# THE CONFORMAL TRANSFORMATIONS OF A SPACE OF FOUR DIMENSIONS AND THEIR APPLICATIONS TO GEOMETRICAL OPTICS

*By* H. BATEMAN.

[Received October 9th, 1908.—Read November 12th, 1908.]

CONTENTS.



1. The method of inversion which was first applied to problems in electrostatics by Lord Kelvin,* and which forms the basis of his theory of electric images, has also been applied with success in other branches of mathematical physics, as, for instance, in hydrodynamics. In geometrical optics, however, the method has been seldom used, probably because the necessary developments are not to be found in books on geometrical optics. The object of this paper is to show that the method can be of real value in both geometrical and physical optics. It is found that the transformation which is really needed is an inversion in a space of four dimensions, the transition to three-dimensional space being made by replacing the fourth coordinate by $ict$, where $t$ is the time and $c$ the velocity of light.

The first part of the paper is devoted to the general conformal transformation of a space of four dimensions. Shortly after Lord Kelvin's discovery of the method of transforming electrostatical problems by means of inversion,† Liouville‡ obtained the most general transformation that can be used for three-dimensional problems in this way.

---

* In a letter to Liouville dated October 8th, 1845. Liouville's *Journal de Mathématiques* 1845).

† The method of inversion had been used in geometry some time before. It apparently originated with Ptolemy. Quetelet used it in 1827 and Bellavatis gave a general statement of it in 1836. In 1843-4 it was propounded afresh by Ingram and Stubbs (*Transactions of the Dublin Philosophical Society*, Vol. I., pp. 58, 145, 159 ; *Philosophical Magazine*, Vol. XXIII., p. 338, Vol. XXV., p. 208).

‡ *Journal de Mathématiques* (1845) ; T. XV. (1850), p. 103.

The group of tranformations of this kind is known as the group of conformal transformations of space,* it preserves the angles between two surfaces and changes a sphere into either a sphere or a plane.†

The property, however, upon which the applications to electrostatical problems depends is that the transformations enable us to pass from one solution of the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$$

to another.‡

Now the group of conformal transformations in a space of four dimensions possesses the analogous property in connection with the two differential equations

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=\frac{1}{c^2}\,\frac{\partial^2 V}{\partial t^2},$$

$$\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2=\frac{1}{c^2}\left(\frac{\partial V}{\partial t}\right)^2,$$

which are fundamental in the wave theory of light.

This has been known for some time,§ but the analysis given in § 2 will be useful in indicating the procedure to be adopted to obtain the relation connecting the two solutions for any transformation of the group.

In § 3 a particular solution of the first of the above equations is

---

* A simple method of obtaining the group of conformal transformations is given in Bianchi's *Vorlesungen über Differential Geometrie*, Leipzig (1899), p. 487. Another investigation is given in Maxwell's *Collected Papers*, Vol. II., p. 297, where reference is made to a paper by J. N. Haton de Goupillière, *Journal de l'Ecole Polytechnique*, T. XXV., p. 188 (1867). See also a paper by Bromwich, *Proc. London Math. Soc.*, Vol. XXXIII., p. 185, and three papers by Tait, *Collected Papers*, Vol. I., pp. 176, 352, Vol. II., p. 329.

† The effect of combining the elementary transformations of the group is discussed by Darboux, *Une Classe remarquable de courbes et de surfaces algébriques*, Paris (1896), pp. 236–241. It is shown that any number of successive inversions can be replaced by a single inversion followed by a displacement. It follows from this that any conformal transformation of the group can be replaced by successive inversions with regard to suitably chosen spheres, Cf. *Math. Tripos*, Part I. (1903).

‡ In this connection see a paper by Forsyth, *Proceedings of the London Mathematical Society*, Vol. XXIX. (1898), p. 165. The tranformations which can be applied to the equation

$$\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2=0$$

are derived by J. E. Campbell, *Messenger of Mathematics*, Vol. XXVIII. (1898), p. 97.

§ Liouville's theorem was extended by Lie to a space of $n$ dimensions in 1871. *Math. Ann.*, Vol. V., p. 145; *Göttinger Nachrichten*, May, 1871. I cannot, however, find any statement with regard to the first of the two equations.

expressed in terms of Riemann's general hypergeometric function, and new light is thrown upon the theory of the transformations of the hypergeometric equation into itself.

In § 4 the applications to geometrical optics are considered. When applied to a symmetrical optical instrument, the transformation reduces to a homographic transformation of the points on the axis.

## 2. *The Conformal Transformations of a Space of Four Dimensions.*

The study of the conformal transformations of a space of four dimensions is simplified by the introduction of the six homogeneous coordinates* :

$$\left.\begin{array}{lll} l = x-iy, & m = z+iw, & n = x^2+y^2+z^2+w^2 \\ \lambda = x+iy, & \mu = z-iw, & \nu = -1 \end{array}\right\} \qquad (1)$$

connected by the identical relation

$$l\lambda+m\mu+n\nu = 0. \qquad (2)$$

A function $F(x, y, z, w)$ can be expressed by means of them as a homogeneous function of arbitrary degree. For instance, we may write

$$V = F(x, y, z, w) = F\left(-\frac{l+\lambda}{2\nu}, -\frac{l-\lambda}{2i\nu}, -\frac{m+\mu}{2\nu}, -\frac{m-\mu}{2i\nu}\right),$$

$$U = f(x, y, z, w) = -\frac{1}{\nu} f\left(-\frac{l+\lambda}{2\nu}, -\frac{l-\lambda}{2i\nu}, -\frac{m+\mu}{2\nu}, -\frac{m-\mu}{2i\nu}\right).$$

In the first representation $V$ is a homogeneous function of degree zero, and in the second $U$ is a homogeneous function of degree $-1$. The coordinate $n$ may be introduced into the representations by means of the identical relation (2), the homogeneity of the expression being thereby unaltered.

Conversely, any homogeneous function of the six variables $(l, m, n, \lambda, \mu, \nu)$ can be expressed as a function of $(x, y, z, w)$. We shall now consider under what circumstances such a function can satisfy one of the differential equations—

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial w}\right)^2 = 0,$$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 U}{\partial w^2} = 0.$$

* These bear the same relation to the hexaspherical coordinates of a point as the ordinary line coordinates of a line bear to the system of coordinates introduced by Klein.

