A NEW DEVELOPMENT OF THE THEORY OF THE HYPERGEOMETRIC FUNCTIONS

By E. W. BARNES.

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1. The differential equation of the hypergeometric series may be written

$$\left[(\vartheta+a_1)(\vartheta+a_2)-\frac{1}{x}\,\vartheta(\vartheta+\rho-1)\right]y=0,$$

where $\vartheta = x\dfrac{d}{dx}$, or

$$\frac{d^2y}{dx^2}+\frac{\rho-(1+a_1+a_2)x}{x(1-x)}\,\frac{dy}{dx}-\frac{a_1a_2}{x(1-x)}\,y=0.$$

It is known to be satisfied by the hypergeometric series

$$F(a_1, a_2;\rho;x)=\frac{\Gamma(\rho)}{\Gamma(a_1)\,\Gamma(a_2)}\sum_{n=0}^{\infty}\frac{\Gamma(a_1+n)\,\Gamma(a_2+n)}{n!\,\Gamma(\rho+n)}\,x^n,$$

which is convergent when $|x|<1$.

This series was first discussed in detail by Gauss(1) in 1812. Kummer(2), in 1836, obtained the twenty-four solutions of the hypergeometric equation usually given in the text-books by a process which can be traced back to Euler.(3) These twenty-four solutions are reducible to six sets of four, each four being identical functions differently expressed. The six sets can be divided into pairs Y_1, Y_2; Y_3, Y_4; Y_5, Y_6, each pair corresponding respectively to one of the three singularities 0, 1, ∞ of the differential equation.

Riemann,(4) in 1857, extended the theory by introducing his P-function, and discussed the general theory of transformation of the variable. Riemann did not connect his theory directly with that of Kummer, and it was reserved for Thomae,(5) in 1879, to work out in detail from the theory of linear differential equations the relations which connect any one of the twenty-four solutions of the hypergeometric equation with the two essentially different solutions which are valid in the neighbourhood of either of the singularities not associated with the particular solution chosen.

The differential equation for Riemann's P-function was first given by Papperitz,(6) in 1885, in the form

$$\frac{d^2y}{dx^2}+\left\{\frac{1-\alpha-\alpha'}{x-a}+\frac{1-\beta-\beta'}{x-b}+\frac{1-\gamma-\gamma'}{x-c}\right\}\frac{dy}{dx}$$
$$+\left\{\frac{\alpha\alpha'(a-b)(a-c)}{x-a}+\frac{\beta\beta'(b-c)(b-a)}{x-b}+\frac{\gamma\gamma'(c-a)(c-b)}{x-c}\right\}\frac{y}{(x-a)(x-b)(x-c)}=0,$$

wherein $\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=1.$

In the supplement[7] to Riemann's collected works published in 1902, some brief extracts from a course of lectures delivered by Riemann in 1858–1859 are given. In these lectures Riemann worked from integrals of the type

$$\int_0^1 s^a (1-s)^b (1-xs)^c \, ds$$

and commenced the development of the theory of polyhedral and elliptic-modular functions. A synopsis of the lectures is given in the volume (pp. 109, 110), and it is a matter for grave regret that they have not been published *in extenso*.

Klein's contribution[8] to the theory with which the present paper is immediately concerned consists in taking Pochhammer's circuit[9] (which seems itself to have been found among Riemann's manuscripts) and giving a very elegant form to integrals of the type

$$x^{\alpha} (1-x)^{\beta} \int s^a (1-s)^b (1-xs)^c \, ds \tag{1}$$

by means of homogeneous variables.

2. It might be thought that, when the theory had been discussed with such vigour, there was no room for a new development.

But anyone who has long worked with integrals of the type (1) taken round Pochhammer's circuits is well acquainted with the labour which is involved in an accurate determination of the many-valued functions which occur. Moreover, Riemann's P-function was only defined by him after the singularities a, b, c had been transformed to 0, 1, and ∞ and for the region of the plane for which $I(x)$ is positive. [Throughout this paper $I(x)$ will be used to denote the imaginary part of x and $R(x)$ the real part of x.] It is obviously of importance to give a definition which will hold for all values of x when the singularities have their most general position.

In this paper I shew that it is possible to meet both these objections or requirements and to simplify substantially the whole theory by means *of contour integrals involving gamma functions of the variable of integration*. It is possible to write down twenty-four contour integrals which satisfy the hypergeometric equation. A typical integral is

$$I_1 = -\frac{1}{2\pi i} \int \frac{\Gamma(\alpha_1 + s)\, \Gamma(\alpha_2 + s)\, \Gamma(-s)}{\Gamma(\rho + s)} (-x)^s \, ds,$$

wherein $|\arg(-x)|^* < \pi$, and the contour of the integral is parallel to the imaginary axis with loops, if necessary to ensure that the points 0, 1, 2, ... are to the right and the points $\left\{ \begin{matrix} -\alpha_1, & -\alpha_1 - 1, & -\alpha_1 - 2, & \ldots \\ -\alpha_2, & -\alpha_2 - 1, & -\alpha_2 - 2, & \ldots \end{matrix} \right.$ are to the left of the contour. *This integral is a valid solution for general values of $|x|$ whether greater than, equal to, or less than unity.*

The complete set of integrals I denote by $I_1, I_2, \ldots, I_{12}$; $I'_1, \ldots, I'_{12}$. Substantially $I_r = I'_r$, and from the set $I_1, \ldots, I_{12}$ we can derive six sets of pairs which are substantially equal. We thus get four times over the fundamental solutions

$$Y_1,\ Y_2,\ Y_3,\ Y_4,\ Y_5,\ Y_6$$

expressed as contour integrals of the specified type.

Now, by connecting any of these solutions Y with the two solutions which arise at one of the two singularities not associated with Y, we have obviously twelve linear relations between $Y_1, Y_2, \ldots, Y_6$.

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* In this way we propose to write briefly the modulus of the argument (or amplitude) of $-x$.

Denoting a linear relation between a, b, c by $(a, b, c) = 0$, these twelve relations will be

$$(Y_1, Y_5, Y_6) = 0, \quad \text{I.}; \qquad (Y_3, Y_4, Y_5) = 0, \quad \text{VII.};$$
$$(Y_2, Y_5, Y_6) = 0, \quad \text{II.}; \qquad (Y_3, Y_4, Y_6) = 0, \quad \text{VIII.};$$
$$(Y_1, Y_2, Y_5) = 0, \quad \text{III.}; \qquad (Y_1, Y_3, Y_4) = 0, \quad \text{IX.};$$
$$(Y_1, Y_2, Y_6) = 0, \quad \text{IV.}; \qquad (Y_2, Y_3, Y_4) = 0, \quad \text{X.};$$
$$(Y_3, Y_5, Y_6) = 0, \quad \text{V.}; \qquad (Y_1, Y_2, Y_3) = 0, \quad \text{XI.};$$
$$(Y_4, Y_5, Y_6) = 0, \quad \text{VI.}; \qquad (Y_1, Y_2, Y_4) = 0, \quad \text{XII.};$$

Now the advantage of the contour integrals which I have introduced is, that the contour integral I_R or the alternative contour integral I'_R gives at once, by an almost obvious transformation, the relation (R). The relations I., ..., XII. thus arise almost intuitively.

