THE PRINCIPLE OF RELATIVITY IN ELECTRODYNAMICS AND AN EXTENSION THEREOF

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Introductory.

1. The absence, as far as experiment can detect, of any phenomenon arising from the Earth's motion relative to the electromagnetic æther has been fully accounted for by Lorentz† and Einstein,‡ provided the hypothesis of electromagnetism as the ultimate basis of matter be accepted, so that the only available means of estimating the distance between two points is the measuring of the time of propagation of effects between the bodies, such propagation taking place in accordance with the equations of the electron theory. It has been proved not only within the limits of experimental accuracy, but exactly, that any actual effect is completely obscured by the fact that the observer necessarily shares in the motion of the earth, and has therefore different measures of time and space from those which he would have if he did not do so. The foundation of this theory of relativity is the set of relations subsisting between the space and time measures of two observers having a uniform relative velocity. If this is v and the axis of x is taken in the direction of v, these relations are

$$T = \beta\left(t - \frac{vx}{c^2}\right), \qquad X = \beta(x - vt), \qquad Y = y, \qquad Z = z. \tag{1}$$

where
$$\beta = (1 - v^2/c^2)^{-\frac{1}{2}}.$$

The analytical result obtained is that, if e, h, u are vectors and ρ a scalar satisfying the equations

$$\frac{1}{c}\left(\frac{\partial e}{\partial t} + \rho u\right) = \operatorname{curl} h,$$

$$-\frac{1}{c}\frac{\partial h}{\partial t} = \operatorname{curl} e,$$

$$\operatorname{div} e = \rho,$$

$$\operatorname{div} h = 0,$$

* This paper contains in an abbreviated form the chief parts of the work contributed by the Author to a joint paper by Mr. Bateman and himself read at the meeting held on February 11th, 1909, and also the work of the paper by the author read at the meeting held on March 11th, 1909.

† Lorentz, *Amsterdam Proceedings*, 1903-4, p. 809.

‡ Einstein, *Ann. der Physik*, 17. Cf. also Larmor, *Aether and Matter*, Ch. XI.

so that e, h are the electric and magnetic intensities, ρ is the density of electricity, and u the velocity of convection in a sequence of electromagnetic phenomena in the æther, the observer being at rest relative to the æther, then the quantities E, H, U, P given by the equations

$$E_X = e_x, \quad E_Y = \beta(e_y - vh_z/c), \quad E_Z = \beta(e_z + vh_y/c);$$
$$H_X = h_x, \quad H_Y = \beta(h_y + ve_z/c), \quad H_Z = \beta(h_z - ve_y/c);$$
$$\mathrm{P} = \beta\rho(1 - vu_x/c^2),$$
$$U = B^{-1}(1 - vu_x/c^2)^{-1}\{\beta(u_x - v),\ u_y,\ u_z\},$$

of which the last is deduced without further assumption from (1), satisfy the same equations in (X, Y, Z, T).

Thus E, H, U, P are quantities which can consistently represent a sequence of phenomena in which the second observer is supposed to be at rest in the æther.

The relations given above are reciprocal, and it is proved that the charges in corresponding elements of volume at corresponding times are the same in magnitude. Thus it is impossible for either observer to deduce from any experimental observation that he is at rest in the true æther and that the other is moving or *vice versa*.

This being so, the relations connecting the values, as estimated by the two observers of all physical quantities, must be such as to leave the constitutive equations for material media invariant also. This aspect of the question has been discussed by Minkowski,* Frank,† and Mirimanoff.‡ The two former agree in stating that the Lorentz equations must be modified if the theorem of relativity is to be maintained in its entirety; but the last-named shews that this may be avoided, and that the relativity is complete and exact provided a change is made in the constitutive equations as ordinarily given. The modification required is of the second order of small quantities only in the equation

$$D = \epsilon E + (\epsilon - 1)[U, B]/c,$$

which has been verified to the first order by H. A. Wilson, and is of the first order in u/c in the equation $B = \mu H$. Thus the suggested equations are not contrary to experiment. The first part of the present paper is a verification of Mirimanoff's results by a process of averaging.§

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* *Gött. Nachr.*, 1908, p. 53.

† *Ann. der Phys.*, 27, 1908, p. 1059.

‡ *Ann. der Phys.*, 28, 1909, p. 192.

§ Frank (*Ann. der Phys.*, 27, 1908) gives a verification of Minkowski's result by a similar process, but the investigation does not appear to take into account the Röntgen current-curl $[\rho u]$.

The second part deals with a new transformation or, rather, group of transformations, which shews that a uniform translation through space is not the only type of motion which it will be impossible to detect so long as our measures of time and space are of electrodynamic origin. It has been pointed out by Minkowski that in a space of four dimensions in which the coordinates are $(x, y, z, ct\sqrt{-1})$, the geometrical transformation employed by Einstein, is simply a finite rotational displacement of the whole space about $y = 0$, $z = 0$. The equation $\nabla^2 V = 0$, *i.e.*,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{p}{c^2}\,\frac{\partial^2}{\partial t^2}\right) V = 0,$$

is known to be invariant for such a transformation.

But this equation is invariant for a larger group of transformations than that of rotations, viz., for the group of conformal transformations in the four dimensional space, which, as is known, is built up out of inversions with respect to the hyperspheres of the space.* The question arises whether the theorem of relativity also holds for the types of motion of an electromagnetic system derived from one another by such a transformation. The following investigation shews that this is so, and develops a scheme of correlation between the physical quantities in the system and its transformation. The constitutive equations take exactly the same form as in the first part of the paper.

The motions that arise in these transformations are naturally a good deal more complicated than in that of Einstein. In that case, a fixed configuration transforms into one every point of which has the same velocity of translation. But, in the present case, under the simplest operation of the group a fixed system becomes one in which the whole is expanding or contracting radially about a point in a certain way, which, though analytically simple, is difficult to describe geometrically. But the important property of it is, that any sphere which is expanding with velocity equal to that of light transforms into a sphere expanding (or contracting) with equal velocity.

