KUMMER'S QUARTIC SURFACE AS A WAVE SURFACE

By H. BATEMAN.

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1. It has been known for some time that Fresnel's wave surface is a case of Kummer's sixteen nodal quartic surface; also it is known that the wave surface of a dynamical medium possessing certain general properties is a type of Kummer's surface which can be derived from Fresnel's surface by means of a homogeneous strain.*

The Kummer's surfaces which occur in this way are of a particular type in which the six nodes on any singular conic belong to an involution. The lines joining corresponding points of the involution meet in a point which is one of the vertices of a certain tetrahedron, each face of which cuts the surface in two conics intersecting in four nodes.† This type of surface is called the tetrahedroid.‡

The general Kummer's surface appears to be the wave surface for a medium of a purely ideal character in which the disturbance at any point is specified by means of four vectors, whose components are connected with one another by six linear relations of a certain type and satisfy a system of partial differential equations analogous to the electrodynamical equations.

These conditions may not correspond to anything occurring in nature; nevertheless their investigation was thought to be of some mathematical interest on account of the connection which is established between line-geometry and the theory of partial differential equations.

The system of partial differential equations under consideration is of special interest on account of the fact that when the four vectors occurring in them are regarded as independent, the equations are invariant for any transformation of coordinates.

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* T. J. I'A. Bromwich, *Proc. London Math. Soc.*, Ser. 1, Vol. XXXIV (1902), p. 307. A less general case had been studied previously by Macdonald. *Ibid.*, Vol. XXXII (1900), p. 311.

† Hudson's *Kummer's Quartic Surface*, Ch. IX.

‡ Cayley, *Coll. Papers*, I, 302; *Liouville's Journal*, t. XI (1846), p. 291.

2. Let a line in space be specified by six coordinates $(l, m, n, \lambda, \mu, \nu)$ satisfying the relations

$$\begin{aligned} \lambda+mz-ny &= 0, \\ \mu+nx-lz &= 0, \\ \nu+ly-mx &= 0, \end{aligned} \qquad \left.\begin{aligned} \lambda x+\mu y+\nu z &= 0 \\ l\lambda+m\mu+n\nu &= 0 \end{aligned}\right\}, \qquad \cdot \qquad (1)$$

where (x, y, z) are the coordinates of any point on the line.

Let the equation of a quadratic complex be

$$\Phi = \kappa\,(l, m, n, \lambda, \mu, \nu)^2 = 0,$$

where κ is a symmetrical matrix of six rows and columns. The point (x, y, z) is a singular point if the cone of lines belonging to the complex which pass through this point consists of two plane pencils. Let

$$(l, m, n, \lambda, \mu, \nu), \quad (l', m', n', \lambda', \mu', \nu')$$

be two lines belonging to the same pencil, then the line

$$(l+\theta l', \; m+\theta m', \; n+\theta n', \; \lambda+\theta\lambda', \; \mu+\theta\mu', \; \nu+\theta\nu')$$

must belong to the complex for all values of θ.

Substituting in the equation of the complex, we obtain

$$l'\frac{\partial\Phi}{\partial l}+m'\frac{\partial\Phi}{\partial m}+n'\frac{\partial\Phi}{\partial n}+\lambda'\frac{\partial\Phi}{\partial\lambda}+\mu'\frac{\partial\Phi}{\partial\mu}+\nu'\frac{\partial\Phi}{\partial\nu} = 0. \qquad (2)$$

Now let $(l, m, n, \lambda, \mu, \nu)$ be the line common to the two pencils having their vertices at the point (x, y, z); then we may obtain three linearly independent equations of the type (2), by choosing successively for $(l', m', n', \lambda', \mu', \nu')$, two lines belonging to one pencil and one line belonging to the other. The solution of these three linear equations is necessarily a linear combination of three particular solutions. Now we may obtain three particular solutions by taking

$$\left(\frac{\partial\Phi}{\partial\lambda}, \frac{\partial\Phi}{\partial\mu}, \frac{\partial\Phi}{\partial\nu}, \frac{\partial\Phi}{\partial l}, \frac{\partial\Phi}{\partial m}, \frac{\partial\Phi}{\partial n}\right)$$