3. We can apply the same process to Riemann's P-function.

Since Papperitz's equation can be transformed into Kummer's, we can, from any contour integral which satisfies the latter equation, deduce a contour integral which satisfies Papperitz's equation for all values of $|x|$ on a suitably dissected plane.

A typical integral obtained in this way is

$$K_1 = -\frac{1}{2\pi\iota}\left(\frac{x-c}{x-a}\,\frac{b-a}{b-c}\right)^{\gamma}\int\frac{\Gamma(\alpha+\gamma-s)}{\Gamma(1-\alpha'-\gamma+s)}\,\Gamma(\beta+s)\,\Gamma(\beta'+s)\left(-\frac{x-a}{c-a}\,\frac{c-b}{x-b}\right)^{s}ds,$$

wherein
$$\left|\arg\left(-\frac{x-a}{c-a}\,\frac{c-b}{x-b}\right)\right| < \pi,$$

and the contour of the integral is parallel to the imaginary axis with loops if necessary to ensure that $\alpha+\gamma,\ \alpha+\gamma+1, \ldots$ are to the right and $\begin{cases}-\beta,\ -\beta-1, \ldots \\ -\beta',\ -\beta'-1, \ldots\end{cases}$ are to the left of the contour.

This integral, by an obvious interchange of $\begin{cases} a,\ b,\ c, \\ \alpha,\ \beta,\ \gamma \\ \alpha',\ \beta',\ \gamma' \end{cases}$ can take twenty-four different forms. The twenty-four solutions of Papperitz's equation are thus in evidence. And the relations between them and Riemann's functions P_α, $P_{\alpha'}$; P_β, $P_{\beta'}$; P_γ, $P_{\gamma'}$, and Riemann's relations between the latter, arise with complete symmetry.*

4. The idea of taking contour integrals involving gamma functions in the subject of integration appears to be due to Pincherle,(10) who has been followed by Mellin,(11) though the type of contour and its use can be traced back to Riemann.(12) The author has made the method fundamental in several recent investigations.(13)

In conclusion, it may be observed that the contour integrals introduced in this paper are valid when any of the quantities α_1, α_2, ρ are integers or differ by integers, and in the case of Riemann's P-function, when $\alpha-\alpha'$, $\beta-\beta'$, or $\gamma-\gamma'$ are integers. The corresponding solutions, even when they involve logarithmic terms, are readily obtained. The results, in general, agree

* Some of the relations I.–XII. are given in Chapter vi. of Forsyth's *Treatise on Differential Equations* (Third Edition, London, 1903); but the forms there given are not in complete accord with the forms obtained in this paper. The twenty-four solutions of Papperitz's equation are given in Whittaker's *Course of Modern Analysis*, but the forms which he gives are not in complete accord with the forms given by Thomae (*loc. cit.*, p. 329). The fact that many-valued functions are involved in the expressions which Whittaker gives would be an obstacle in the way of determining Riemann's coefficients a_γ, $a_{\gamma'}$ by the method which he suggests.

with those of Lindelöf.[14] As an example of this particularization, the differential equation of the quarter-periods of the Jacobian elliptic functions is discussed in Part III.

(1) Gauss, *Göttinger Commentationes Recentiores* (1812), T. II.; *Ges. Werke*, T. III., pp. 123–163.

(2) Kummer, *Crelle* (1836), T. XV., pp. 39–83 and 127–172.

(3) Euler, *Nova Acta Acad. Petropol*, T. XII. (1778), p. 58.

(4) Riemann, *Abh. d. Ges. d. Wiss. zu Göttingen*, T. VII. (1857); *Mathematische Werke* (2te *Auf.*), (1892), pp. 67 *et seq.*

(5) Thomae, *Crelle*, T. LXXXVII. (1879), pp. 222–349, especially pp. 306–333.

(6) Papperitz, *Mathematische Annalen*, T. XXV. (1885), p. 213.

(7) Riemann, *Mathematische Werke : Nachträge*, Edited by Noether and Wirtinger (1902), pp. 69–94.

(8) Klein, *Vorlesungen über die hypergeometrische Funktion* (lithographed), Göttingen (1894); *Mathematische Annalen* (1891), T. XXXVIII., pp. 144–152.

(9) Pochhammer, *Mathematische Annalen*, T. XXXV. (1890), pp. 470–494.

(10) Pincherle, *Atti d. R. Accademia dei Lincei*, Series IV., *Rendiconti*, Vol. IV., pp. 694–700 and 792–799.

(11) Mellin, *Acta Societatis Scientiarum Fennicae* (1895), T. XX., No. 12, p. 78.

(12) Riemann, *Mathematische Werke* (1892), pp. 145–147, or *Œuvres Mathématiques* (1898), pp. 166, 167.

(13) Barnes, *Proceedings of the London Mathematical Society* (1905), Ser. 2, Vol. 3, pp. 253–272; *Philosophical Transactions of the Royal Society*, (A) (1906), Vol. CCVI., pp. 249–297; *Quarterly Journal of Mathematics* (1907), Vol. XXXVIII., pp. 108–116 and pp. 116–140.

(14) Lindelöf, *Acta Societatis Scientiarum Fennicae* (1893), T. XIX., No. 1, pp. 1–31.

PART I.

The ordinary Hypergeometric Equation.

5. We take the differential equation of the ordinary hypergeometric functions in the form

$$\left[(\vartheta+a_1)(\vartheta+a_2)-\frac{1}{x}\vartheta(\vartheta+\rho-1)\right]y=0,$$

where $x\dfrac{d}{dx}\equiv\vartheta$. We shall refer to this as Kummer's equation. It may be written

$$\frac{d^2y}{dx^2}+\frac{\rho-(a_1+a_2+1)x}{x(1-x)}\frac{dy}{dx}-\frac{a_1a_2}{x(1-x)}y=0.$$

Suppose now that x has any value, real or complex, such that

$$|\arg(-x)|<\pi.$$

Then I say that the integrals

$$I_1=-\frac{1}{2\pi\iota}\int\frac{\Gamma(a_1+s)\,\Gamma(a_2+s)}{\Gamma(\rho+s)}\Gamma(-s)(-x)^s\,ds,$$

$$I_2=-\frac{1}{2\pi\iota}\int\frac{\Gamma(a_1+s)\,\Gamma(a_2+s)\,\Gamma(1-\rho-s)}{\Gamma(1+s)}(-x)^s\,ds,$$

$$I_3=-\frac{1}{2\pi\iota}\int\frac{\Gamma(a_1+s)\,\Gamma(1-\rho-s)\,\Gamma(-s)}{\Gamma(1-a_2-s)}(-x)^s\,ds,$$

$$I_4=-\frac{1}{2\pi\iota}\int\frac{\Gamma(a_2+s)\,\Gamma(1-\rho-s)\,\Gamma(-s)}{\Gamma(1-a_1-s)}(-x)^s\,ds,$$

exist and are solutions of Kummer's equation. Each integral is taken along a contour which is parallel to the imaginary axis with loops if necessary to ensure that those sequences of poles of the respective subjects of integration which are ultimately positive are to the right of the contour, while those sequences which are ultimately negative lie to the left.