It may be remarked here that it appears to be impossible for a uniform velocity of rotation of an electromagnetic system to be obscured in the way in which the types of motion above mentioned are. For, without considering the electromagnetic equations at all, if a disturbance propagated with velocity c equally in all directions is to transform into the like, it follows that the space $(x, y, z, \iota ct)$ of the one system must be conformal

* This was pointed out to me by Mr. Bateman, a conversation with whom suggested the present investigation.

with the space $(X, Y, Z, \iota cT)$ of the other: inasmuch as

$$\delta X^2+\delta Y^2+\delta Z^2-c^2\delta T^2 = 0$$

must be a consequence of

$$\delta x^2+\delta y^2+\delta z^2-c^2\delta t^2 = 0.$$

We may, however, prove that no conformal transformation in this space exists which will transform every point at rest in the space (x, y, z) into a point moving with angular velocity about a fixed line in the (XYZ) space. Thus, as in material dynamics, the equations of electrodynamics will not be preserved in the same form for a set of axes rotating uniformly relatively to the space in which they are satisfied by actual phenomena.

I.

Some considerations concerning the Lorentz-Einstein Transformation.

2. The following relations arising immediately from the Lorentz-Einstein transformation are used in what follows.

The velocities of a moving point in the two systems are connected thus:

$$\left.\begin{aligned} W_X &= (w_x-v)\Big/\left(1-\frac{vw_x}{c^2}\right) \\ W_N &= w_n\Big/\left(1-\frac{vw_x}{c^2}\right) \end{aligned}\right\}, \tag{2}$$

the suffixes n and N denoting components in any direction perpendicular to v.

The relative coordinates of two moving points (x, y, z), (x', y', z')*:

$$\left.\begin{aligned} X'-X &= (x'-x)\Big/\beta\left(1-\frac{vw'_x}{c^2}\right) \\ N'-N &= (n'-n)+vw'_n(x'-x)\Big/c^2\left(1-\frac{vw'_x}{c^2}\right) \end{aligned}\right\}. \tag{3}$$

The relative velocity of two moving points:

$$\left.\begin{aligned} W'_X-W_X &= (w'_x-w_x)\Big/\beta^2\left(1-\frac{vw_x}{c^2}\right)\left(1-\frac{vw'_x}{c^2}\right) \\ W'_N-W_N &= (w'_n-w_n)\Big/\beta\left(1-\frac{vw_x}{c^2}\right) \\ &\qquad -vw'_n(w'_x-w_x)\Big/\beta c^2\left(1-\frac{vw_x}{c^2}\right)\left(1-\frac{vw'_x}{c^2}\right) \end{aligned}\right\}. \tag{4}$$

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* Inasmuch as the relative coordinates are changing, it is necessary to specify exactly when they are measured. The expressions given are the values at the instant at which the second point is at $(x'y'z')$. $n, n', \ldots$, here stand for either coordinate perpendicular to $x, x', \ldots$.

If a denote a volume which is moving with velocity (w_x, w_y, w_z), the corresponding volume A is given by

$$A = a\Big/\beta\left(1-\frac{vw_x}{c^2}\right). \tag{5}$$

In applying the transformation to the consideration of the number of free electrons per unit volume, it must be remembered that they are moving with different velocities, and that this velocity affects the relative volumes which they occupy in the two systems of coordinates.

But the magnitude of the charge of each electron and the number of them are unchanged. If, therefore, we confine our attention to a group of electrons all having the same component of velocity w_x' in the one system and W_x' in the other, the volumes occupied, δa, δA, are connected by the relation (5).

Thus this group contributes to the density of the free electricity as measured in the two systems different amounts, $\delta\rho = \Sigma e/\delta a$, $\delta \mathrm{P} = \Sigma e/\delta A$. Using (5), we have

$$\delta \mathrm{P} = \frac{\Sigma e}{\delta A} = \frac{\beta \Sigma e}{\delta a}\left(1-\frac{vw_x'}{c^2}\right).$$

Let w_x be the mass velocity of the moving medium as distinguished from w_x' the velocity of the individual electron.

Then

$$\delta \mathrm{P} = \frac{\beta\Sigma e}{\delta a}\left\{\left(1-\frac{vw_x}{c^2}\right) - \frac{v}{c^2}(w_x'-w_x)\right\}$$

$$= \beta\left(1-\frac{vw_x}{c^2}\right)\delta\rho - \frac{\beta v}{c^2}\,\frac{\Sigma e(w_x'-w_x)}{\delta a}$$

$$= \beta\left(1-\frac{vw_x}{c^2}\right)\delta\rho - \frac{\beta v}{c^2}\,\delta j_x,$$

where δj_x is the contribution of the same group of electrons to the conduction current through the medium.

This last relation is true for each particular value of w_x', so that on summing for all electrons, we have

$$\mathrm{P} = \beta\left(1-\frac{vw_x}{c^2}\right)\rho - \frac{\beta v}{c^2}j_x, \tag{6}$$

an equation identical with Minkowski's and Mirimanoff's.

Similarly the contributions of the same group of electrons to the current density in the two systems are,

$$\delta j = \frac{\Sigma e(w'-w)}{\delta a},$$

$$\delta J = \frac{\Sigma e(W'-W)}{\delta A},$$

these being vector equations.