First, let $V$ be a homogeneous function of degree zero. We evidently have

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial l} + \frac{\partial V}{\partial \lambda} + 2x\frac{\partial V}{\partial n},$$

$$\frac{\partial V}{\partial y} = i\frac{\partial V}{\partial l} - i\frac{\partial V}{\partial \lambda} + 2y\frac{\partial V}{\partial n},$$

$$\frac{\partial V}{\partial z} = \frac{\partial V}{\partial m} + \frac{\partial V}{\partial \mu} + 2z\frac{\partial V}{\partial n},$$

$$\frac{\partial V}{\partial w} = i\frac{\partial V}{\partial m} - i\frac{\partial V}{\partial \mu} + 2w\frac{\partial V}{\partial n},$$

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial w}\right)^2$$

$$= 4\frac{\partial V}{\partial l}\frac{\partial V}{\partial \lambda} + 4\frac{\partial V}{\partial m}\frac{\partial V}{\partial \mu} + 4n\left(\frac{\partial V}{\partial n}\right)^2 + 4l\frac{\partial V}{\partial l}\frac{\partial V}{\partial n}$$

$$+ 4\lambda\frac{\partial V}{\partial \lambda}\frac{\partial V}{\partial n} + 4\mu\frac{\partial V}{\partial \mu}\frac{\partial V}{\partial n} + 4m\frac{\partial V}{\partial m}\frac{\partial V}{\partial n}.$$

But, since $V$ is a homogeneous function of degree zero,

$$l\frac{\partial V}{\partial l} + \lambda\frac{\partial V}{\partial \lambda} + m\frac{\partial V}{\partial m} + \mu\frac{\partial V}{\partial \mu} + n\frac{\partial V}{\partial n} + \nu\frac{\partial V}{\partial \nu} = 0,$$

and $\nu = -1$, therefore

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial w}\right)^2 \equiv 4\left[\frac{\partial V}{\partial l}\frac{\partial V}{\partial \lambda} + \frac{\partial V}{\partial m}\frac{\partial V}{\partial \mu} + \frac{\partial V}{\partial n}\frac{\partial V}{\partial \nu}\right].$$

If, instead of $(l, m, n, \lambda, \mu, \nu)$, we use the usual hexaspherical coordinates defined by the relations

$$a_1 = x, \quad a_2 = y, \quad a_3 = z, \quad a_4 = w, \quad a_5 = \frac{r^2-1}{2}, \quad a_6 = \frac{r^2+1}{2i};$$

$$l = a_1 + ia_2, \qquad m = a_3 + ia_4, \qquad n = a_5 + ia_6;$$

$$\lambda = a_1 - ia_2, \qquad \mu = a_3 - ia_4, \qquad \nu = a_5 - ia_6,$$

the equation takes the more symmetrical form

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial w}\right)^2 \equiv \sum_{k=1}^{6}\left(\frac{\partial V}{\partial a_k}\right)^2.$$

This relation shows that a homogeneous function of degree zero, which is a solution of

$$\frac{\partial V}{\partial l}\frac{\partial V}{\partial \lambda} + \frac{\partial V}{\partial m}\frac{\partial V}{\partial \mu} + \frac{\partial V}{\partial n}\frac{\partial V}{\partial \nu} = 0,$$

*i.e.*, of

$$\sum_{k=1}^{6}\left(\frac{\partial V}{\partial a_k}\right)^2 = 0$$

is a solution of $\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2+\left(\frac{\partial V}{\partial w}\right)^2=0$

when expressed in terms of $x, y, z, w$.

Next, let $U$ be a homogeneous function of degree $-1$ in ($l, m, n, \lambda, \mu, \nu$), then we can show in a similar way that

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}+\frac{\partial^2 U}{\partial w^2}\equiv 4\left[\frac{\partial^2 U}{\partial l\,\partial\lambda}+\frac{\partial^2 U}{\partial m\,\partial\mu}+\frac{\partial^2 U}{\partial n\,\partial\nu}\right]\equiv\sum_{k=1}^{6}\frac{\partial^2 U}{\partial a^2}.$$

Hence, if $U$ is a solution of

$$\frac{\partial^2 U}{\partial l\,\partial\lambda}+\frac{\partial^2 U}{\partial m\,\partial\mu}+\frac{\partial^2 U}{\partial n\,\partial\nu}=0,$$

*i.e.*, of

$$\sum_{k=1}^{6}\frac{\partial^2 U}{\partial a_k^2}=0,$$

it is a solution of

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}+\frac{\partial^2 U}{\partial w^2}=0$$

when expressed in terms of ($x, y, z, w$).

When ($a_1, a_2, \ldots, a_6$) are interpreted as the coordinates of a point in a space of six dimensions, the expressions

$$\sum_{k=1}^{6} a_k^2,\qquad \sum_{k=1}^{6}\left(\frac{\partial U}{\partial a_k}\right)^2,\qquad \sum_{k=1}^{6}\frac{\partial^2 U}{\partial a_k^2}$$

remain unaltered in form after any change of rectangular axes in which the origin remains the same. Any change of this kind corresponds to a transformation in the ($x, y, z, w$) space, enabling us to pass from one solution of the equation

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}+\frac{\partial^2 U}{\partial w^2}=0$$

to another, and a similar remark applies to the equation

$$\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2+\left(\frac{\partial V}{\partial w}\right)^2=0.$$

To illustrate the method of formation of the transformation, we may consider the effect of simply interchanging $n$ and $-\nu$. The functions $V$ and $U$ of formulæ (3) and (4) then transform into

$$F\left(\frac{l+\lambda}{2n},\ \frac{l-\lambda}{2in},\ \frac{m+\mu}{2n},\ \frac{m-\mu}{2in}\right)=\quad F\left(\frac{x}{r^2},\ \frac{y}{r^2},\ \frac{z}{r^2},\ \frac{w}{r^2}\right)$$

and $$\frac{1}{n}f\left(\frac{l+\lambda}{2n},\ \frac{l-\lambda}{2in},\ \frac{m+\mu}{2n},\ \frac{m-\mu}{2in}\right)=\frac{1}{r^2}f\left(\frac{x}{r^2},\ \frac{y}{r^2},\ \frac{z}{r^2},\ \frac{w}{r^2}\right)$$

respectively, $r^2$ being written in place of $x^2+y^2+z^2+w^2$.