In the first place, the integrals exist. For, when s tends to infinity along a parallel to the imaginary axis in the finite part of the plane

$$|\Gamma(s+a)| \exp\left\{\left(\frac{\pi}{2}-\epsilon\right\} |v|\right.$$

tends uniformly to zero if $s = u+\iota v$, where u and v are real and $\epsilon > 0$. The integrals therefore exist if $|\arg(-x)| < \pi$.

In the second place, the integrals satisfy Kummer's equation. Take, for instance, the integral I_1. We may obviously differentiate it with regard to x by differentiating under the sign of integration.

Hence

$$(\vartheta+a_1)(\vartheta+a_2)I_1 = -\frac{1}{2\pi\iota}\int (s+a_1)(s+a_2)\frac{\Gamma(a_1+s)\,\Gamma(a_2+s)\,\Gamma(-s)}{\Gamma(\rho+s)}(-x)^s ds$$

$$= -\frac{1}{2\pi\iota}\int \frac{\Gamma(a_1+s)\,\Gamma(a_2+s)\,\Gamma(1-s)}{\Gamma(\rho+s-1)}(-x)^{s-1} ds$$

taken along a contour which is derived from the former by moving it through a distance unity in a positive direction parallel to the real axis.

The original contour was evidently so chosen that we may take the last integral along the original contour: we may therefore write it

$$-\frac{1}{2\pi\iota}\int \frac{s(\rho+s-1)}{x}\;\frac{\Gamma(a_1+s)\,\Gamma(a_2+s)\,\Gamma(-s)}{\Gamma(\rho+s)}(-x)^s ds = \frac{d}{dx}(\vartheta+\rho-1)I_1.$$

Thus the integral satisfies Kummer's equation.

Evidently an almost identical proof will apply to the other integrals I_2, I_3, I_4.

6. Let us denote the hypergeometric series

$$1+\frac{a_1 a_2}{1.\rho}x+\frac{a_1(a_1+1)\,a_2(a_2+1)}{1.2.\rho.\rho+1}x^2+\ldots$$

by $F\{a_1, a_2;\, \rho;\, x\}$.

The series is convergent when $|x| < 1$. When $|x| > 1$,

$$F\{a_1, a_2;\, \rho;\, x\}$$

represents the continuation of the function represented by the series when $|x| < 1$.

We may now shew that, when $|x| < 1$, $F\{\alpha_1, \alpha_2; \rho; x\}$ *is a solution of Kummer's equation and that, when* $|x| > 1$, *the function* $F\{\alpha_1, \alpha_2; \rho; x\}$ *can be expressed in the form*

$$\frac{\Gamma(\rho)\,\Gamma(\alpha_2-\alpha_1)}{\Gamma(\alpha_2)\,\Gamma(\rho-\alpha_1)}(-x)^{-\alpha_1}F\{\alpha_1,\ 1+\alpha_1-\rho;\ 1+\alpha_1-\alpha_2;\ 1/x\}$$
$$+\frac{\Gamma(\rho)\,\Gamma(\alpha_1-\alpha_2)}{\Gamma(\alpha_1)\,\Gamma(\rho-\alpha_2)}(-x)^{-\alpha_2}F\{\alpha_2,\ 1+\alpha_2-\rho;\ 1+\alpha_2-\alpha_1;\ 1/x\}, \quad \text{(I.)}$$

provided $|\arg(-x)| < \pi$. This proviso uniquely prescribes $(-x)^{-\alpha_1}$ and $(-x)^{-\alpha_2}$, and indicates that $F\{\alpha_1, \alpha_2; \rho; x\}$ outside the circle $|x| = 1$ needs a cross-cut from 1 to $+\infty$ along the real axis to make it one-valued.

Take the integral

$$I_1 = -\frac{1}{2\pi\iota}\int \frac{\Gamma(\alpha_1+s)\,\Gamma(\alpha_2+s)\,\Gamma(-s)}{\Gamma(\rho+s)}(-x)^s\,ds,$$

wherein $|\arg(-x)| < \pi$, and suppose that $|x| < 1$. Then we may bend the contour of the integral round until it embraces the positive half of the real axis and encloses the poles of $\Gamma(-s)$ but no other poles of the subject of integration. And by the asymptotic expansion of the gamma function this alteration of the contour will not affect the value of the integral, provided $|x| < 1$.

But, by Cauchy's theorem, the value of the new integral is given by the sum of the residues within the contour. The residue of $\Gamma(-s)$ at $s = n$ is $(-)^{n-1}/n!$. We therefore have

$$I_1 = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1+n)\,\Gamma(\alpha_2+n)}{\Gamma(\rho+n)}\frac{(-)^n}{n!}(-x)^n,$$
$$= \frac{\Gamma(\alpha_1)\,\Gamma(\alpha_2)}{\Gamma(\rho)}F\{\alpha_1, \alpha_2; \rho; x\}, \quad (\text{if } |x| < 1).$$

Return to the original integral I_1 and suppose that $|x| > 1$. Then we may bend the contour of the integral round till it becomes a contour which encloses the poles of $\Gamma(\alpha_1+s)$ and $\Gamma(\alpha_2+s)$, but not those of $\Gamma(-s)$.

By Cauchy's theory of residues we get, if $|x| > 1$,

$$I_1 = \sum_{n=0}^{\infty} \frac{(-)^n}{n!}\frac{\Gamma(\alpha_2-\alpha_1-n)}{\Gamma(\rho-\alpha_1-n)}\Gamma(\alpha_1+n)(-x)^{-\alpha_1-n}$$

$+$ a similar series obtained by interchanging α_1 and α_2

$$= (-x)^{-\alpha_1}\frac{\Gamma(\alpha_2-\alpha_1)\,\Gamma(\alpha_1)}{\Gamma(\rho-\alpha_1)}F\{\alpha_1,\ 1+\alpha_1-\rho;\ 1+\alpha_1-\alpha_2;\ 1/x\}$$
$$+(-x)^{-\alpha_2}\frac{\Gamma(\alpha_1-\alpha_2)\,\Gamma(\alpha_2)}{\Gamma(\rho-\alpha_2)}F\{\alpha_2,\ 1+\alpha_2-\rho;\ 1+\alpha_2-\alpha_1;\ 1/x\}.$$

By equating the two values of I_1 we obtain the given theorem.

