\pi^2}\right)} = 2[z\chi_1(x-1) + z^3\chi_3(x-1) + \dots],$$

siquidem in producto praefixo Π denotato ipsi p valores 1, 2, 3, ... ∞ tribuis.

Ponamus

$$y = \frac{-z^2}{4p^2\pi^2},$$

erit expressio sub signo multiplicatorio in (28.),

$$29.\quad \frac{(1-x^2y)(1-x'^2y)}{(1-y)} = 1+(1-x^2-x'^2)y+\frac{(1-x^2)(1-x'^2)y^2}{1-y}$$
$$= 1+2xx'y+xx'(2+xx')\frac{y^2}{1-y}.$$

Quae expressio evoluta in seriem secundum dignitates ascendentes ipsius y seu $(-z^2)$, coëfficientes omnes habet positivos, si xx' positivum est. Quo casu igitur etiam productum Π, e factoribus (29.) conflatum, si ad dignitates ipsius $(-z^2)$ evolvitur, coëfficientes omnes habebit positivos; sive cum in expressione (28.) productum Π adhuc ducatur in $-xx'z$, coëfficientes expressionis illius evolutae, $2\chi_{2m+1}(x-1)$, erunt positivi, si m est impar, negativi, si m est numerus par.

Fit autem $xx'=x(1-x)$ positivum pro iis valoribus ipsius x omnibus, qui sunt inter 0 et 1 positi, neque pro illis aliis. Unde

„*erit* $\chi_{2m+1}(x-1)$ *pro valoribus ipsius* x *omnibus inter* 0 *et* 1 *po-*
„*sitis positivum, si* m *est numerus impar, negativum, si* m *est par.*"

Unde, cum posito $x=\frac{t}{h}$, sit t inter 0 et h, si x inter 0 et 1, sequitur e (17.) incremento h semper positivo accepto,

„*pro omnibus ipsius* t *valoribus inter* 0 *et* h *positis, esse* T_m *posi-*
„*tivum, si* m *sit numerus impar, negativum, si* m *sit par.*"

5.

Et hinc profecti sine ulla negotio iam de formula nostra (10.) deducimus hoc theorema.

Theorema.

„*Proposita summa*

$$„\Sigma_a^x f(x) = f(a+h)+f(a+2h)+f(a+3h)\;\ldots\; +f(x),$$

„*quoties expressio*

$$„\Sigma_a^x f^{(2m+2)}(x-t) = \Sigma_a^x \frac{\partial^{2m+2} f(x-t)}{\partial x^{2m+2}},$$

„*pro valoribus omnibus ipsius* t *inter* 0 *et* h *positis neque in infinitum* „*abit, neque signum mutat: excessus seriei summatoriae usque ad* „$(m+2)^{\text{tum}}$ *terminum productae super valorem summae propositae,*

$$„\int_0^x dx\left\{\frac{f(x)}{h}+\tfrac{1}{2}f'(x)+\alpha_1 f''(x)h-\alpha_2 f^{\text{IV}}(x)h^3\ldots\right.$$
$$\left.\ldots(-1)^{m+1}\alpha_m f^{(2m)}(x)h^{2m-1}\right\}-\Sigma_a^x f(x)$$

„*idem signum habet atque* $\sum\limits_a^x f^{(2m+2)}(x-t)$, *si m est numerus impar,*
„*signum contrarium, si m est numerus par.*"

Quod est de re, quae satis vagis ratiociniis tractari solet, theorema rigorosum et accuratum.

Vocemus S_m valorem seriei Maclaurinianae usque ad $(m+2)$tum terminum productae,

$$S_m = \int_u^x \partial x \left[\frac{f(x)}{h} + \tfrac{1}{2} f'(x) + \alpha_1 f''(x) h - \alpha_2 f^{IV}(x) h^3 \dots (-1)^{m+1} \alpha_m f^{(2m)}(x) h^{2m-1}\right].$$

Sequitur e theoremate invento hoc:

„*Si utraque expressio*

$$„\sum_a^x f^{(2m)}(x-t),\quad \sum_a^x f^{(2m+2)}(x-t)$$

„*pro valoribus omnibus ipsius t inter* 0 *et h positis neque in infi-*
„*nitum abit neque signum mutat, idemque utrique signum suppetit,*
„*summae propositae* $\sum\limits_a^x f(x)$ *valor inclusus est inter valores* G_{m-1}
„*et* G_m."

Idem extenditur ad casum generaliorem, quo indicum m differentia est numerus quilibet impar.