Accordingly, from a solution $V = F(x, y, z, w)$ of the equation

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial w}\right)^2 = 0,$$

we may derive a second solution

$$v = F\left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}, \frac{w}{r^2}\right),$$

and from the solution $U = f(x, y, z, w)$ of

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 U}{\partial w^2} = 0,$$

we may derive another solution

$$u = \frac{1}{r^2} f\left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}, \frac{w}{r^2}\right).$$

Putting $w = ict$, where $c$ is the velocity of light and $t$ is the time, the equations take the well known form*

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 = \frac{1}{c^2}\left(\frac{\partial V}{\partial t}\right)^2,$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2},$$

and the transformation may be written

$$X = \frac{x}{r^2 - c^2t^2}, \quad Y = \frac{y}{r^2 - c^2t^2}, \quad Z = \frac{z}{r^2 - c^2t^2}, \quad T = \frac{t}{r^2 - c^2t^2},$$

where now $$r^2 = x^2 + y^2 + z^2.$$

The study of this transformation will be taken up later.

* It may be mentioned here that Lorentz's fundamental equations of the electron theory, viz.,

$$\frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z} = 4\pi u, \qquad \frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} = \frac{\partial \alpha}{\partial t},$$

$$u = \frac{\partial f}{\partial t} + \rho\xi, \qquad P = 4\pi c^2 f,$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho\xi) + \frac{\partial}{\partial y}(\rho\eta) + \frac{\partial}{\partial z}(\rho\zeta) = 0,$$

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = \rho, \qquad \frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z} = 0,$$

may be reduced to a symmetrical form by writing $s = ict$ and putting

$$\alpha + i\frac{P}{c} = p, \qquad \beta + i\frac{Q}{c} = q, \qquad \gamma + i\frac{R}{c} = r.$$

The four mutually orthogonal vectors $(A, B, C, D)$ whose components are respectively

$$(0, r, -q, -p), \quad (-r, 0, p, -q), \quad (q, -p, 0, -r), \quad (p, q, r, 0)$$

A second transformation of some interest is obtained by interchanging $m$ with $n$ and $\mu$ with $\nu$. This changes

$$-\frac{1}{\nu} f\left(-\frac{l+\lambda}{2\nu}, -\frac{l-\lambda}{2i\nu}, -\frac{m+\mu}{2\nu}, -\frac{m-\mu}{2i\nu}\right)$$

into
$$-\frac{1}{\mu} f\left(-\frac{l+\lambda}{2\mu}, -\frac{l-\lambda}{2i\mu}, -\frac{n+\nu}{2\mu}, -\frac{n-\nu}{2i\mu}\right),$$

that is
$$f(x, y, z, w)$$

into
$$-\frac{1}{z-iw} f\left(-\frac{x}{2(z-iw)}, -\frac{y}{2(z-iw)}, -\frac{1-r^2}{2(z-iw)}, -\frac{1+r^2}{2i(z-iw)}\right).$$

Putting $w = ict$ and changing the sign of $z$, the formulæ for the transformation are

$$X = \frac{x}{z-ct}, \quad Y = \frac{y}{z-ct}, \quad Z = \frac{r^2-1}{2(z-ct)}, \quad cT = \frac{r^2+1}{2(z-ct)},$$

where, now,
$$r^2 = x^2+y^2+z^2-c^2t^2.$$

If $V = F(x, y, z, t)$ is a solution of

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 = \frac{1}{c^2}\left(\frac{\partial V}{\partial t}\right)^2,$$

the function $F(X, Y, Z, T)$ is also a solution, and, if $U = f(x, y, z, t)$ is

satisfy the equations

$$\operatorname{div} A = 4\pi\rho\xi, \quad \operatorname{div} B = 4\pi\rho\eta, \quad \operatorname{div} C = 4\pi\rho\zeta, \quad \operatorname{div} D = 4\pi\rho i c,$$

where
$$\operatorname{div} M \equiv \frac{\partial M_1}{\partial x} + \frac{\partial M_2}{\partial y} + \frac{\partial M_3}{\partial z} + \frac{\partial M_4}{\partial s} \quad \text{if} \quad M \equiv (M_1, M_2, M_3, M_4).$$

Again, if we put $4\pi\rho\xi = X, \quad 4\pi\rho\eta = Y, \quad 4\pi\rho\eta = Z, \quad 4\pi\rho i c = S,$

and introduce four new vectors $A_1, B_1, C_1, D_1$ whose components are respectively

$$(S, -Z, Y, -X) \quad (Z, S, -X, -Y) \quad (-Y, X, S, -Z) \quad (X, Y, Z, S),$$

we find
$$\operatorname{div} A_1 = \nabla^2 p, \quad \operatorname{div} B_1 = \nabla^2 q, \quad \operatorname{div} C_1 = \nabla^2 r, \quad \operatorname{div} D_1 = 0,$$

where
$$\nabla^2\phi \equiv \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} + \frac{\partial^2\phi}{\partial s^2}.$$

Finally, if $X, Y, Z, S$ can be derived from a potential function $n$ so that

$$X = -\frac{\partial n}{\partial x}, \quad Y = -\frac{\partial n}{\partial y}, \quad Z = -\frac{\partial n}{\partial z}, \quad S = -\frac{\partial n}{\partial s},$$

we can form four mutually orthogonal vectors $\theta, \phi, \psi, \chi$ whose components are respectively

$$(n, r, -q, -p), \quad (-r, n, p, -q), \quad (q, -p, n, -r), \quad (p, q, r, n),$$

and the equations then take the simple form

$$\operatorname{div}\theta = \operatorname{div}\phi = \operatorname{div}\psi = \operatorname{div}\chi = 0,$$
$$\nabla^2 p = \nabla^2 q = \nabla^2 r = \nabla^2 n = 0,$$
$$\operatorname{div} A_1 = \operatorname{div} B_1 = \operatorname{div} C_1 = \operatorname{div} D_1 = 0.$$

a solution of $$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2},$$

the function $$\frac{1}{z-ct} f\left[\frac{x}{z-ct}, \frac{y}{z-ct}, \frac{r^2-1}{2(z-ct)}, \frac{r^2+1}{2c(z-ct)}\right]$$

is also a solution.