Using equations (3) and (5) these give

$$\delta J_x = \frac{\Sigma e\,(w_x'-w_x)}{\delta a\,\beta\left(1-\frac{vw_x}{c^2}\right)} = \frac{\delta j_x}{\beta\left(1-\frac{vw_x}{c^2}\right)} = \beta\left(1+\frac{vW_x}{c^2}\right)\delta j_x,$$

$$\delta J_y = \frac{\Sigma e}{\delta a}\left\{(w_y'-w_y) - \frac{vw_y(w_x'-w_x)}{c^2\left(1-\frac{vw_x}{c^2}\right)}\right\}$$

$$= \delta j_y - \frac{\beta v}{c^2} W_y \delta j_x,$$

$$\delta J_z = \delta j_z - \frac{\beta v}{c^2} W_z \delta j_x.$$

Summing for all electrons, we obtain Minkowski's equation,

$$\left.\begin{aligned} J_x &= \beta j_x\left(1+\frac{vW_x}{c^2}\right) \\ J_y &= j_y - \frac{\beta v W_y}{c^2} j_x \\ J_z &= j_z - \frac{\beta v W_z}{c^2} j_x \end{aligned}\right\}. \tag{7}$$

The Polarization p and Magnetization m.

So far no distinction has been made between various types of motion of the electrons; but now they must be differentiated.

Consider first the so-called polarization electrons which move with the body, but have an electric moment. Taking δa, δA to be the volumes occupied by such a polarized element, its contributions to the polarization in the two systems are related as follows:

$$\delta P_x = \frac{\Sigma eX}{\delta A},$$

where X is measured relative to the centre of the element.

Using (4) and (5) this becomes

$$\left.\begin{aligned}
\delta P_x &= \frac{\Sigma ex}{\delta a} = \delta p_x, \ \ldots \\
\text{Similarly,} \quad \delta P_y &= \frac{\Sigma eY}{\delta A} \\
&= \frac{\Sigma e}{\delta a}\left\{\beta\left(1-\frac{vw_x}{c^2}\right)y+\frac{\beta vw_y x}{c^2}\right\} \\
&= \beta\left(1-\frac{vw_x}{c^2}\right)\delta p_y+\frac{\beta vw_y}{c^2}\delta p_x, \ \ldots \\
\text{and} \quad \delta P_z &= \beta\left(1-\frac{vw_x}{c^2}\right)\delta p_z+\frac{\beta vw_z}{c^2}\delta p_x, \ \ldots
\end{aligned}\right\}. \qquad (a)$$

Since the polarization electrons have no velocity relative to the body, they contribute nothing to the magnetization in either system.

Consider next the *magnetization electrons*, that is, systems of electrons spinning round centres fixed in the body and having a magnetic moment; possibly also an electric moment in a moving system.

If these have an electric moment, consider the contribution to the polarization

$$\delta P_X = \frac{\Sigma eX}{\delta A} = \frac{\Sigma e}{\delta a}\beta\left(1-\frac{vw'_x}{c^2}\right)X = \frac{\Sigma ex}{\delta a},$$

where we consider in the first place only those electrons having a definite value of w'_x and afterwards effect a summation.

Similarly δP_Y, *i.e.*, $\Sigma eY/\delta A$, becomes, after reduction,

$$\beta\left(1-\frac{vw_x}{c^2}\right)\delta p_y+\frac{\beta vw_y}{c^2}\delta p_x+\frac{\beta v}{c}\delta m_z,$$

and likewise $$\delta P_z = \beta\left(1-\frac{vw_x}{c^2}\right)\delta p_z+\frac{\beta vw_z}{c^2}\delta p_x-\frac{\beta v}{c}\delta m_y.$$

In the same manner for the same group of electrons

$$\delta M_X = \frac{1}{2c}\Sigma e\left\{Y(W'_Z-W_Z)-Z(W'_Y-W_Y)\right\}/\delta A,$$

which, on reduction, becomes

$$\delta m_x-(\beta v W_Y\delta m_y+\beta v W_Z\delta m_z)/c^2,$$

and, similarly, $$\delta M_N = \beta\left(1+\frac{vW_x}{c^2}\right)\delta m_n.$$

From these, by addition, the formulæ of transformation are seen to be exactly those given by Mirimanoff, viz.,

$$\left.\begin{aligned} P_X &= p_x \\ P_Y &= \beta\left(1-\frac{vw_x}{c^2}\right)p_y+\frac{\beta v w_y}{c^2}p_x+\frac{\beta v}{c}m_z \\ P_Z &= \beta\left(1-\frac{vw_x}{c^2}\right)p_z+\frac{\beta v w_z}{c^2}p_x-\frac{\beta v}{c}m_y \end{aligned}\right\}, \tag{8}$$

$$\left.\begin{aligned} M_X &= m_x-\beta v(W_Y\delta m_y+W_Z\delta m_z)/c^2 \\ M_N &= \beta\left(1+\frac{vW_x}{c^2}\right)m_n \end{aligned}\right\}. \tag{9}$$

It is thus verified, by direct method, that the equations of the field as given by Lorentz for moving ponderable bodies are left unchanged by a transformation in which the pairs of vectors (e, b), $(d, h-[pw]/c)$ are correlated in the same manner as were (e, h) for the free æther.*

The constitutive equations now become

$$\left.\begin{aligned} D_X &= \epsilon E_X \\ \left(1-\frac{v^2}{c^2}\right)D_N &= \left(\epsilon-\frac{v^2}{c^2}\right)E_N+\frac{1}{c}\,[v,\,\epsilon B-H]_N \end{aligned}\right\}, \tag{10}$$

$$\mu H = B+(\mu-1)[v,\,E]/c, \tag{11}$$

$$\left.\begin{aligned} J_X &= \sigma E_X/\beta \\ J_N &= \beta\sigma(E_n+[v,\,B]_n/c) \end{aligned}\right\}. \tag{12}$$

The first and last of these appear to be the exact forms of the better known approximate equations

$$D = \epsilon E+(\epsilon-1)[w,\,B]/c,$$

$$J = \sigma(E+[w,\,B]),$$

and reduce to these, if $\mu = 1$ and if v^2/c^2 is neglected.

The middle equation has as yet no experimental corroboration.