7. We may evidently apply the previous method to the integrals I_2, I_3, I_4 of § 5.

From I_2 we see that the series which represents

$$(-x)^{1-\rho} F\{1+a_1-\rho,\ 1+a_2-\rho\ ;\ 2-\rho\ ;\ x\}$$

is a solution of Kummer's equation, when $|x|<1$, and that this function can, when $|x|>1$ and $|\arg(-x)|<\pi$, be expressed in the form

$$\frac{\Gamma(a_2-a_1)\,\Gamma(2-\rho)}{\Gamma(1-a_1)\,\Gamma(1+a_2-\rho)}\cdot(-x)^{-a_1} F\{a_1,\ 1+a_1-\rho\ ;\ 1+a_1-a_2\ ;\ 1/x\}$$

$$+\frac{\Gamma(a_1-a_2)\,\Gamma(2-\rho)}{\Gamma(1-a_2)\,\Gamma(1+a_1-\rho)}\,(-x)^{-a_2} F\{a_2,\ 1+a_2-\rho\ ;\ 1+a_2-a_1\ ;\ 1/x\}. \qquad \text{(II.)}$$

From I_3 we see that the series which represents

$$(-x)^{-a_1} F\{a_1,\ 1+a_1-\rho\ ;\ 1+a_1-a_2\ ;\ 1/x\}$$

is a solution of Kummer's equation when $|x|>1$, and that this function can, when $|x|<1$ and $|\arg(-x)|<\pi$, be expressed in the form

$$\frac{\Gamma(1-\rho)\,\Gamma(1+a_1-a_2)}{\Gamma(1-a_2)\,\Gamma(1+a_1-\rho)}\,F\{a_1,\ a_2\ ;\ \rho\ ;\ x\}$$

$$+\frac{\Gamma(\rho-1)\,\Gamma(1-a_2+a_1)}{\Gamma(a_1)\,\Gamma(\rho-a_2)}\,(-x)^{1-\rho} F\{1+a_1-\rho,\ 1+a_2-\rho\ ;\ 2-\rho\ ;\ x\}. \qquad \text{(III.)}$$

From I_4 we see that the series which represents

$$(-x)^{-a_2} F\{a_2,\ 1+a_2-\rho\ ;\ 1+a_2-a_1\ ;\ 1/x\}$$

is a solution of Kummer's equation when $|x|>1$, and that this function can, when $|x|<1$ and $|\arg(-x)|<\pi$, be expressed in the form

$$\frac{\Gamma(1-\rho)\,\Gamma(1+a_2-a_1)}{\Gamma(1-a_1)\,\Gamma(1+a_2-\rho)}\,F\{a_1,\ a_2\ ;\ \rho\ ;\ x\}$$

$$+\frac{\Gamma(\rho-1)\,\Gamma(1-a_1+a_2)}{\Gamma(a_2)\,\Gamma(\rho-a_1)}\,(-x)^{1-\rho} F\{1+a_1-\rho,\ 1+a_2-\rho\ ;\ 2-\rho\ ;\ x\}. \qquad \text{(IV.)}$$

8. We may similarly shew that the integrals

$$I_5 = -\frac{1}{2\pi\iota}\int \frac{\Gamma(a_1+s)\,\Gamma(a_2+s)\,\Gamma(-s)}{\Gamma(1+a_1+a_2-\rho+s)}\,(x-1)^s\,ds,$$

$$I_6 = -\frac{1}{2\pi\iota}\int \frac{\Gamma(a_1+s)\,\Gamma(a_2+s)\,\Gamma(\rho-a_1-a_2-s)}{\Gamma(1+s)}\,(x-1)^s\,ds,$$

$$I_7 = -\frac{1}{2\pi\iota}\int \frac{\Gamma(\rho-a_1-a_2-s)\,\Gamma(a_1+s)\,\Gamma(-s)}{\Gamma(1-a_2-s)}\,(x-1)^s\,ds,$$

$$I_8 = -\frac{1}{2\pi\iota}\int \frac{\Gamma(\rho-a_1-a_2-s)\,\Gamma(a_2+s)\,\Gamma(-s)}{\Gamma(1-a_1-s)}\,(x-1)^s\,ds,$$

which exist when $|\arg(x-1)| < \pi$ for all values of $|x|$, are solutions of Kummer's equation. These integrals, when treated in the manner of § 6, lead to the relations

$$F\{\alpha_1, \alpha_2;\ 1+\alpha_1+\alpha_2-\rho;\ 1-x\}$$
$$= \frac{\Gamma(\alpha_2-\alpha_1)\,\Gamma(\alpha_1+\alpha_2-\rho)}{\Gamma(\alpha_2)\,\Gamma(1+\alpha_2-\rho)}(x-1)^{-\alpha_1}F\left\{\alpha_1, \rho-\alpha_2;\ 1+\alpha_1-\alpha_2;\ \frac{1}{1-x}\right\}$$
$$+\frac{\Gamma(\alpha_1-\alpha_2)\,\Gamma(\alpha_1+\alpha_2-\rho)}{\Gamma(\alpha_1)\,\Gamma(1+\alpha_1-\rho)}(x-1)^{-\alpha_2}F\left\{\alpha_2, \rho-\alpha_1;\ 1+\alpha_2-\alpha_1;\ \frac{1}{1-x}\right\}, \quad \text{(V.)}$$

$$(x-1)^{\rho-\alpha_1-\alpha_2}F\{\rho-\alpha_1, \rho-\alpha_2;\ 1+\rho-\alpha_1-\alpha_2;\ 1-x\}$$
$$= \frac{\Gamma(\alpha_2-\alpha_1)\,\Gamma(1+\rho-\alpha_1-\alpha_2)}{\Gamma(1-\alpha_1)\,\Gamma(\rho-\alpha_1)}(x-1)^{-\alpha_1}F\left\{\alpha_1, \rho-\alpha_2;\ 1+\alpha_1-\alpha_2;\ \frac{1}{1-x}\right\}$$
$$+\frac{\Gamma(\alpha_1-\alpha_2)\,\Gamma(1+\rho-\alpha_1-\alpha_2)}{\Gamma(1-\alpha_2)\,\Gamma(\rho-\alpha_2)}(x-1)^{-\alpha_2}F\left\{\alpha_2, \rho-\alpha_1;\ 1+\alpha_2-\alpha_1;\ \frac{1}{1-x}\right\}, \quad \text{(VI.)}$$

$$(x-1)^{-\alpha_1}F\left\{\rho-\alpha_2, \alpha_1;\ 1-\alpha_2+\alpha_1;\ \frac{1}{1-x}\right\}$$
$$= \frac{\Gamma(\rho-\alpha_1-\alpha_2)\,\Gamma(1-\alpha_2+\alpha_1)}{\Gamma(\rho-\alpha_2)\,\Gamma(1-\alpha_2)}F\{\alpha_1, \alpha_2;\ 1-\rho+\alpha_1+\alpha_2;\ 1-x\}$$
$$+\frac{\Gamma(\rho-\alpha_1)\,\Gamma(1-\alpha_2+\alpha_1)}{\Gamma(\alpha_1)\,\Gamma(1+\rho-\alpha_1-\alpha_2)}(x-1)^{\rho-\alpha_1-\alpha_2}F\{\rho-\alpha_1, \rho-\alpha_2;\ 1+\rho-\alpha_1-\alpha_2;\ 1-x\}, \quad \text{(VII.)}$$