In following up the connection between different solutions, it is convenient to use polar coordinates. Putting

$$x = r\cos\theta\cos\phi, \quad y = r\cos\theta\sin\phi, \quad z = r\sin\theta\cos\psi,$$
$$w = ict = r\sin\theta\sin\psi;$$
$$X = R\cos\Theta\cos\Phi, \quad Y = R\cos\Theta\sin\Phi, \quad Z = R\sin\Theta\cos\Psi,$$
$$W = icT = R\sin\Theta\sin\Psi:$$

we obtain the relations

$$r^2 = -e^{-2i\Psi}, \quad R^2 = -e^{-2i\psi}, \quad \sin\Theta = \operatorname{cosec}\theta, \quad \Phi = \phi.$$

There is a similar transformation for Laplace's equation.*

If $$X = \frac{r^2-a^2}{2(x-iy)}, \quad Y = \frac{r^2+a^2}{2i(x-iy)}, \quad Z = \frac{az}{x-iy},$$

a solution $f(x, y, z)$ corresponds to a second solution

$$\frac{1}{\sqrt{x-iy}} f\left(\frac{r^2-a^2}{2(x-iy)}, \frac{r^2+a^2}{2i(x-iy)}, \frac{az}{x-iy}\right).$$

Putting $x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta,$

$$X = R\sin\Theta\cos\Phi, \quad Y = R\sin\Theta\sin\Phi, \quad Z = R\cos\Theta,$$

the formulæ of transformation become

$$R = iae^{i\phi}, \quad r = iae^{i\Phi}, \quad \sin\Theta = \operatorname{cosec}\theta.$$

The transition from one solution of Laplace's equation to another is now easily effected.

The effects of combining the different transformations belonging to a group of conformal transformations is most easily studied by interpreting

* This transformation was given by the author in a Smith's Prize Essay of 1905; it was deduced from a result given by Brill, *Messenger of Mathematics* (1891), pp. 135–137. If $V = f(x_1, \ldots, x\ , t)$ is a solution of the differential equation

$$\frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} + \ldots + \frac{\partial^2 V}{\partial x_n^2} = \frac{1}{a}\frac{\partial V}{\partial t},$$

another solution is given by

$$t^{-\frac{1}{2}n} e^{-(x_1^2+x_2^2+\ldots+x_n^2)/4at} \left(\frac{x_1}{t}, \frac{x_2}{t}, \ldots, \frac{x_n}{t}, -\frac{1}{t}\right).$$

the transformation as a change of axes in a space in which the coordinates are the spherical coordinates $\alpha_r$.* It is important to notice that the angle between two manifolds in this space is equal to the angle between the corresponding manifolds in the space to which the conformal transformations are applied. In the case of a space of four dimensions, we have, in fact,

$$d\alpha_1^2+d\alpha_2^2+d\alpha_3^2+d\alpha_4^2+d\alpha_5^2+d\alpha_6^2 \equiv dx^2+dy^2+dz^2+dw^2,$$

$$d\alpha_1\, d\alpha_1'+d\alpha_2 d\alpha_2'+d\alpha_3 d\alpha_3'+d\alpha_4 d\alpha_4'+d\alpha_5 d\alpha_5'+d\alpha_6 d\alpha_6' \equiv dx\, dx'+dy\, dy'+dz\, dz'+dw\, dw',$$

from which the result easily follows.

A change in the sign of $\alpha_6$ corresponds to an inversion, a change in the sign of $\alpha_6$ coupled with a change in the sign of $\alpha_4$ corresponds to the other transformation we have mentioned. It is evident that each of these transformations is of period 2. In general, a reflexion in a linear manifold in the $\alpha$ space corresponds to an inversion with regard to the corresponding circle, sphere, or hypersphere, in the space of four dimensions. A displacement of period $n$ in the $\alpha$ space may be obtained by taking successive reflexions in two plane five-folds which cut at an angle $\pi/n$.† It evidently corresponds to a periodic conformal transformation made up of inversions with regard to two hyperspheres which cut at an angle $\pi/n$.

### 3. *The Relation between Riemann's General Hypergeometric Function and the Group of Conformal Transformations of a Space of Four Dimensions.*

We shall now endeavour to satisfy the differential equation

$$\frac{\partial^2 U}{\partial l\, \partial\lambda}+\frac{\partial^2 U}{\partial m\, \partial\mu}+\frac{\partial^2 U}{\partial n\, \partial\nu}=0,$$

by means of a function of the form

$$U = l^{-\alpha}\lambda^{-\alpha'}m^{-\beta}\mu^{-\beta'}n^{-\gamma}\nu^{-\gamma'}P, \tag{1}$$

* Reference should be made to Darboux, *Théorie des Surfaces*, Tome I., p. 213. Jessop's *Treatise on the Line Complex*, p. 251. Koenig's *La Géometrie réglée*, p. 125, and to an article by Borel on the "Transformations of Geometry in Niewenglowsky's Solid Geometry."

† This is equivalent to a rotation through an angle $2\pi/n$ just as successive reflexions in two planes are equivalent to a rotation about their line of intersection.

where $P$ is a function of the ratio of any two of the quantities $l\lambda$, $m\mu$, $n\nu$. The quantity $U$ will then be a solution of the equation

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2}.$$

if the relation $$\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=1$$

is satisfied, for it will then be a homogeneous function of degree $-1$ in $(l, m, n, \lambda, \mu, \nu)$.

Let us put
$$l\lambda = (b-c)(z-a) = \xi,$$
$$m\mu = (c-a)(z-b) = \eta,$$
$$n\nu = (a-b)(z-c) = \zeta,$$

where $a$, $b$ and $c$ are arbitrary constants. The relation

$$l\lambda+m\mu+n\nu = 0$$

is then satisfied, and $P$ becomes a function of $z$ alone. We may thus write

$$P(\xi, \eta, \zeta) \equiv P(z) \equiv H(\xi^{b-c}\eta^{c-a}\zeta^{a-b}) \equiv H(\theta),$$

the particular functional form in terms of $\xi$, $\eta$, $\zeta$ being chosen to facilitate the calculations. $H$ is clearly a homogeneous function of degree zero in $\xi$, $\eta$, $\zeta$, and therefore in $l$, $m$, $n$, $\lambda$, $\mu$, $\nu$.