*8Cf. Mirimanoff, *loc. cit.*

3. *The Mechanical Action of Radiation on a Bounding Surface under the Lorentz-Einstein Transformation.*

M. Planck* in a recent paper has shewn, from thermodynamic considerations, that the pressure of radiation in equilibrium in an enclosure must be independent of the motion of that enclosure, and obtains relations between the energy per unit volume, and the temperature of the radiation when considered at rest and in motion in turn. Inasmuch as energy and pressure are independent of thermodynamics, it seems appropriate to shew here that Planck's results for those quantities are an immediate result of the foregoing transformation.

Taking the expression for the force per unit area on a moving surface,†

$$p' = \frac{1}{8\pi}\{2e'e_\nu + 2h'h_\nu - n(ee' + hh')\},$$

this being a vector equation, we adapt it to the case where the surface is moving in the direction of the axis of x with velocity v.

The equations of the transformation are

$$e'_x = E_X, \quad e'_y = E_Y/\beta, \quad e'_z = E_Z/\beta,$$
$$h'_x = H_X, \quad h'_y = H_Y/\beta, \quad h'_z = H_Z/\beta.$$

Let the element of area upon which p' acts be ds, the normal having direction cosines (l, m, n), and let the transformed area be dS, with normal (L, M, N).

Then $\quad lds = LdS, \quad \beta mds = MdS, \quad \beta nds = NdS.$

Making these substitutions in p' and reducing, we obtain

$$p'_x ds = P_X dS - \frac{v}{4\pi c}[EH]_N dS,$$

where P_X is the component pressure per unit area on dS at rest as given by the electric and magnetic intensities E, H.

Likewise $$p'_y ds = \frac{1}{\beta} P_Y dS,$$

$$p'_z ds = \frac{1}{\beta} P_Z dS.$$

* *Annalen der Physik*, XXVI, No. 6 (1908).

† Cf. Abraham, *Theorie der Elektricität*, II, p. 333. Notice that in this expression

$$e' = e + [wh]/c, \quad h' = h - [we]/c,$$

so that e', h' are *not* the electric and magnetic intensities in the Lorentz-Einstein transformation.

Now pass to the consideration of the equilibrium radiation within a cavity in a body. The pressure is normal to the surface and the vector $\frac{c}{4\pi}[EH]$ vanishes at all points.

Thus $$P_X = LP, \quad P_Y = MP, \quad P_Z = NP.$$

Hence
$$p'_x ds = LPdS = Plds,$$
$$p'_y ds = \frac{1}{\beta} MPdS = Pmds,$$
$$p'_z ds = \frac{1}{\beta} NPdS = Pnds.$$

Thus the pressure in the moving cavity is in the direction (l, m, n), and is equal to P.

Let the expression for the energy of the field be now similarly treated.

$$\frac{1}{8\pi}(e^2+h^2) = \frac{1}{8\pi}\left\{E_X^2+H_X^2+\beta^2\left(1+\frac{v^2}{c^2}\right)(E_Y^2+H_Y^2+E_Z^2+H_Z^2) + \frac{4\beta^2 v}{c}(E_Y H_Z - E_Z H_Y)\right\}.$$

In the cavity at rest when the radiation is in equilibrium, the mean values of $(E_X^2+H_X^2)$, $(E_Y^2+H_Y^2)$, $(E_Z^2+H_Z^2)$ are each equal to $\frac{1}{3}(E^2+H^2)$, and that of $(E_Y H_Z - E_Z H_Y)$ is zero, the average being taken over any interval of time very small, but large compared with the periods of the constituent radiation.

Hence, denoting mean values by a stroke,

$$\frac{1}{8\pi}\overline{(e^2+h^2)} = \frac{1}{8\pi}\overline{(E^2+H^2)}\left\{\tfrac{1}{3}+\frac{2(c^2+v^2)}{3(c^2-v^2)}\right\},$$

or $$\bar{\epsilon} = \frac{3c^2+v^2}{3(c^2-v^2)}\bar{E}.$$

In identical manner, Planck's equation

$$\bar{g} = \frac{4v}{3(c^2-v^2)}\bar{E}$$

is obtained.

Finally, $$\bar{p} = \bar{P} = \tfrac{1}{3}\bar{E} = \frac{c^2-v^2}{3c^2+v^2}\bar{\epsilon},$$

$$\bar{g} = \frac{4v}{3c^2+v^2}\bar{\epsilon}.$$

II.

4. The group of conformal transformations in four dimensions referred to in the introduction can be built up of transformations by inversion in the hyperspheres of the space.

To establish the new theorem of relativity it is sufficient to consider a single member of the group, *i.e.*, a single inversion. Let the centre of inversion be taken as origin.

The analysis is best carried out in spherical polar coordinates, so that the geometrical correlation of the two systems is

$$R = \frac{k^2 r}{r^2 - c^2 t^2}, \qquad T = \frac{k^2 t}{r^2 - c^2 t^2},$$

the angular coordinates remaining unaltered.

Geometrical relations arising immediately are first given.

If u, U are the velocities of a moving point and the corresponding point in the other system,

$$U_R = \frac{(r^2 + c^2 t^2)\, u_r - 2c^2 rt}{2rtu_r - (r^2 + c^2 t^2)},$$

$$U_N = -\frac{(r^2 - c^2 t^2)\, u_n}{2rtu_r - r^2 + c^2 t^2}.$$

In particular, if v is the velocity in either system of a point corresponding to a stationary point in the other,

$$v = \frac{2c^2 rt}{r^2 + c^2 t^2},$$

and is a radial velocity.