and

$$(x-1)^{-\alpha_2}F\left\{\rho-\alpha_1, \alpha_2;\ 1-\alpha_1+\alpha_2;\ \frac{1}{1-x}\right\}$$
$$= \frac{\Gamma(\rho-\alpha_1-\alpha_2)\,\Gamma(1-\alpha_1+\alpha_2)}{\Gamma(\rho-\alpha_1)\,\Gamma(1-\alpha_1)}F\{\alpha_1, \alpha_2;\ 1-\rho+\alpha_1+\alpha_2;\ 1-x\}$$
$$+\frac{\Gamma(\rho-\alpha_2)\,\Gamma(1-\alpha_1+\alpha_2)}{\Gamma(\alpha_2)\,\Gamma(1+\rho-\alpha_1-\alpha_2)}(x-1)^{\rho-\alpha_1-\alpha_2}F\{\rho-\alpha_1, \rho-\alpha_2;\ 1+\rho-\alpha_1-\alpha_2;\ 1-x\}. \quad \text{(VIII.)}$$

Each of the series which represents any one of the four hypergeometric functions involved in these formulæ is, when it is convergent, a solution of Kummer's equation. The many-valued functions involved are limited by the cross-cut defined by $|\arg(x-1)| < \pi$.

9. In a similar manner we see that the integrals

$$I_9 = -\frac{1}{2\pi\iota}(1-x)^{-\alpha_1}\int\frac{\Gamma(\alpha_1+s)\,\Gamma(\rho-\alpha_2+s)\,\Gamma(-s)}{\Gamma(\rho+s)}\left(\frac{x}{1-x}\right)^s ds,$$

$$I_{10} = -\frac{1}{2\pi\iota}(1-x)^{-\alpha_1}\int\frac{\Gamma(\alpha_1+s)\,\Gamma(\rho-\alpha_2+s)\,\Gamma(1-\rho-s)}{\Gamma(1+s)}\left(\frac{x}{1-x}\right)^s ds,$$

$$I_{11} = -\frac{1}{2\pi\iota}(1-x)^{-a_1}\int \frac{\Gamma(a_1+s)\,\Gamma(1-\rho-s)\,\Gamma(-s)}{\Gamma(1-\rho+a_2-s)}\left(\frac{x}{1-x}\right)^s ds,$$

$$I_{12} = -\frac{1}{2\pi\iota}(1-x)^{-a_1}\int \frac{\Gamma(-s)\,\Gamma(1-\rho-s)\,\Gamma(\rho-a_2+s)}{\Gamma(1-a_1-s)}\left(\frac{x}{1-x}\right)^s ds$$

are solutions of Kummer's equation which exist for all values of $|x|$ when $|\arg\{x/(1-x)\}| < \pi$. These integrals are therefore defined for the whole plane with a cross-cut along the entire real axis except between 0 and 1. These integrals, when treated in the manner of § 6, lead to the relations

$$\begin{aligned}
&(1-x)^{-a_1}F\left\{a_1, \rho-a_2; \rho; \frac{x}{x-1}\right\} \\
&= \frac{\Gamma(\rho-a_1-a_2)\,\Gamma(\rho)}{\Gamma(\rho-a_1)\,\Gamma(\rho-a_2)}x^{-a_1}F\left\{a_1, 1+a_1-\rho; 1+a_1+a_2-\rho; \frac{x-1}{x}\right\} \\
&\quad+\frac{\Gamma(a_1+a_2-\rho)\,\Gamma(\rho)}{\Gamma(a_1)\,\Gamma(a_2)}(1-x)^{\rho-a_1-a_2}x^{a_2-\rho} \\
&\qquad\times F\left\{1-a_2, \rho-a_2; 1+\rho-a_1-a_2; \frac{x-1}{x}\right\}, \qquad \text{(IX.)}
\end{aligned}$$

$$\begin{aligned}
&(1-x)^{\rho-a_1-1}x^{1-\rho}F\left\{1-a_2, 1+a_1-\rho; 2-\rho; \frac{x}{x-1}\right\} \\
&= \frac{\Gamma(\rho-a_1-a_2)\,\Gamma(2-\rho)}{\Gamma(1-a_1)\,\Gamma(1-a_2)}x^{-a_1}F\left\{1-\rho+a_1, a_1; 1-\rho+a_1+a_2; \frac{x-1}{x}\right\} \\
&\quad+\frac{\Gamma(a_1+a_2-\rho)\,\Gamma(2-\rho)}{\Gamma(1+a_1-\rho)\,\Gamma(1+a_2-\rho)}x^{a_2-\rho}(1-x)^{\rho-a_1-a_2} \\
&\qquad\times F\left\{\rho-a_2, 1-a_2; 1+\rho-a_1-a_2; \frac{x-1}{x}\right\}, \qquad \text{(X.)}
\end{aligned}$$

$$\begin{aligned}
&x^{-a_1}F\left\{1+a_1-\rho, a_1; 1-\rho+a_1+a_2; \frac{x-1}{x}\right\} \\
&= \frac{\Gamma(1-\rho)\,\Gamma(1-\rho+a_1+a_2)}{\Gamma(1-\rho+a_1)\,\Gamma(1-\rho+a_2)}(1-x)^{-a_1}F\left\{a_1, \rho-a_2; \rho; \frac{x}{x-1}\right\} \\
&\quad+\frac{\Gamma(1-\rho+a_1+a_2)\,\Gamma(\rho-1)}{\Gamma(a_1)\,\Gamma(a_2)}x^{1-\rho}(1-x)^{\rho-a_1-1} \\
&\qquad\times F\left\{1-\rho+a_1, 1-a_2; 2-\rho; \frac{x}{x-1}\right\}, \qquad \text{(XI.)}
\end{aligned}$$

$$\begin{aligned}
&x^{a_2-\rho}(1-x)^{\rho-a_1-a_2}F\left\{1-a_2, \rho-a_2; 1+\rho-a_1-a_2; \frac{x-1}{x}\right\} \\
&= \frac{\Gamma(1-\rho)\,\Gamma(1+\rho-a_1-a_2)}{\Gamma(1-a_1)\,\Gamma(1-a_2)}(1-x)^{-a_1}F\left\{a_1, \rho-a_2; \rho; \frac{x}{x-1}\right\} \\
&\quad+\frac{\Gamma(\rho-1)\,\Gamma(1+\rho-a_1-a_2)}{\Gamma(\rho-a_1)\,\Gamma(\rho-a_2)}x^{1-\rho}(1-x)^{\rho-a_1-1} \\
&\qquad\times F\left\{1-\rho+a_1, 1-a_2; 2-\rho; \frac{x}{x-1}\right\}. \qquad \text{(XII.)}
\end{aligned}$$

10. We have now given twelve integrals which are solutions of Kummer's equation.

If we consider the integrals I_1 and I_9, we see that they both give rise to hypergeometric series convergent near $x=0$ of the same (zero) exponent. By adjustment of the constant multiplier, we therefore see that

$$I_1\,\Gamma(\rho-a_2) = I_9\,\Gamma(a_2)$$

or

$$F\{a_1, a_2;\ \rho;\ x\} = (1-x)^{-a_1} F\left\{a_1,\ \rho-a_2;\ \rho;\ \frac{x}{x-1}\right\}. \tag{1}$$

Similarly we obtain other relations set forth in § 13 *infra*.