On differentiating equation (1), we obtain

$$\frac{\partial U}{\partial l} = -\frac{\alpha}{l}U+\lambda\frac{U}{P}\frac{\partial P}{\partial \xi},$$

$$\frac{\partial^2 U}{\partial l\partial\lambda} = \frac{\alpha\alpha'}{\xi}U+\frac{1-\alpha-\alpha'}{P}U\frac{\partial P}{\partial\xi}+\frac{\xi}{P}U\frac{\partial^2 P}{\partial\xi^2}.$$

The differential equation

$$\frac{\partial^2 U}{\partial l\partial\lambda}+\frac{\partial^2 U}{\partial m\partial\mu}+\frac{\partial^2 U}{\partial n\partial\nu}=0$$

will thus be satisfied, if

$$\xi\frac{\partial^2 P}{\partial\xi^2}+\eta\frac{\partial^2 P}{\partial\eta^2}+\zeta\frac{\partial^2 P}{\partial\zeta^2}+(1-\alpha-\alpha')\frac{\partial P}{\partial\xi}+(1-\beta-\beta')\frac{\partial P}{\partial\eta}$$
$$+(1-\gamma-\gamma')\frac{\partial P}{\partial\zeta}+\left(\frac{\alpha\alpha'}{\xi}+\frac{\beta\beta'}{\eta}+\frac{\gamma\gamma'}{\zeta}\right)P = 0.$$

Now
$$\frac{\partial P}{\partial \xi} = \frac{b-c}{\xi}\,\theta H'(\theta) = \frac{\theta}{z-a}\,H'(\theta),$$

$$\frac{dP}{dz} = \frac{\partial P}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial P}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial P}{\partial \zeta}\frac{\partial \zeta}{\partial z} = (b-c)\frac{\partial P}{\partial \xi} + (c-a)\frac{\partial P}{\partial \eta} + (a-b)\frac{\partial P}{\partial \zeta}$$

$$= \left(\frac{b-c}{z-a} + \frac{c-a}{z-b} + \frac{a-b}{z-c}\right)\theta H'(\theta)$$

$$= -\frac{(b-c)(c-a)(a-b)}{(z-a)(z-b)(z-c)}\,\theta H'(\theta).$$

Hence
$$\frac{\partial P}{\partial \xi} = \frac{(z-b)(z-c)}{(a-b)(a-c)}\,\frac{1}{b-c}\,\frac{dP}{dz}.$$

Also, since
$$\xi\frac{\partial P}{\partial \xi} = (b-c)\,\theta H'(\theta) = \phi(\theta),$$

we have
$$\xi\frac{\partial^2 P}{\partial \xi^2} + \frac{\partial P}{\partial \xi} = \frac{\partial}{\partial \xi}\,\phi(\theta) = \frac{(z-b)(z-c)}{(a-b)(a-c)}\,\frac{1}{b-c}\,\frac{d}{dz}\,\phi(\theta),$$

this relation being obtained in the same way as the one above.

The last expression may be written

$$\frac{(z-b)(z-c)}{(a-b)(a-c)}\,\frac{d}{dz}\,\theta H'(\theta),$$

*i.e.*,
$$-\frac{(z-b)(z-c)}{(a-b)(a-c)}\,\frac{d}{dz}\,\frac{(z-a)(z-b)(z-c)}{(b-c)(c-a)(a-b)}\,\frac{dP}{dz}.$$

or
$$\frac{(z-a)(z-b)^2(z-c)^2}{(b-c)(c-a)^2(a-b)^2}\left[\frac{d^2P}{dz^2} + \left(\frac{1}{z-a} + \frac{1}{z-b} + \frac{1}{z-c}\right)\frac{dP}{dz}\right].$$

Now
$$\frac{(z-b)(z-c)}{(c-a)(a-b)} + \frac{(z-c)(z-a)}{(a-b)(b-c)} + \frac{(z-a)(z-b)}{(b-c)(c-a)} = -1,$$

and
$$\frac{\partial P}{\partial \xi} + \frac{\partial P}{\partial \eta} + \frac{\partial P}{\partial \zeta} = \left(\frac{1}{z-a} + \frac{1}{z-b} + \frac{1}{z-c}\right)\theta H'(\theta)$$

$$= -\frac{(z-a)(z-b)(z-c)}{(b-c)(c-a)(a-b)}\left(\frac{1}{z-a} + \frac{1}{z-b} + \frac{1}{z-c}\right)\frac{dP}{dz};$$

consequently,

$$\xi\frac{\partial^2 P}{\partial \xi^2} + \eta\frac{\partial^2 P}{\partial \eta^2} + \zeta\frac{\partial^2 P}{\partial \zeta^2} = -\frac{(z-a)(z-b)(z-c)}{(b-c)(c-a)(a-b)}\,\frac{d^2P}{dz^2}.$$

The differential equation thus reduces to

$$\frac{d^2P}{dz^2}+\left(\frac{1-a-a'}{z-a}+\frac{1-\beta-\beta'}{z-b}+\frac{1-\gamma-\gamma'}{z-c}\right)\frac{dP}{dz}$$
$$+\left[\frac{aa'(a-b)(a-c)}{z-a}+\frac{\beta\beta'(b-a)(b-c)}{z-b}+\frac{\gamma\gamma'(c-a)(c-b)}{z-c}\right]$$
$$\times\frac{P}{(z-a)(z-b)(z-c)}=0.$$