successively proportional to the six coordinates of any three lines through the point (x, y, z), for then equations (2) are satisfied identically in virtue of the fact that two lines intersect. A linear combination of the coordinates of three lines through the point (x, y, z), however, represent the coordinates of some line through this point, and so every set of values of

$$\left(\frac{\partial\Phi}{\partial\lambda}, \frac{\partial\Phi}{\partial\mu}, \frac{\partial\Phi}{\partial\nu}, \frac{\partial\Phi}{\partial l}, \frac{\partial\Phi}{\partial m}, \frac{\partial\Phi}{\partial n}\right)$$

which satisfy the three equations of the type (2) represent the coordinates of a line through the point (x, y, z).

Putting $$l_0 = \frac{\partial \Phi}{\partial \lambda}, \quad \lambda_0 = \frac{\partial \Phi}{\partial l}, \quad \ldots,$$

we have the system of equations

$$\begin{aligned} \lambda + mz - ny &= 0, & \lambda_0 + m_0 z - n_0 y &= 0, \\ \mu + nx - lz &= 0, & \mu_0 + n_0 x - l_0 z &= 0, \\ \nu + ly - mx &= 0, & \nu_0 + l_0 y - m_0 x &= 0. \end{aligned}$$

These are the conditions that (x, y, z) should be a singular point of the complex, and $(l, m, n, \lambda, \mu, \nu)$ a singular line.

The equation of the complex to which $(l_0, m_0, n_0, \lambda_0, \mu_0, \nu_0)$ belongs may be obtained by solving the equations

$$l_0 = \frac{\partial \Phi}{\partial \lambda}, \quad \lambda_0 = \frac{\partial \Phi}{\partial l}, \quad \ldots,$$

for $(l, m, n, \lambda, \mu, \nu)$, and substituting in the equation

$$l\lambda_0 + l_0\lambda + m\mu_0 + m_0\mu + n\nu_0 + n_0\nu = 0.$$

Denoting the equation of this complex by Ω, we have

$$\frac{\partial \Omega}{\partial \lambda_0} = l + \lambda_0 \frac{\partial l}{\partial \lambda_0} + l_0 \frac{\partial \lambda}{\partial \lambda_0} + \mu_0 \frac{\partial m}{\partial \lambda_0} + m_0 \frac{\partial \mu}{\partial \lambda_0} + \nu_0 \frac{\partial n}{\partial \lambda_0} + n_0 \frac{\partial \nu}{\partial \lambda_0};$$

but, if $$(l, \lambda) = K(l_0, \lambda_0),$$

we have $$\Omega = K(l_0, m_0, n_0, \lambda_0, \mu_0, \nu_0)^2,$$

and it follows that

$$\lambda_0 \frac{\partial l}{\partial \lambda_0} + l_0 \frac{\partial \lambda}{\partial \lambda_0} + \mu_0 \frac{\partial m}{\partial \lambda_0} + m_0 \frac{\partial \mu}{\partial \lambda_0} + \nu_0 \frac{\partial n}{\partial \lambda_0} + n_0 \frac{\partial \nu}{\partial \lambda_0} = \tfrac{1}{2} \frac{\partial \Omega}{\partial \lambda_0};$$

hence $$\frac{\partial \Omega}{\partial \lambda_0} = 2l,$$

and so $$\left(\frac{\partial \Omega}{\partial \lambda_0}, \frac{\partial \Omega}{\partial \mu_0}, \frac{\partial \Omega}{\partial \nu_0}, \frac{\partial \Omega}{\partial l_0}, \frac{\partial \Omega}{\partial m_0}, \frac{\partial \Omega}{\partial n_0}\right)$$

are proportional to the six coordinates of a line. This shows that (x, y, z) is a singular point of the complex Ω, and so the two complexes Ω and Φ have the same singular surface.

It is known that the singular line which is the intersection of the planes of the two pencils associated with a singular point (x, y, z) of the complex Φ is a tangent to the Kummer's surface which is the locus of the singular points of Φ;* hence it follows that $(l, m, n, \lambda, \mu, \nu)$, $(l_0, m_0, n_0, \lambda_0, \mu_0, \nu_0)$ are both tangents to this surface.