As in the former transformation,

$$\beta = (1 - v^2/c^2)^{-\frac{1}{2}} = \frac{r^2 + c^2 t^2}{r^2 - c^2 t^2};$$

and, using this, the above equations become

$$U_R = -\frac{u_r - v}{1 - u_r v/c^2},$$

$$\beta U_N = \frac{u_n}{1 - u_r v/c^2}.$$

If we write $k^2 \Lambda = (R^2 - c^2 T^2)$, and consider a *small* volume at rest in the (r, t) system, we find that to an observer in the (R, T) system its dimensions radially are diminished in the ratio $-1 : \beta\Lambda$, while transversely they are diminished in the ratio $1 : \Lambda$.

Thus a minute sphere, in the one system at rest, is observed in the other to be an oblate spheroid in the other, the axis of revolution being in the direction of the radius R, and the ratio of the axes being $1 : \beta$.

Passing to the field equations of the electron theory, and putting them in terms of polar coordinates, it is found by direct transformation that the following transformation leaves them invariant, λ being written for $(r^2-c^2t^2)/k^2$:

$$E_R = \lambda^2 e_r,$$

$$E_\theta = \lambda^2\beta(-e_\theta+vh_\phi/c),$$

$$E_\phi = \lambda^2\beta(e_\phi-vh_\theta/c),$$

$$H_R = -\lambda^2 h_r,$$

$$H_\theta = \lambda^2\beta(h_\theta+ve_\phi/c),$$

$$H_\phi = \lambda^2\beta(h_\phi-ve_\theta/c),$$

$$\mathrm{P} = -\lambda^3\beta(1-u_r v/c^2)\,\rho,$$

together with the transformations in velocity and coordinates given above.

It is a consequence of this result that, if e, h, u, ρ are the electric and magnetic intensities, the velocity of convection, and the density of electricity in a given set of phenomena, then another set of phenomena consistent with the fundamental equations is expressed by the quantities E, H, U, P, so obtained if expressed in terms of the variables R and T. The relations between the two sets of quantities are completely reciprocal.

Further, there is exact correspondence of charges. For, if the ratio of a small element of volume $r^2 \sin\theta\,\delta r\,\delta\theta\,\delta\phi$ moving with velocity u, to the corresponding element $R^2 \sin\theta\,\delta R\,\delta\theta\,\delta\phi$ moving with velocity U be calculated, it is found to be

$$-\lambda^3(1-vu_r/c^2),$$

that is P/ρ.

Hence it is impossible for an observer to discriminate between the sequence of electromagnetic phenomena as he knows them and the sequence obtained therefrom by this transformation. So long as his measures of time and space are dependent on electrically constituted apparatus, the change in the space coordinates will be obscured by the change in the measure of time which will be automatically made.

It may be worth while to give the following forms of the transformation:

$$E_X = k^{-4}(r^2-c^2t^2)\left[(r^2-c^2t^2)\,e_x-2\left\{(y^2+z^2)\,e_x-xye_y-xze_z\right\}\right.$$
$$\left.-2ct\,(yh_z-zh_y)\right],\ \ldots,$$
$$H_X = k^{-4}(r^2-c^2t^2)\left[-(r^2-c^2t^2)\,h_x+2\left\{(y^2+z^2)\,h_x-xyh_y-xzh_z\right\}\right.$$
$$\left.-2ct\,(ye_z-ze_y)\right],$$

or, in vector notation,

$$E = k^{-4}(r^2-c^2t^2)\left\{(r^2-c^2t^2)\,e+2\,[r\,[re]]-2ct\,[rh]\right\},$$
$$H = k^{-4}(r^2-c^2t^2)\left\{-(r^2-c^2t^2)\,h-2\,[r\,[rh]]-2ct\,[re]\right\}.*$$

Invariants and Constitutive Equations under the Transformation.

5. The theorem of relativity being established, it is possible to deduce from any known set of phenomena another which geometrically is the transformation of it. The values of physical quantities in the new phenomena will be obtained from those in the old by the equations obtained in the last section. The results so obtained are quite analogous to those in § 2, and will be given below. Two quantities that are invariant will be mentioned first.†

The following equation is immediately established:

$$[E_R^2+E_\theta^2+E_\phi^2-H_R^2-H_\theta^2-H_\phi^2] = \lambda^4\,[e_r^2+e_\theta^2+e_\phi^2-h_r^2-h_\theta^2-h_\phi^2].$$

Thus, if l, L denote the differences of the magnetic energies in the two systems per unit volume,

$$L = \lambda^4 l.$$

Now $$\frac{\partial(R,\,T)}{\partial(r,\,t)} = \frac{1}{\lambda^2} \quad \text{and} \quad \frac{R}{r} = \frac{1}{\lambda^2}.$$

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* Mr. Bateman has suggested to me that the formulæ of transformation for the scalar and vector potentials are:—

$$\Phi = \beta\lambda\,\{va_r/c-\phi\},$$
$$A_R = \beta\lambda\,\{a_r-v\phi/c\},$$
$$A_N = -\lambda a_n.$$

† These invariant equations hold equally in the case of the Lorentz-Einstein transformation. Cf. Planck, *loc. cit.*, in reference to the invariance of the kinetic potential.

Hence $$LR^2 \sin\theta\, dR\, d\theta\, d\phi\, dT = lr^2 \sin\theta\, dr\, d\theta\, d\phi\, dt;$$

and therefore for a closed system, if we integrate through the whole volumes and corresponding times,

$$\iint L\, dV dT = \iint l\, dv\, dt.$$

Thus, if the electrodynamic equations are based on the principle of least action, the action being the time integral of the difference of the magnetic aad electric energies of the system, we may say that the *action* is invariant under the transformation.

The other invariant is expressed in the equation

$$E_R H_R + E_\theta H_\theta + E_\phi H_\phi = -\lambda^4 (e_r h_r + e_\theta h_\theta + e_\phi h_\phi).$$

Thus, if the electric and magnetic vectors are at right angles at any point in the one system, they are also at right angles at the corresponding point and instant in the other.