11. All the twenty-four solutions of Kummer's equation can now be displayed as six sets of four equivalent solutions.

For, from the relation (1) of § 10, we have

$$(1-x)^{-a_1} F\left\{a_1,\ \rho-a_2;\ \rho;\ \frac{x}{x-1}\right\} = F\{a_1, a_2;\ \rho;\ x\}:$$

and therefore by symmetry each is equal to

$$(1-x)^{-a_2} F\left\{a_2,\ \rho-a_1;\ \rho;\ \frac{x}{x-1}\right\}.$$

Take now the equality

$$F\left\{a_1,\ \rho-a_2;\ \rho;\ \frac{x}{x-1}\right\} = (1-x)^{a_1-a_2} F\left\{a_2,\ \rho-a_1;\ \rho;\ \frac{x}{x-1}\right\},$$

and in it put x for $x/(x-1)$, a_2 for $\rho-a_2$, and we get

$$F\{a_1, a_2;\ \rho;\ x\} = (1-x)^{\rho-a_1-a_2} F\{\rho-a_1,\ \rho-a_2;\ \rho;\ x\}.$$

We therefore have the first set of four equivalent solutions:

$$\begin{aligned} Y_1 = F\{a_1, a_2;\ \rho;\ x\} &= (1-x)^{\rho-a_1-a_2} F\{\rho-a_1,\ \rho-a_2;\ \rho;\ x\} \\ &= (1-x)^{-a_1} F\left\{a_1,\ \rho-a_2;\ \rho;\ \frac{x}{x-1}\right\} \\ &= (1-x)^{-a_2} F\left\{a_2,\ \rho-a_1;\ \rho;\ \frac{x}{x-1}\right\}. \end{aligned} \tag{A}$$

Change a_1 into $1+a_1-\rho$, a_2 into $1+a_2-\rho$; ρ into $2-\rho$, and we get the second set

$$\begin{aligned} Y_2 &= x^{1-\rho} F\{1+a_1-\rho,\ 1+a_2-\rho;\ 2-\rho;\ x\} \\ &= x^{1-\rho}(1-x)^{\rho-a_1-a_2} F\{1-a_1,\ 1-a_2;\ 2-\rho;\ x\} \\ &= x^{1-\rho}(1-x)^{\rho-a_1-1} F\left\{1+a_1-\rho,\ 1-a_2;\ 2-\rho;\ \frac{x}{x-1}\right\} \\ &= x^{1-\rho}(1-x)^{\rho-a_2-1} F\left\{1+a_2-\rho,\ 1-a_1;\ 2-\rho;\ \frac{x}{x-1}\right\}. \end{aligned} \tag{B}$$

Thus Y_1 and Y_2 are the two linearly independent solutions of Kummer's equation valid near $x=0$.

If Y_3 and Y_4 are the two linearly independent solutions valid near $x=1$, we have from (A), by obvious transformations,

$$\begin{aligned} Y_3 &= F\{\alpha_1, \alpha_2;\ 1+\alpha_1+\alpha_2-\rho;\ 1-x\} \\ &= x^{1-\rho} F\{1+\alpha_1-\rho, 1+\alpha_2-\rho;\ 1+\alpha_1+\alpha_2-\rho;\ 1-x\} \\ &= x^{-\alpha_1} F\left\{\alpha_1, 1+\alpha_1-\rho;\ 1+\alpha_1+\alpha_2-\rho;\ \frac{x-1}{x}\right\} \\ &= x^{-\alpha_2} F\left\{\alpha_2, 1+\alpha_2-\rho;\ 1+\alpha_1+\alpha_2-\rho;\ \frac{x-1}{x}\right\}, \end{aligned} \tag{C}$$

$$\begin{aligned} Y_4 &= (1-x)^{\rho-\alpha_1-\alpha_2} F\{\rho-\alpha_1, \rho-\alpha_2;\ 1+\rho-\alpha_1-\alpha_2;\ 1-x\} \\ &= x^{1-\rho}(1-x)^{\rho-\alpha_1-\alpha_2} F\{1-\alpha_1, 1-\alpha_2;\ 1+\rho-\alpha_1-\alpha_2;\ 1-x\} \\ &= x^{\alpha_2-\rho}(1-x)^{\rho-\alpha_1-\alpha_2} F\left\{\rho-\alpha_2, 1-\alpha_2;\ 1+\rho-\alpha_1-\alpha_2;\ \frac{x-1}{x}\right\} \\ &= x^{\alpha_1-\rho}(1-x)^{\rho-\alpha_1-\alpha_2} F\left\{\rho-\alpha_1, 1-\alpha_1;\ 1+\rho-\alpha_1-\alpha_2;\ \frac{x-1}{x}\right\}. \end{aligned} \tag{D}$$

And, finally, if Y_5 and Y_6 are the two linearly independent solutions valid near $x=\infty$, we have

$$\begin{aligned} Y_5 &= x^{-\alpha_1} F\{\alpha_1, 1+\alpha_1-\rho;\ 1+\alpha_1-\alpha_2;\ 1/x\} \\ &= x^{\alpha_2-\rho}(x-1)^{\rho-\alpha_1-\alpha_2} F\{1-\alpha_2, \rho-\alpha_2;\ 1+\alpha_1-\alpha_2;\ 1/x\} \\ &= (x-1)^{-\alpha_1} F\left\{\alpha_1, \rho-\alpha_2;\ 1+\alpha_1-\alpha_2;\ \frac{1}{1-x}\right\} \\ &= x^{1-\rho}(x-1)^{\rho-1-\alpha_1} F\left\{1-\alpha_2, 1+\alpha_1-\rho;\ 1+\alpha_1-\alpha_2;\ \frac{1}{1-x}\right\}, \end{aligned} \tag{E}$$

$$\begin{aligned} Y_6 &= x^{-\alpha_2} F\{\alpha_2, 1+\alpha_2-\rho;\ 1+\alpha_2-\alpha_1;\ 1/x\} \\ &= x^{\alpha_1-\rho}(x-1)^{\rho-\alpha_1-\alpha_2} F\{1-\alpha_1, \rho-\alpha_1;\ 1+\alpha_2-\alpha_1;\ 1/x\} \\ &= (x-1)^{-\alpha_2} F\left\{\alpha_2, \rho-\alpha_1;\ 1+\alpha_2-\alpha_1;\ \frac{1}{1-x}\right\} \\ &= x^{1-\rho}(x-1)^{\rho-1-\alpha_2} F\left\{1-\alpha_1, 1+\alpha_2-\rho;\ 1+\alpha_2-\alpha_1;\ \frac{1}{1-x}\right\}, \end{aligned} \tag{F}$$

We assume that in each case the principal values (whose argument lies between $\pm n\pi$) of n-th powers of x and $(x-1)$ are taken, where n is any quantity.