Facile patet, esse generaliter

$$30.\quad \int_a^x \varphi(x)\partial x = \int_0^h \sum_a^x \varphi(x-t)\partial t.$$

Unde si $\sum\limits_a^x f^{(2m+2)}(x-t)$, si t inter 0 et h, neque signum mutat neque in infinitum abit, idem etiam signum erit integrali

$$\int_a^x f^{(2m+2)}(x)\partial x;$$

porro e theoremate invento idem signum est expressioni

$$(-1)^{m+1}[G_m - \sum_a^x f(x)].$$

Hinc habemus theorema:

„*Si* $\sum\limits_a^x f^{(2m+2)}(x-t)$, *quoties t inter* 0 *et h, neque signum mutat*
„*neque in infinitum abit, excessus* $G_m - \sum\limits_a^x f(x)$ *signum contrarium*
„*habet atque terminus seriei Maclaurinianae, qui ipsam* G_m *proxime*
„*continuat,*

$$„(-1)^m \alpha_m \int_a^x f^{(2m+2)}(x)\partial x."$$

Casibus, quibus prae ceteris applicatur series summatoria Maclauriniana, conditionibus antecedentibus stabilitis satisfieri solet. Quibus igitur casibus de erroris limitibus tibi constabit, atque seriei tutus et legitimus usus erit.

Corollarium.

Apponam summas dignitatum imparium numerorum naturalium sive functionum $\Pi_{(2m+1)}\,\chi_{2m+1}(x)$, expressas per quantitatem

$$u = x(x+1).$$

Fit

$$\sum_0^x x^3 = \tfrac{1}{4}.u^2$$

$$\sum_0^x x^5 = \tfrac{1}{6}u^2\left(u-\tfrac{1}{2}\right)$$

$$\sum_0^x x^7 = \tfrac{1}{8}u^2\left(u^2-\tfrac{4}{3}u+\tfrac{2}{3}\right)$$

$$\sum_0^x x^9 = \tfrac{1}{10}u^2\left(u^3-\tfrac{5}{2}u^2+3u-\tfrac{3}{2}\right)$$

$$\sum_0^x x^{11} = \tfrac{1}{12}u^2\left(u^4-4u^3+\tfrac{17}{2}u^2-10u+5\right)$$

$$\sum_0^x x^{13} = \tfrac{1}{14}u^2\left(u^5-\tfrac{35}{6}u^4+\tfrac{287}{15}u^3-\tfrac{118}{3}u^2+\tfrac{691}{15}u-\tfrac{691}{30}\right)$$

etc. etc.

Quae expressiones maxime in inferiorum dignitatum summis eo se commendant, quod earum terminorum numerus duobus minor sit atque vulgarium formularum.

Ad continuandas expressiones observo, si

$$\sum_a^x x^{2p-3} = \frac{1}{2p-2}\left[u^{p-1}-a_1u^{p-2}+a_2u^{p-3}\ldots.(-1)^{p-1}a_{p-3}u^2\right],$$

$$\sum_a^x x^{2p-1} = \frac{1}{2p}\left[u^p-b_1u^{p-1}+b_2u^{p-2}\ldots.(-1)^p b_{p-2}u^2\right],$$

haberi:

$$2p(2p-1)a_1 = (2p-2)(2p-3)b_1-p(p-1)$$

$$2p(2p-1)a_2 = (2p-4)(2p-5)b_2-(p-1)(p-2)b_1$$

$$2p(2p-1)a_3 = (2p-6)(2p-7)b_3-(p-2)(p-3)b_2$$

. . . .

$$2p(2p-1)a_{p-3} = 5.6\,b_{p-3}-3.4\,b_{p-4}$$

$$0 = 3.4\,b_{p-2}-2.3\,b_{p-3}.$$

Harum relationum ope, cognitis a_m, coefficientes b_m aliae post alias computantur. Calculus et retro institui potest, cum coëfficientem postremum eandem habeas atque in forma vulgari, quae secundum dignitates ipsius x procedit.

Expressiones similes summarum parium dignitatum obtines ex antecedentibus differentiando, cum sit

$$\sum_0^x x^{2p} = \frac{1}{2p+1} \frac{\partial \sum_0^x x^{2p+1}}{\partial x}.$$

Relationes antecc. inter quantitates a et b facile e noto theoremate inveniuntur, quod summa numerorum naturalium ad dignitatem imparem elatorum bis differentiata, reiectaque constante et per constantem divisione facta, prodeat summa numerorum naturalium ad dignitatem imparem proxime minorem elatorum.

Ex iisdem relationibus ipso conspectu demonstratur, expressiones propositas, sicuti in exemplis appositis videre est, alternantibus signis procedere. Quippe quod, ubi in ulla valet, e natura relationum istarum etiam de subsequentibus omnibus valebit. Unde quoties u est quantitas negativa, expressionum termini omnes signum idem habent, quod e signo dignitatis supremae determinatur. Hinc petitur demonstratio nova magis elementaris theorematis supra propositi, expressionem T_m pro ipsius t valoribus omnibus inter 0 et h positis signum idem servare.

Residui seriei summatoriae Maclaurinianae expressionem a nostra diversam dedit ill. *Poisson* in commentatione egregia *„Sur le calcul numérique des Intégrales définies"* (Acad. des Sciences Vol. VI. pag. 571 sqq.).

D. 2. Junii 1834.