Consider now the conditions which hold at a reflecting surface which is at rest in the (r, t) system, viz.,

$$\frac{e_r}{l} = \frac{e_\theta}{m} = \frac{e_\phi}{n},$$

$$lh_r + mh_\theta + nh_\phi = 0,$$

where (l, m, n) are the direction cosines of the normal to the surface referred to the directions of (r, θ, ϕ) respectively.

The mechanical force per unit charge moving with the surface in the transformed system is

$$F = \left(E_r,\ E_\theta - \frac{v}{c} H_\phi,\ E_\phi + \frac{v}{c} H_\theta\right) = \lambda^2 \left(e_r,\ -\frac{e_\theta}{\beta},\ -\frac{e_\phi}{\beta}\right).$$

Hence $$\frac{F_r}{l} = \frac{F_\theta}{-m/\beta} = \frac{F_\phi}{-n/\beta}.$$

But if we consider corresponding areas ds, dS and their projections, we obtain

$$\frac{l\, ds}{L\, dS} = \frac{r^2}{R^2} = \lambda^2,$$

$$\frac{m\, ds}{M\, dS} = \frac{n\, ds}{N\, dS} = \frac{r\, \delta r}{R\, \delta R},$$

δr being an element of length at rest.

Hence $$\frac{m\,ds}{M\,dS} = \frac{n\,ds}{N\,dS} = -\lambda^2\beta,$$

so that $$\frac{l}{L} = \frac{-m/\beta}{M} = \frac{-n/\beta}{N} = \frac{1}{\kappa} \text{ (say)};$$

and therefore $$\frac{F_R}{L} = \frac{F_\theta}{M} = \frac{F_\phi}{N}.$$

Again, $$LH_r + MH_\theta + NH_\phi = -\lambda^2\kappa(lh_r + mh_\theta + nh_\phi),$$

since $$me_\phi - ne_\theta = 0.$$

Hence $$LH_r + MH_\theta + NH_\phi = 0.$$

Thus the conditions for a moving reflecting surface are satisfied at the transformed surface.

Thus a perfect reflector transforms into a perfect reflector.

6. There is no difficulty in adapting the analysis of § 2 to the new transformation if the following form of it is noticed.

The differential elements in the transformation are connected by the equation
$$\delta R = \lambda^{-1}\beta(-\delta r + v\,\delta t),$$
$$\delta T = \lambda^{-1}\beta\left(\delta t - \frac{v}{c^2}\,\delta r\right),$$

where, as before,
$$\lambda = (r^2 - c^2t^2)/k^2, \qquad v = 2c^2rt/(r^2+c^2t^2), \qquad \beta = (1 - v^2/c^2)^{-\frac{1}{2}}.$$

Thus within a small element of volume the space time coordinates are changed by a transformation of the same form as the Lorentz-Einstein, save for the magnification factor λ^{-1}, and a difference of sign, which itself disappears in a sequence of two such transformations, *e.g.*, in an infinitesimal transformation of the group.

The same is true of the fundamental equations connecting the magnetic and electric intensities in the two systems.

If, therefore, we confine our attention to a portion of matter contained within a volume which is small, but large enough to allow of the process of averaging commonly employed in molecular physics, we shall obtain results similar to those of the last section. It will be enough here to give the equations without the analysis, referring to the corresponding equations of previous sections.

The velocity of a moving point

$$\left.\begin{aligned} W_R &= -\frac{w_r - v}{1-\dfrac{vw_r}{c^2}} \\ W_N &= \frac{w_n}{\beta\left(1-\dfrac{vw_r}{c^2}\right)} \end{aligned}\right\}, \qquad (2')$$

where W_N is the velocity in any direction perpendicular to R.

The relative velocity of two moving points

$$\left.\begin{aligned} W'_R - W_R &= -\frac{w'_r - w_r}{\beta^2\left(1-\dfrac{vw_r}{c^2}\right)\left(1-\dfrac{vw'_r}{c^2}\right)} \\ W'_N - W_N &= \frac{w'_n - w_n}{\beta\left(1-\dfrac{vw_r}{c^2}\right)} - \frac{w'_n v\,(w'_r - w_r)}{\beta c^2\left(1-\dfrac{vw_r}{c^2}\right)\left(1-\dfrac{vw'_r}{c^2}\right)} \end{aligned}\right\}. \qquad (4')$$

The distance between two points (r, t), (r', t') at the time T corresponding to (r, t),

$$\left.\begin{aligned} \lambda\delta R &= -\frac{\delta r}{\beta(1-vw'_r/c^2)} \\ \lambda\delta N &= \delta n + \frac{vw'_n\,\delta r}{c^2(1-vw'_r/c^2)} \end{aligned}\right\}. \qquad (3')$$

The element of volume $\lambda^3\delta A = -\dfrac{\delta a}{\beta(1-vw_r/c^2)}$. (5')

The density of free electricity

$$-\lambda^{-3}\rho = \beta(1-vw_r/c^2)\,\rho - \beta v j_r/c^2. \qquad (6')$$

The current density

$$\left.\begin{aligned} \lambda^{-3}J_R &= \beta(1+vW_R/c^2)\,j_r \\ -\lambda^{-3}J_N &= j_n - \beta v W_N j_r/c^2 \end{aligned}\right\}. \qquad (7')$$

The polarization

$$\left.\begin{aligned} \lambda^{-2}P_R &= p_r \\ \lambda^{-2}P_\theta &= -\beta(1-vw_r/c^2)\,p_\theta - \beta v w_\theta p_r/c^2 - \beta v m_\phi/c \\ \lambda^{-2}P_\phi &= -\beta(1-vw_r/c^2)\,p_\phi - \beta v w_\phi p_r/c^2 + \beta v m_\theta/c \end{aligned}\right\}. \qquad (8')$$