12. An examination of the transformation formulæ I., ..., XII. shews us that these formulæ are precisely the relations between the solutions $Y_1, \ldots, Y_6$ set forth in § 2.

Thus the relation (I.) may be written

$$Y_1 = \frac{\Gamma(\rho)\,\Gamma(a_2-a_1)}{\Gamma(a_2)\,\Gamma(\rho-a_1)}\,e^{\pm\pi\iota a_1}\,Y_5 + \frac{\Gamma(\rho)\,\Gamma(a_1-a_2)}{\Gamma(a_1)\,\Gamma(\rho-a_2)}\,e^{\pm\pi\iota a_2}\,Y_6,$$

also

$$Y_2 e^{\mp\pi\iota(1-\rho)} = \frac{\Gamma(a_2-a_1)\,\Gamma(2-\rho)}{\Gamma(1-a_1)\,\Gamma(1+a_2-\rho)}\,e^{\pm\pi\iota a_1}\,Y_5 + \frac{\Gamma(a_1-a_2)\,\Gamma(2-\rho)}{\Gamma(1-a_2)\,\Gamma(1+a_1-\rho)}\,e^{\pm\pi\iota a_2}\,Y_6, \qquad \text{II.}$$

$$Y_5 e^{\pm\pi\iota a_1} = \frac{\Gamma(1-\rho)\,\Gamma(1+a_1-a_2)}{\Gamma(1-a_2)\,\Gamma(1+a_1-\rho)}\,Y_1 + \frac{\Gamma(\rho-1)\,\Gamma(1-a_2+a_1)}{\Gamma(a_1)\,\Gamma(\rho-a_2)}\,e^{\mp\pi\iota(1-\rho)}\,Y_2, \qquad \text{III.}$$

$$Y_6 e^{\pm\pi\iota a_2} = \frac{\Gamma(1-\rho)\,\Gamma(1+a_2-a_1)}{\Gamma(1-a_1)\,\Gamma(1+a_2-\rho)}\,Y_1 + \frac{\Gamma(\rho-1)\,\Gamma(1-a_1+a_2)}{\Gamma(a_2)\,\Gamma(\rho-a_1)}\,e^{\mp\pi\iota(1-\rho)}\,Y_2, \qquad \text{IV.}$$

$$Y_3 = \frac{\Gamma(a_2-a_1)\,\Gamma(a_1+a_2-\rho)}{\Gamma(a_2)\,\Gamma(1+a_2-\rho)}\,Y_5 + \frac{\Gamma(a_1-a_2)\,\Gamma(a_1+a_2-\rho)}{\Gamma(a_1)\,\Gamma(1+a_1-\rho)}\,Y_6, \qquad \text{V.}$$

$$Y_4 e^{\pm\pi\iota(\rho-a_1-a_2)} = \frac{\Gamma(a_2-a_1)\,\Gamma(1+\rho-a_1-a_2)}{\Gamma(1-a_1)\,\Gamma(\rho-a_1)}\,Y_5 + \frac{\Gamma(a_1-a_2)\,\Gamma(1+\rho-a_1-a_2)}{\Gamma(1-a_2)\,\Gamma(\rho-a_2)}\,Y_6, \qquad \text{VI.}$$

$$Y_5 = \frac{\Gamma(\rho-a_1-a_2)\,\Gamma(1-a_2+a_1)}{\Gamma(\rho-a_2)\,\Gamma(1-a_2)}\,Y_3 + \frac{\Gamma(\rho-a_1)\,\Gamma(1-a_2+a_1)}{\Gamma(a_1)\,\Gamma(1+\rho-a_1-a_2)}\,e^{\pm\pi\iota(\rho-a_1-a_2)}\,Y_4, \qquad \text{VII.}$$

$$Y_6 = \frac{\Gamma(\rho-a_1-a_2)\,\Gamma(1-a_1+a_2)}{\Gamma(\rho-a_1)\,\Gamma(1-a_1)}\,Y_3 + \frac{\Gamma(\rho-a_2)\,\Gamma(1-a_1+a_2)}{\Gamma(a_2)\,\Gamma(1+\rho-a_1-a_2)}\,e^{\pm\pi\iota(\rho-a_1-a_2)}\,Y_4, \qquad \text{VIII.}$$

$$Y_1 = \frac{\Gamma(\rho-a_1-a_2)\,\Gamma(\rho)}{\Gamma(\rho-a_1)\,\Gamma(\rho-a_2)}\,Y_3 + \frac{\Gamma(a_1+a_2-\rho)\,\Gamma(\rho)}{\Gamma(a_1)\,\Gamma(a_2)}\,Y_4, \qquad \text{IX.}$$

$$Y_2 = \frac{\Gamma(\rho-a_1-a_2)\,\Gamma(2-\rho)}{\Gamma(1-a_1)\,\Gamma(1-a_2)}\,Y_3 + \frac{\Gamma(a_1+a_2-\rho)\,\Gamma(2-\rho)}{\Gamma(1+a_1-\rho)\,\Gamma(1+a_2-\rho)}\,Y_4, \qquad \text{X.}$$

$$Y_3 = \frac{\Gamma(1-\rho)\,\Gamma(1-\rho+a_1+a_2)}{\Gamma(1-\rho+a_1)\,\Gamma(1-\rho+a_2)}\,Y_1 + \frac{\Gamma(1-\rho+a_1+a_2)\,\Gamma(\rho-1)}{\Gamma(a_1)\,\Gamma(a_2)}\,Y_2, \qquad \text{XI.}$$

$$Y_4 = \frac{\Gamma(1-\rho)\,\Gamma(1+\rho-a_1-a_2)}{\Gamma(1-a_1)\,\Gamma(1-a_2)}\,Y_1 + \frac{\Gamma(\rho-1)\,\Gamma(1+\rho-a_1-a_2)}{\Gamma(\rho-a_1)\,\Gamma(\rho-a_2)}\,Y_2. \qquad \text{XII.}$$

In each case the upper or lower sign is taken as $I(x)$ is positive or negative.

We have, however, hitherto only written down the integrals which lead directly to twelve of the twenty-four solutions of Kummer's equation.