This is Papperitz's form* of the differential equation satisfied by Riemann's general hypergeometric function†

$$P\left\{\begin{matrix} a & b & c & \\ a & \beta & \gamma & z \\ a' & \beta' & \gamma' & \end{matrix}\right\};$$

hence we have the result that

$$U=l^{-a}\lambda^{-a'}m^{-\beta}\mu^{-\beta'}n^{-\gamma}\nu^{-\gamma'}P\left\{\begin{matrix} a & b & c & \\ a & \beta & \gamma & z \\ a' & \beta' & \gamma' & \end{matrix}\right\}$$

is a homogeneous function of $(l, m, n, \lambda, \mu, \nu)$ of degree $-1$, satisfying the equation

$$\frac{\partial^2 U}{\partial l\partial\lambda}+\frac{\partial^2 U}{\partial m\partial\mu}+\frac{\partial^2 U}{\partial n\partial\nu}=0.$$

When expressed in terms of $x$, $y$, $z$ and $w$, it will thus be a solution of the equation

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}+\frac{\partial^2 U}{\partial w^2}=0.$$

The various transformations‡ of the general hypergeometric function are easily obtained from this result. If we write $U$ in the form

$$U=l^{-a-\theta}\lambda^{-a'-\theta}m^{-\beta-\phi}\mu^{-\beta'-\phi}n^{-\gamma-\psi}\nu^{-\gamma'-\psi}(l\lambda)^{\theta}(m\mu)^{\phi}(n\nu)^{\psi}P\left\{\begin{matrix} a & b & c & \\ a & \beta & \gamma & z \\ a' & \beta' & \gamma' & \end{matrix}\right\},$$

* *Mathematische Annalen*, T. xxv. (1885), p. 213.

† *Abhandlungen d. K. Gesell. d. Wissenschaften zu Göttingen*, Band vii. (1857), *Gesammelte Werke*, p. 63.

‡ See Whittaker's *Analysis*, p. 246. Forsyth's *Theory of Linear Differential Equations*, Vol. iv., p. 135.

where
$$\theta+\phi+\psi = 0,$$

we see that
$$(z-a)^{\theta}(z-b)^{\phi}(z-c)^{\psi} P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$$

is a multiple of
$$P\begin{Bmatrix} a & b & c & \\ \alpha+\theta & \beta+\phi & \gamma+\psi & z \\ \alpha'+\theta & \beta'+\phi & \gamma'+\psi & \end{Bmatrix}.$$

Again, if we write
$$a' = \frac{Aa+B}{Ca+D}, \quad b' = \frac{Ab+B}{Cb+D},$$
$$c' = \frac{Ac+B}{Cc+D}, \quad z' = \frac{Az+B}{Cz+D},$$

we have
$$(b'-c')(z'-a') = \frac{(AD-BC)^2}{(Ca+D)(Cb+D)(Cc+D)(Cd+D)}(b-c)(z-a),$$

so that
$$\frac{l\lambda}{(b'-c')(z'-a')} = \frac{m\mu}{(c'-a')(z'-b')} = \frac{n\nu}{(a'-b')(z'-c')},$$

This shows that $P$ is the same function of the quantities $a'$, $b'$, $c'$, $z'$ as it is of $a$, $b$, $c$, $z$; that is

$$P\begin{Bmatrix} a' & b' & c' & \\ \alpha & \beta & \gamma & z' \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}.$$

Hence the general hypergeometric function is unaltered if the quantities $a$, $b$, $c$, $z$ are replaced by quantities $a'$, $b'$, $c'$, $z'$ which are derived from them by the same homographic substitution.

### 4. *Applications to Geometrical Optics.*

Let us consider a series of waves of light traversing a homogeneous or heterogeneous medium, and let

$$V = \int \mu\, ds$$

be the reduced path from a standard orthotomic surface or wave front to the point $(x, y, z)$. Let us suppose, moreover, that $V$ is expressed only in terms of the coordinates $(x, y, z)$, and the constants of the standard wave front. Then $V$ satisfies the differential equation*

$$\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2=\mu^2,$$

and is, in fact, the characteristic function introduced by Hamilton. If it is expressed as a function of $x$, $y$, $z$, and the coordinates of the initial point $x_0 y_0 z_0$, it is the Eikonal according to the nomenclature of Bruns.†

Since $V$ is proportional to the time this differential equation may be replaced by

$$\left(\frac{\partial t}{\partial x}\right)^2+\left(\frac{\partial t}{\partial y}\right)^2+\left(\frac{\partial t}{\partial z}\right)^2=\frac{1}{C^2},$$

where $C$ is the velocity of radiation at the point $(x, y, z)$.

Now suppose that the surfaces $t=$ const. are obtained by solving an equation

$$F(x, y, z, t)=0$$

for $t$; then, since

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial t}\,\frac{\partial t}{\partial x}=0,$$

the function $F$ must satisfy the differential equation

$$\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2+\left(\frac{\partial F}{\partial z}\right)^2=\frac{1}{C^2}\left(\frac{\partial F}{\partial t}\right)^2.$$

Confining ourselves to the case in which $C$ is constant, we may use the results of § 2 to obtain new solutions of this differential equation.

Let

$$X=X(x, y, z, t), \qquad Z=Z(x, y, z, t),$$

$$Y=Y(x, y, z, t), \qquad T=T(x, y, z, t)$$

be the formulæ giving a transformation which enables us to pass from one solution of the above equation to another; then

$$F(X, Y, Z, T),$$

when expressed in terms of $x$, $y$, $z$, $t$, is a second solution of the equation, and if the equation

$$F=0$$

be solved for $t$, the surfaces $t=$ const. will form a system of parallel wave

* See Herman's *Optics*, p. 253.

† Cf. Schwarzschild's *Untersuchungen zur Geometrischen Optik*, Göttingen Abhandlungen (2), 4.

surfaces and $t$ considered as a function of $(x, y, z)$ will be the characteristic function for them.

The transformation

$$X = \frac{x}{r^2-c^2t^2}, \quad Y = \frac{y}{r^2-c^2t^2}, \quad Z = \frac{z}{r^2-c^2t^2}, \quad T = \frac{t}{r^2-c^2t^2}$$

is of special importance because it makes the standard wave surface $t = 0$ in the original system correspond to a standard wave surface $t = 0$ in the new system; also, since the equations of the surfaces are

$$F(x, y, z, 0) = 0,$$

and

$$F\left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}, 0\right) = 0,$$

respectively, it is clear that one is the inverse of the other with regard to a unit sphere whose centre is at the origin. Our theorem tells us that if the surfaces parallel to the first are given by

$$F(x, y, z, t) = 0,$$

the surfaces parallel to the second are given by

$$F\left(\frac{x}{r^2-c^2t^2}, \frac{y}{r^2-c^2t^2}, \frac{z}{r^2-c^2t^2}, \frac{t}{r^2-c^2t^2}\right) = 0.$$

Applying this result to the family of right circular cones parallel to a given one, we may obtain the family of Dupin's cyclides parallel to a given one.