The magnetization

$$\left.\begin{aligned} -\lambda^{-2} M_R &= m_r - \beta v\,(W_\theta m_\theta + W_\phi m_\phi)/c^2 \\ \lambda^{-2} M_N &= \beta(1 + v W_R/c^2)\, m_n \end{aligned}\right\}. \tag{9'}$$

7. The question now arises as to whether in a *ponderable* body the transformations above given between (E, H) and (e, h) stand as they are, or whether they should be really between (E, B) and (e, b). Examination shews that relativity is fully maintained if the latter is adopted in accordance with Mirimanoff's paper on the Lorentz-Einstein transformation. We know, changing (H, h) into (B, b) that the transformation obtained leaves invariant the equations

$$-\frac{1}{c}\frac{\partial b}{\partial t} = \text{curl } e,$$

$$0 = \text{div } b.$$

The transformation being now

$$\left.\begin{aligned} E_R &= \lambda^2 e_r, & B_R &= -\lambda^2 b_r \\ E_\theta &= \lambda^2\beta\,(-e_\theta + v b_\phi/c), & B_\theta &= \lambda^2\beta(b_\theta + v e_\phi/c) \\ E_\phi &= \lambda^2\beta\,(-e_\phi - v b_\theta/c), & B_\phi &= \lambda^2\beta(b_\phi - v e_\theta/c) \end{aligned}\right\}, \tag{10'}$$

with the use of (8′), (9′), the following equations are deduced :

$$\left.\begin{aligned} D_R &= \lambda^2 d_r, & Q_R &= -\lambda^2 q_r \\ D_\theta &= \lambda^2\beta(-d_\theta + v q_\phi/c), & Q_\theta &= \lambda^2\beta(q_\theta + v d_\phi/c) \\ D_\phi &= \lambda^2\beta(d_\phi - v q_\theta/c), & Q_\phi &= \lambda^2\beta(q_\phi - v d_\theta/c) \end{aligned}\right\}, \tag{11'}$$

where the new quantities are defined by

$$D = E + P, \qquad d = e + p,$$

$$B = M + H, \qquad b = m + h,$$

$$Q = H - \frac{1}{c}[PU], \quad q = h - \frac{1}{c}[pu].$$

These together with (6′), (7′) will, by comparison with the transformation in its original form, leave unchanged the equations

$$\frac{1}{c}\left(\frac{\partial d}{\partial t} + \rho u\right) = \text{curl } q,$$

$$\rho = \text{div } d.$$

These equations expanded become the second and third of Lorentz' equations for ponderable bodies.

If, now, at a point $(xyzt)$ the velocity u vanishes, and we assume that consequently $d = \epsilon e$, $b = \mu h$, we at once obtain from (7′), (10′), (11′) equations identical with (10), (11), (12), § 2.

Thus, if we assume those equations together with Lorentz' fundamental equations, to be the correct scheme of the electrodynamics of moving bodies, we arrive at the conclusion that that scheme is invariant, not only under the extension of the transformation of Lorentz and Einstein given by Mirimanoff, but also under the transformation obtained by inversion in the four dimensional space here developed, and therefore also under a transformation obtained by combining any number of such operations.

The Transformation of the Conditions holding at a Surface of Discontinuity.

8. In a paper in the *Annalen der Physik*,* starting from the equation

$$\frac{1}{c}\left(\frac{\partial D}{\partial t}+S\right) = \operatorname{curl} H,$$

Einstein and Laub deduce that the conditions to be satisfied at a surface of discontinuity in the material media are that the following quantities must be continuous in crossing the surface :

$$D_n,\quad \left\{H+\frac{1}{c}[DW]\right\}_{\bar{n}},\quad B_n,\quad \left\{E-\frac{1}{c}[BW]\right\}_{\bar{n}},$$

n denoting the direction of the normal, and $\bar{n}$ of a tangent line to the surface. S, as in Minkowski's paper, denotes the sum of the conduction and convection currents.

If we apply the extended transformation to these conditions, we find that they are not conserved. But if we take the Lorentz equation which is obtained from that above by the substitution of the vector

$$Q = H-\frac{1}{c}[PW] \text{ for } Q,$$

the conditions are that

$$D_n,\quad \left\{Q+\frac{1}{c}[DW]\right\}_{\bar{n}},\quad B_n,\quad \left\{E-\frac{1}{c}[BW]\right\}_{\bar{n}}$$

* A. Einstein und J. Laub, 1909, 2, p. 446.

shall be continuous, provided there is no true electricity on the surface of separation.

There is no difficulty in proving that these conditions are conserved under Mirimanoff's extended transformation.

It will be proved now that they are also conserved under the transformation by inversion, and therefore by the most general transformation of the whole group which leaves the fundamental equations invariant.

From the equations (13'), we obtain if we write

$$H' = H+\frac{1}{c}[EW] = Q+\frac{1}{c}[DW],$$

$$\left.\begin{aligned} H'_R &= -\frac{\lambda^2}{1-\frac{vw_r}{c^2}}\left\{h'_r-\frac{v}{c^2}(w_r h'_r+w_\theta h'_\theta+w_\phi h'_\phi)\right\} \\ H'_\theta &= \frac{\lambda^2}{\beta\left(1-\frac{vw_r}{c^2}\right)}h'_\theta \\ H'_\phi &= \frac{\lambda^2}{\beta\left(1-\frac{vw_r}{c^2}\right)}h'_\phi \end{aligned}\right\}. \tag{14'}$$

If we consider an element of length (δs, δS) of which the components are (δr, δn), (δR, δN) in the two systems respectively, δn, δN being the components perpendicular to r, R respectively, we have equations analogous to (4),

$$\delta R = -\frac{\lambda^{-1}\delta r}{\beta\left(1-\frac{vw_r}{c^2}\right)},$$

$$\delta N = \lambda^{-1}\left\{\delta n+\frac{vw_n\delta_r}{c^2\left(1-\frac{vw_r}{c^2}\right)}\right\}.$$