3. Let B, E, H, D be four vectors whose components satisfy a system of linear partial differential equations of the type

$$\left.\begin{aligned}
\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} &= \frac{\partial D_x}{\partial t}, & \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -\frac{\partial B_x}{\partial t} \\
\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} &= \frac{\partial D_y}{\partial t}, & \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -\frac{\partial B_y}{\partial t} \\
\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= \frac{\partial D_z}{\partial t}, & \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -\frac{\partial B_z}{\partial t} \\
\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} &= 0, & \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} &= 0.
\end{aligned}\right\}. \tag{I}$$

These equations are invariant for any transformation of coordinates from (x, y, z, t) to (x', y', z', t') provided we define the corresponding vectors B', E', H', D' by means of the equations

$$B_x dy\,dz + B_y dz\,dx + B_z dx\,dy + E_x dx\,dt + E_y dy\,dt + E_z dz\,dt$$
$$= B'_x dy'dz' + B'_y dz'dx' + B'_z dx'dy' + E'_x dx'dt' + E'_y dy'dt' + E'_z dz'dt',$$

$$D_x dy\,dz + D_y dz\,dx + D_z dx\,dy - H_x dx\,dt - H_y dy\,dt - H_z dz\,dt$$
$$= D'_x dy'dz' + D'_y dz'dx' + D'_z dx'dy' - H'_x dx'dt' - H'_y dy'dt' - H'_z dz'dt',$$

where a term such as $dy\,dz$ is interpreted to mean

$$\frac{\partial(y, x)}{\partial(\alpha, \beta)}\, d\alpha\, d\beta,$$

when x, y, z, t are expressed in an arbitrary manner as functions of two parameters α, β.

The invariance of the fundamental equations (I) is then a direct consequence of the equations of type†

$$\iint (B_x dy\,dz + B_y dz\,dx + B_z dx\,dy + E_x dx\,dt + E_y dy\,dt + E_z dz\,dt)$$
$$= \iiint \left[\left(\frac{\partial B_x}{\partial t} + \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) dy\,dz\,dt + \left(\frac{\partial B_y}{\partial t} + \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) dz\,dx\,dt\right.$$
$$\left. + \left(\frac{\partial B_z}{\partial t} + \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) dx\,dy\,dt + \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) dx\,dy\,dz\right].$$

* Jessop's *Treatise on the Line Complex*, p. 91.

† H. F. Baker, *Camb. Phil. Trans.*, Vol. XVIII, p. 442.

We shall now study the system of differential equations (I) in connection with a set of six constitutive relations connecting the components of the different vectors. These relations may be regarded as defining the properties of a medium through which disturbances can be propagated.

The form which these relations take after a transformation of coordinates depends on the nature of the transformation; this is only natural because, in general, a transformation of coordinates alters the properties of a medium.

The constitutive relations which will be studied here are of the type

$$\left.\begin{aligned} -B_x &= \kappa_{11}H_x+\kappa_{12}H_y+\kappa_{13}H_z+\kappa_{14}D_x+\kappa_{15}D_y+\kappa_{16}D_z \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ E_x &= \kappa_{41}H_x+\kappa_{42}H_y+\kappa_{43}H_z+\kappa_{44}D_x+\kappa_{45}D_y+\kappa_{46}D_z \end{aligned}\right\}, \tag{II}$$

where the coefficients κ_{rs} are arbitrary constants satisfying the relations

$$\kappa_{rs} = \kappa_{sr} \quad (r = 1, 2, 3, 4, 5, 6;\ s = 1, 2, 3, 4, 5, 6).$$

To obtain the form of an elementary wave issuing from a point of the medium specified by these equations, we write $H+\frac{\partial H}{\partial \theta}\delta\theta$, instead of H, $\frac{\partial H}{\partial x}+\frac{\partial H}{\partial \theta}\frac{\partial \theta}{\partial x}$, instead of $\frac{\partial H}{\partial x}$, and similarly for the other quantities.