To obtain a geometrical interpretation of the transformation we describe a sphere of radius $ct$ round the point $(x, y, z)$ as centre. The inverse sphere is then of radius $cT$ and its centre is at the point $(XYZ)$.

A more general result is that the sphere

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = c^2(t-t_0)^2$$

corresponds in the transformation to the sphere

$$(X-X_0)^2+(Y-Y_0)^2+(Z-Z_0)^2 = c^2(T-T_0)^2,$$

the centres of the two spheres being corresponding points at the times $t_0$, $T_0$ respectively.

To show that the laws of reflection and refraction remain unchanged in the transformation, we take the surface at which the light is incident as the standard one from which the time is measured. Let $(x_0, y_0, z_0)$ be a point on this surface; then we must associate with this point the time $t_0 = 0$.

Consider a ray of light travelling from $(x_0, y_0, z_0)$ in a direction $(l, m, n)$ with velocity $c$. At time $t$ the wave has reached a point $(x, y, z)$ on the ray, where

$$x = x_0 + l.ct, \quad y = y_0 + m.ct, \quad z = z_0 + n.ct.$$

The corresponding point $(X, Y, Z)$ derived from this by the transformation also travels along a straight line, for

$$\begin{aligned} X &= \frac{x_0 + lct}{(x_0+lct)^2 + (y_0+mct)^2 + (z_0+nct)^2 - c^2t^2} \\ &= \frac{x_0}{x_0^2+y_0^2+z_0^2} + \left[\frac{ct}{(x_0+lct)^2+(y_0+mct)^2+(z_0+nct)^2-c^2t^2}\right] \\ &\qquad\qquad \times \left[l - \frac{2x_0}{x_0^2+y_0^2+z_0^2}(lx_0+my_0+nz_0)\right] \\ &= X_0 + LcT, \end{aligned}$$

where

$$T = \frac{t}{r^2 - c^2t^2}.$$

The corresponding ray thus passes through the inverse point $(X_0 Y_0 Z_0)$ on the inverse surface at which it may be supposed to be incident. Its direction cosines $(L, M, N)$ are connected with those of the former ray by means of the equations

$$L = l - \frac{2x_0}{x_0^2+y_0^2+z_0^2}(lx_0+my_0+nz_0),$$

$$M = m - \frac{2y_0}{x_0^2+y_0^2+z_0^2}(lx_0+my_0+nz_0),$$

$$N = n - \frac{2z_0}{x_0^2+y_0^2+z_0^2}(lx_0+my_0+nz_0).$$

These relations establish a correspondence between the sheafs of rays through the points $(x_0, y_0, z_0)$, $(X_0, Y_0, Z_0)$ respectively. This correspondence is such that the angle between two rays $(l, m, n)$, $(l', m', n')$ is equal to the angle between the two corresponding rays $(L, M, N)$, $(L', M', N')$, for we have identically

$$LL' + MM' + NN' \equiv ll' + mm' + nn'.$$

Since the transformation enables us to derive the surfaces which are parallel to one surface from the surfaces which are parallel to the inverse surface, it is natural to expect that the above relation between the direction cosines will make the normals to the two surfaces correspond.

Now $$dX_0 = \frac{dx_0}{x_0^2+y_0^2+z_0^2} - \frac{2(x_0dx_0+y_0dy_0+z_0dz_0)\,x_0}{(x_0^2+y_0^2+z_0^2)^2};$$

hence, if $(l_0, m_0, n_0)$ are the direction cosines of a tangent to the first surface, the direction cosines of the tangent for the corresponding displacement on the inverse surface are given by

$$L_0 = l_0 - \frac{2x_0}{x_0^2+y_0^2+z_0^2}(l_0x_0+m_0y_0+n_0z_0).$$

The tangents to the two surfaces can thus be paired with one another in the correspondence.

Now, if $(l, m, n)$ are the direction cosines of the normal to one surface, $(l_0, m_0, n_0)$ those of a tangent,

$$ll_0+mm_0+nn_0 = 0.$$

It follows, then, that for the corresponding ray $(L, M, N)$

$$LL_0+MM_0+NN_0 = 0.$$

This ray, being perpendicular to all the tangents to the inverse surface, is the normal at $(X_0, Y_0, Z_0)$ to this surface.

Since the angles between lines remain unaltered by the transformation, it follows that if $(l, m, n)$, $(l', m', n')$, $(p, q, r)$ are the direction cosines of an incident ray, the refracted ray and the normal at a point $(x_0, y_0, z_0)$ on a surface $f$, the corresponding quantities $(L, M, N)$, $(L', M', N')$, $(P, Q, R)$ are the direction cosines of an incident ray, refracted ray and the normal at the point $(X_0, Y_0, Z_0)$ of the inverse surface $F$. This is also easy to verify from the analytical formulæ.

The method of inversion can thus be applied to problems in geometrical optics. A medium of refractive index $\mu$ inverts into a medium of refractive index $\mu$, a ray through the origin corresponds to a ray through the origin, and a ray through a fixed point $(x, y, z)$ not on the surface of separation of the two media corresponds to a ray intersecting the line joining $(x, y, z)$ to the origin.

A spherical shell of radiation emitted by an electron whose velocity was suddenly changed at time $-t$, corresponds to a shell of radiation emitted by an electron whose velocity was suddenly changed at time $-T$.