Hence

$$H'_R\delta R+H'_N\delta N$$

$$=\frac{\lambda}{\beta\left(1-\frac{vw_r}{c^2}\right)}\left\{\frac{h'_r\delta_r}{1-\frac{vw_r}{c^2}}+h'_n\delta_n-\frac{v(w_r h'_r+w_n h'_n)}{c^2\left(1-\frac{vw_r}{c^2}\right)}+\frac{vw_n h'_n}{c^2\left(1-\frac{vw_r}{c^2}\right)}\right\}$$

$$=\frac{\lambda}{\beta\left(1-\frac{vw_r}{c^2}\right)}\{h'_r\delta_r+h'_r\delta_n\},$$

or
$$H'_S\,\delta S = \frac{\lambda}{\beta\left(1-\frac{vw_r}{c^2}\right)}\,h'_s\,\delta s.$$

Thus, since $\delta S/\delta s$ does not depend on the physical properties of the medium, and is consequently continuous in crossing the surface of discontinuity, H' will be continuous in those directions in which h' is continuous, and discontinuous in those directions in which h' is discontinuous.

Exactly the same is to be proved of $E' = E - \frac{1}{c}\,[BW]$ by means of (10′). Taking the component of D perpendicular to the same element, we have

$$D_R\,\delta N - D_N\,\delta R = \lambda\,d_r\left\{\delta_n + \frac{vw_n\,\delta_r}{c^2\left(1-\frac{vw_r}{c^2}\right)}\right\}$$
$$-\frac{\lambda\,\delta r}{1-\frac{vw_r}{c^2}}\left\{d_n\left(1-\frac{vw_r}{c^2}\right)+\frac{v}{c^2}\,d_r w_n - \frac{v}{c}\,h'_{n_1}\right\},$$

where h'_{n_1} denotes the component of h' in the direction perpendicular to r and n.

Thus, on simplifying,

$$D_R\,\delta N - D_N\,\delta R = \lambda(d_r\,\delta_n - d_n\,\delta r) - \frac{\lambda v\,\delta r\,h'_{n_1}}{c\left(1-\frac{vw_r}{c^2}\right)}.$$

Thus D will be continuous in any direction in which d is continuous, provided the component of h' perpendicular to that direction is continuous.

Hence, with what has just been proved as to the continuity of h', it follows that the condition of the continuity of the normal component of d is conserved; and in the same manner it may be shewn of the normal component of b.

The Pressure, Energy, and Momentum, under the Generalized Conformal Transformation.

9. Let the physical quantities considered in § 3 be now treated in the light of the inversion in four-dimensional space.

If the elements ds, dS are corresponding elements of area of which dS is at rest in the (R, T) system, and (l, m, n), (L, M, N) are the direction cosines of the respective normals, by considering the projections of the areas, we have

$$\frac{l\,ds}{L\,dS} = \frac{r^2}{R^2} = \frac{1}{\Lambda^2},$$

where

$$\Lambda = \frac{r^2 - c^2t^2}{k^2},$$

$$\frac{n\,ds}{N\,dS} = \frac{m\,ds}{M\,dS} = \frac{r\,\delta r}{R\,\delta R},$$

in which δr, δR are corresponding elements of a radius, δR being at rest in the (R, T) system.

Hence

$$\frac{\delta r}{\delta R} = -\frac{1}{\Lambda\beta},$$

and

$$\frac{m\,ds}{M\,dS} = \frac{n\,ds}{N\,dS} = -\frac{1}{\Lambda^2\beta}.$$

Using these and the final form of the transformation of the electric vectors in § 3, the following equations are obtained exactly as in the last section:

$$p_r\,ds = \Lambda^2 \left\{ P_r - \frac{v}{4\pi c}[EH]_\nu \right\} dS,$$

$$p_\theta\,ds = -\Lambda^2 P_\theta\,dS/\beta,$$

$$p_\phi\,ds = -\Lambda^2 P_\phi\,dS/\beta,$$

p and P being the mechanical forces per unit area according as the element is observed in the (r, t) or (R, T) system.

If the element dS belongs to the wall of a region in which there is radiation in equilibrium $[EH]_\nu$ is zero as before if the time-average be taken, and P is normal to dS.

Thus

$$p_r\,ds = \Lambda^2 PL\,dS = \Lambda^4 Pl\,ds,$$

$$p_\theta\,ds = -\frac{\Lambda^2}{\beta} PM\,dS = \Lambda^4 Pm\,ds,$$

$$p_\phi\,ds = -\frac{\Lambda^2}{\beta} PN\,dS = \Lambda^4 Pn\,ds.$$

Thus p is normal to ds and is equal to $\Lambda^4 P$.

For the energy the transformation gives

$$\epsilon = \frac{1}{8\pi}\{e^2 + h^2\}$$

$$= \frac{\Lambda^4}{8\pi}\left\{E_R^2 + H_R^2 + \beta^2\left(1 + \frac{v^2}{c^2}\right)(E_\theta^2 + E_\phi^2 + H_\theta^2 + H_\phi^2) - \frac{4v\beta^2}{c}(E_\theta H_\phi - E_\phi H_\theta)\right\},$$

so that for the radiation in equilibrium

$$\epsilon = \frac{\Lambda^4}{3} E \left\{ 1+2\frac{c^2+v^2}{c^2-v^2} \right\} = \Lambda^4 \frac{3c^2+v^2}{3(c^2-v^2)} E.$$

Thus
$$p = \Lambda^4 P = \tfrac{1}{3}\Lambda^4 E = \frac{c^2-v^2}{3c^2+v^2}\epsilon.$$

This is the same equation as that obtained in § 3. Exactly in the same way for the momentum is found the result

$$g = \tfrac{4}{3}\frac{\Lambda^4 v}{c^2-v^2} E = \frac{4v}{3c^2+v^2}\epsilon.$$