Putting

$$(l, m, n, \lambda, \mu, \nu) \equiv \left(\frac{\partial H_x}{\partial \theta}, \frac{\partial H_y}{\partial \theta}, \frac{\partial H_z}{\partial \theta}, \frac{\partial D_x}{\partial \theta}, \frac{\partial D_y}{\partial \theta}, \frac{\partial D_z}{\partial \theta}\right),$$

$$(l', m', n', \lambda', \mu', \nu') \equiv \left(\frac{\partial E_x}{\partial \theta}, \frac{\partial E_y}{\partial \theta}, \frac{\partial E_z}{\partial \theta}, -\frac{\partial B_x}{\partial \theta}, -\frac{\partial B_y}{\partial \theta}, -\frac{\partial B_z}{\partial \theta}\right),$$

$$\xi = \frac{\partial \theta}{\partial x}, \quad \eta = \frac{\partial \theta}{\partial y}, \quad \zeta = \frac{\partial \theta}{\partial z}, \quad \tau = \frac{\partial \theta}{\partial \tau},$$

after the other substitutions have been made in (I) and (II), we obtain the system of equations

$$\lambda+mz-ny = 0, \qquad \lambda'+m'z-n'y = 0,$$

$$\lambda x+\mu y+\nu z = 0, \qquad \lambda' x+\mu' y+\nu' z = 0,$$

where $$x = \frac{\xi}{\tau}, \quad y = \frac{\eta}{\tau}, \quad z = \frac{\zeta}{\tau},$$

and $$(\lambda', \mu', \nu', l', m', n') = \kappa(l, m, n, \lambda, \mu, \nu), \tag{III}$$

or $$l' = \frac{\partial \Phi}{\partial \lambda}, \quad \lambda' = \frac{\partial \Phi}{\partial l}.$$

Now, as was shown in § 2, these equations imply that the point (x, y, z) is a singular point of the complex Φ, and accordingly that $\left(\frac{\partial\theta}{\partial x}, \frac{\partial\theta}{\partial y}, \frac{\partial\theta}{\partial z}, \frac{\partial\theta}{\partial t}\right)$ satisfy a homogeneous equation of degree 4, which is the equation of Kummer's surface.

This equation represents the differential equation of the characteristics of the system of equations which determine the components of the vectors H and D. For, if $\theta(x, y, z, t) = \text{const.}$ represents a system of characteristics, and ϕ, ψ, χ are three other functions of x, y, z, t, which are chosen so that (x, y, z, t) can be expressed as functions of the four coordinates θ, ϕ, ψ, χ, then the problem of determining the values of the vectors H and D, when their values are given on the characteristic, must be indeterminate. In other words, when the equations are expressed in terms of the variables $(\theta, \phi, \psi, \chi)$ in place of (x, y, z, t), it must not be possible to solve the equations and obtain the values of the derivatives

$$\frac{\partial H_x}{\partial\theta}, \quad \frac{\partial D_x}{\partial\theta}, \quad \ldots .$$

Now the condition that this should be the case is that the determinant of the coefficients of these quantities in the transformed equations should vanish.

Putting
$$\frac{\partial H_x}{\partial\theta} = h_x, \quad \frac{\partial D_x}{\partial\theta} = d_x, \quad \ldots,$$

and substituting in equations (I) and (II), we obtain

$$h_z\frac{\partial\theta}{\partial y} - h_y\frac{\partial\theta}{\partial z} + \ldots = d_x\frac{\partial\theta}{\partial t} + \ldots,$$

$$(\kappa_{61}h_x + \kappa_{62}h_y + \kappa_{63}h_z + \kappa_{64}d_x + \kappa_{65}d_y + \kappa_{66}d_z)\frac{\partial\theta}{\partial y}$$

$$-(\kappa_{51}h_x + \kappa_{52}h_y + \kappa_{53}h_z + \kappa_{54}d_x + \kappa_{55}d_y + \kappa_{56}d_z)\frac{\partial\theta}{\partial z} + \ldots$$

$$= (\kappa_{11}h_x + \kappa_{12}h_y + \kappa_{13}h_z + \kappa_{14}d_x + \kappa_{15}d_y + \kappa_{16}d_z)\frac{\partial\theta}{\partial t} + \ldots .$$

The determinant of the coefficient of $h_x, h_y, h_z, d_x, d_y, d_z$ vanishes, if quantities $(l, m, n, \lambda, \mu, \nu)$, $(l', m', n', \lambda', \mu', \nu')$ can be found, so that equations (III) are satisfied. The reasoning is then the same as before.