This method of inversion promises to be of great importance in the theory of radiation. It will be noticed that the spheres of radii $ct$ whose centres are at the points

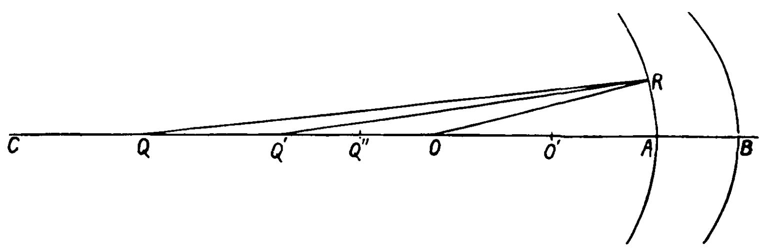
$$x = x_0+lct, \qquad y = y_0+mct, \qquad z = z_0+nct,$$

all touch one another at the point $(x_0, y_0, z_0)$ which may be regarded as a *node* of the radiation.

From a study of the nodes in the ether produced by a vibrating atom or molecule, it appears that various systems of nodes may be obtained from one another by inversions which correspond to the same frequencies of vibration. The investigation will be reserved, however, for another paper.

## 5. *Application of the Preceding Results to a Symmetrical Optical Instrument.*

Let $QR$ (Fig.) be the incident ray, $Q'R$ the refracted ray, $OR$ the normal to the spherical interface, and let $C$ be the centre of inversion.




We shall suppose that the incident ray makes a small angle with the axis.

Let $\qquad AC = z, \qquad AO = a, \qquad AQ = x, \qquad AQ' = x',$

and let the velocity of light for the first medium be represented by $1/\mu$, the times corresponding to the points $Q$, $Q'$, $O$, $A$ respectively may then be taken to be $-\mu x$, $-\mu x'$, $-\mu a$, 0, respectively, and the corresponding quantities $ct$ are simply the reduced distances $(-x, -x', -a, 0)$.

Let $Q_1$, $Q_1'$, $O_1$, $A_1$ be the points corresponding to $Q$, $Q'$, $O$, $A$ in the transformation, $x_1$, $x_1'$, $z_1-a_1$, $z_1$ their distances from $C$.

Now the sphere centre $Q$ and radius $QA$ inverts into the sphere centre $Q_1$ and radius $QA_1$; hence, if the radius of inversion be equal to $k^2$, we have for the points in which these spheres meet the axis

$$z_1 = \frac{k^2}{z}, \qquad z_1 - 2x_1 = \frac{k^2}{z-2x}.$$

We also have the relations

$$A_1 Q_1 = -x_1 = \frac{k^2 x}{z(z-2x)},$$

$$A_1 Q_1' = -x_1' = \frac{k^2 x'}{z(z-2x')},$$

$$A_1 O_1 = -a_1 = \frac{k^2 a}{z(z-2a)}.$$

The linear magnification in the new system is

$$m_1 = \frac{\dfrac{A_1Q_1'}{\mu'}}{\dfrac{A_1Q_1}{\mu}} = \frac{\mu x'(z-2x)}{\mu' x(z-2x')} = m\,\frac{z-2x}{z-2x'}.$$

Let $c$ be the distance of the second interface from the first; then we associate the length $-c$ with the second interface $B$, and the formulæ of the transformation become

$$B_1Q_1' = -\xi_1' = \frac{k^2\xi'}{(\zeta'-\xi')^2-(\xi'-c)^2} = \frac{k^2\xi'}{(\zeta'-c)(\zeta'+c-2\xi')},$$

$$\zeta_1' = \frac{k^2\zeta}{\zeta^2-c^2}, \qquad c_1 = -\frac{k^2c}{\zeta^2-c^2},$$

$$\zeta_1'-\xi_1' = \frac{k^2(\zeta'-\xi')}{(\zeta'-\xi')^2-(\xi'-c)^2} = \frac{k^2(\zeta'-\xi')}{(\zeta'-c)(\zeta'+c-2\xi')},$$

$$\xi_1'-c_1 = \frac{k^2(c-\xi')}{(\zeta'-c)(\zeta'+c-2\xi')},$$

$$\zeta_1'+c_1-2\xi_1' = \frac{k^2}{\zeta'+c-2\xi'}.$$

The last of which may be written

$$\zeta_1'-c_1-2(\xi_1'-c_1) = \frac{k^2}{\zeta'-c-2(\xi'-c)},$$

*i.e.*,

$$z_1-2x_1' = \frac{k^2}{z-2x'},$$

where $z$ and $x'$ are measured, as before, from the first surface. This is exactly the former relation between two corresponding points; consequently the whole course of a ray in one instrument corresponds in the transformation to that of the corresponding ray in the corresponding instrument.*

* We can verify the equation

$$\frac{\mu''}{B_1Q_1''} - \frac{\mu'}{B_1Q_1'} = \frac{\mu''-\mu'}{B_1O_1'},$$

connecting two conjugate points in the second instrument as follows

$$\frac{1}{B_1Q_1''} = \frac{(\zeta'-c)(\zeta'+c-2\xi'')}{k^2\xi''}, \qquad \frac{1}{B_1Q_1'} = \frac{(\zeta'-c)(\zeta'+c-2\xi')}{k^2\xi'},$$

$$B_1O_1' = \frac{(\zeta'-c)(\zeta'+c-2a')}{k^2a'},$$

and

$$\frac{\mu''}{\xi''} - \frac{\mu'}{\xi'} = \frac{\mu''-\mu'}{a'},$$

since $Q''$ and $Q'$ are conjugate points in the first instrument; hence the relation is satisfied.

The linear magnification is given by

$$m_1' = \frac{\dfrac{B_1Q_1''}{\mu''}}{\dfrac{B_1Q_1'}{\mu'}} = \frac{\mu'\xi''(\zeta'+c-2\xi')}{\mu''\xi'(\zeta'+c-2\xi'')};$$

hence

$$m_1' = m'\frac{\zeta'+c-2\xi'}{\zeta'+c-2\xi''} = m'\frac{z-2x'}{z-2x''},$$

which is of the same form as before.

If $M$ and $M_1$ be the total linear magnifications for the two instruments, we have

$$M_1 = M\frac{z-2x}{z-2x^{(n)}}.$$

The relation between two corresponding points is evidently a homographic one; hence we have the following theorem :—

*If the points on the axis of a symmetrical optical instrument be transformed by means of a homographic transformation, any pair of conjugate points for the instrument are transformed into a pair of points which are conjugate with regard to a second instrument. The centres of curvature of the interfaces and the points in which the interfaces meet the axis correspond in the two instruments, and the refractive indices of corresponding media are the same.*