4. Having obtained the differential equation for the characteristics, we must now proceed to integrate it. Regarding (x, y, z, t) for the moment

as the coordinates of a point in a space of four dimensions, the equation

$$\theta(x, y, z, t) = \text{const.}$$

represents a variety, and

$$(X-x)\frac{\partial\theta}{\partial x}+(Y-y)\frac{\partial\theta}{\partial y}+(Z-z)\frac{\partial\theta}{\partial z}+(T-t)\frac{\partial\theta}{\partial t}=0,$$

the equation of the tangent threefold at the point (x, y, z, t). The coefficients $\frac{\partial\theta}{\partial x}, \frac{\partial\theta}{\partial y}, \frac{\partial\theta}{\partial z}, \frac{\partial\theta}{\partial t}$, however, are connected by a *homogeneous* equation of the fourth degree; hence the equation $\theta = \text{const.}$ represents a cone, and since the reciprocal of a Kummer's surface is a Kummer's surface, it follows that the sections of this cone by a system of parallel three folds given by $t = \text{const.}$ are a system of similar Kummer's surfaces.

Now it is known that the series of surfaces obtained by giving different constant values to t in the equation of a characteristic

$$\theta(x, y, z, t) = \text{const.}$$

are the successive wave fronts of an elementary disturbance in the medium whose properties are specified by the given system of differential equations;* hence in the case under consideration the elementary wave surfaces are Kummer's quartics.

To make matters clear, we shall assume that at the wave front the derivatives of the vectors D, H, E, B are discontinuous. This implies that there is more than one set of possible values for the quantities $\frac{\partial D}{\partial\theta}, \frac{\partial H}{\partial\theta}, \ldots$, and this is exactly what was assumed when we formed the equations of the characteristics.

The two sets of values of $\frac{\partial H_x}{\partial\theta}, \ldots, \frac{\partial D_x}{\partial\theta}, \ldots$, differ by quantities $l, m, n, \lambda, \mu, \nu$, which are the coordinates of a line belonging to a certain quadratic complex, while the two sets of values of $\frac{\partial E_x}{\partial\theta}, \ldots, \frac{\partial B_x}{\partial\theta}, \ldots$, differ by quantities $l'', m'', n'', \lambda'', \mu'', \nu''$, which are the coordinates of a line belonging to a cosingular quadratic complex. If we are dealing with a pulse of small thickness $\delta\theta$, the quantities $(l\delta\theta, m\delta\theta, n\delta\theta)$, $(\lambda\delta\theta, \mu\delta\theta, \nu\delta\theta)$, ..., may be interpreted as the vectors belonging to the pulse. Denoting these by (H'_x, H'_y, H'_z), (D'_x, D'_y, D''_z), ..., we see that

* See a paper by T. H. Havelock, "Wave Fronts considered as the Characteristics of Partial Differential Equations," *Proc. London Math. Soc.*, Ser. 2, Vol. 2 (1904), p. 297.

they satisfy the relations

$$D'_x H'_x + D'_y H'_y + D'_z H'_z = 0\,;$$

and are therefore at right angles. The vector D' also satisfies the relation

$$D'_x \frac{\partial \theta}{\partial x} + D'_y \frac{\partial \theta}{\partial y} + D'_z \frac{\partial \theta}{\partial z} = 0,$$

and so lies in the tangent plane to the wave. In the same way it can be shown that the vector B lies in the tangent plane. On account of the correspondence between the Kummer's surface forming the wave front and the Kummer's surface described by the point $\left(\frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}, \frac{\partial \theta}{\partial z}, \frac{\partial \theta}{\partial t}\right)$, it appears that the two vectors in the wave front correspond to singular lines of two cosingular complexes.