We now proceed to give the other set of twelve, and we number them in such a way that I'_R, when treated in the manner of § 6, leads to the relation (R). The integrals are

$$I'_1 = -\frac{1}{2\pi\iota}(1-x)^{\rho-\alpha_1-\alpha_2}\int\frac{\Gamma(-s)\,\Gamma(\rho-\alpha_2+s)\,\Gamma(\rho-\alpha_1+s)}{\Gamma(\rho+s)}(-x)^s\,ds,$$

$$I'_2 = -\frac{1}{2\pi\iota}(1-x)^{\rho-\alpha_1-\alpha_2}\int\frac{\Gamma(1-\rho-s)\,\Gamma(\rho-\alpha_2+s)\,\Gamma(\rho-\alpha_1+s)}{\Gamma(1+s)}(-x)^s\,ds,$$

$$I'_3 = -\frac{1}{2\pi\iota}(1-x)^{\rho-\alpha_1-\alpha_2}\int\frac{\Gamma(1-\rho-s)\,\Gamma(-s)\,\Gamma(\rho-\alpha_2+s)}{\Gamma(1-\rho+\alpha_1-s)}(-x)^s\,ds,$$

$$I'_4 = -\frac{1}{2\pi\iota}(1-x)^{\rho-\alpha_1-\alpha_2}\int\frac{\Gamma(-s)\,\Gamma(\rho-\alpha_1+s)\,(1-\rho-s)}{\Gamma(1-\rho+\alpha_2-s)}(-x)^s\,ds,$$

$$I'_5 = -\frac{1}{2\pi\iota}x^{1-\rho}\int\frac{\Gamma(-s)\,\Gamma(1-\rho+\alpha_1+s)\,\Gamma(1-\rho+\alpha_2+s)}{\Gamma(1-\rho+\alpha_1+\alpha_2+s)}(x-1)^s\,ds,$$

$$I'_6 = -\frac{1}{2\pi\iota}x^{1-\rho}\int\frac{\Gamma(\rho-\alpha_1-\alpha_2-s)\,\Gamma(1-\rho+\alpha_1+s)\,\Gamma(1-\rho+\alpha_2+s)}{\Gamma(1+s)}(x-1)^s\,ds,$$

$$I'_7 = -\frac{1}{2\pi\iota}x^{1-\rho}\int\frac{\Gamma(-s)\,\Gamma(1-\rho+\alpha_1+s)\,\Gamma(\rho-\alpha_1-\alpha_2-s)}{\Gamma(\rho-\alpha_2-s)}(x-1)^s\,ds,$$

$$I'_8 = -\frac{1}{2\pi\iota}x^{1-\rho}\int\frac{\Gamma(-s)\,\Gamma(1-\rho+\alpha_2+s)\,\Gamma(\rho-\alpha_1-\alpha_2-s)}{\Gamma(\rho-\alpha_1-s)}(x-1)^s\,ds,$$

$$I'_9 = -\frac{1}{2\pi\iota}(1-x)^{-\alpha_2}\int\frac{\Gamma(-s)\,\Gamma(\alpha_2+s)\,\Gamma(\rho-\alpha_1+s)}{\Gamma(\rho+s)}\left(\frac{x}{1-x}\right)^s ds,$$

$$I'_{10} = -\frac{1}{2\pi\iota}(1-x)^{-\alpha_2}\int\frac{\Gamma(\alpha_2+s)\,\Gamma(\rho-\alpha_1+s)\,\Gamma(1-\rho-s)}{\Gamma(1+s)}\left(\frac{x}{1-x}\right)^s ds,$$

$$I'_{11} = -\frac{1}{2\pi\iota}(1-x)^{-\alpha_2}\int\frac{\Gamma(-s)\,\Gamma(\alpha_2+s)\,\Gamma(1-\rho-s)}{\Gamma(1-\rho+\alpha_1-s)}\left(\frac{x}{1-x}\right)^s ds,$$

$$I'_{12} = -\frac{1}{2\pi\iota}(1-x)^{-\alpha_2}\int\frac{\Gamma(\rho-\alpha_1+s)\,\Gamma(-s)\,\Gamma(1-\rho-s)}{\Gamma(1-\alpha_2-s)}\left(\frac{x}{1-x}\right)^s ds.$$

13. By considering the hypergeometric series to which the integrals give rise, we obtain an extension of the equalities indicated in § 10.

Thus, if $J_1, \ldots, J_6$ be suitable multiples of $Y_1, \ldots, Y_6$ respectively, we have

$$
\begin{aligned}
J_1 &= I_1\Gamma(\rho-\alpha_1)\Gamma(\rho-\alpha_2) = I_9\Gamma(\rho-\alpha_1)\Gamma(\alpha_2) = I_1'\Gamma(\alpha_1)\Gamma(\alpha_2) \\
&= I_9'\Gamma(\alpha_1)\Gamma(\rho-\alpha_2), \\
J_2 &= I_2\Gamma(1-\alpha_1)\Gamma(1-\alpha_2) = I_{10}e^{\pm\pi i(\rho-1)}\Gamma(1-\alpha_1)\Gamma(1+\alpha_2-\rho) \\
&= I_2'\Gamma(1+\alpha_1-\rho)\Gamma(1+\alpha_2-\rho) = I_{10}'e^{\pm\pi i(\rho-1)}\Gamma(1-\alpha_2)\Gamma(1+\alpha_1-\rho), \\
J_3 &= I_5\Gamma(1-\rho+\alpha_1)\Gamma(1-\rho+\alpha_2) = I_{11}\Gamma(\alpha_2)\Gamma(1-\rho+\alpha_2) \\
&= I_5'\Gamma(\alpha_1)\Gamma(\alpha_2) = I_{11}'\Gamma(\alpha_1)\Gamma(1-\rho+\alpha_1), \\
J_4 &= I_6\Gamma(1-\alpha_1)\Gamma(1-\alpha_2) = I_{12}e^{\pm\pi i(\rho-\alpha_1-\alpha_2)}\Gamma(\rho-\alpha_1)\Gamma(1-\alpha_1) \\
&= I_6'\Gamma(\rho-\alpha_1)\Gamma(\rho-\alpha_2) = I_{12}'e^{\pm\pi i(\rho-\alpha_1-\alpha_2)}\Gamma(\rho-\alpha_2)\Gamma(1-\alpha_2), \\
J_5 &= I_3\Gamma(1-\alpha_2)\Gamma(\rho-\alpha_2) = I_7e^{\pm\pi i\alpha_1}\Gamma(1-\alpha_2)\Gamma(1-\rho+\alpha_1) \\
&= I_3'\Gamma(\alpha_1)\Gamma(1-\rho+\alpha_1) = I_7'e^{\pm\pi i\alpha_1}\Gamma(\alpha_1)\Gamma(\rho-\alpha_2), \\
J_6 &= I_4\Gamma(1-\alpha_1)\Gamma(\rho-\alpha_1) = I_8e^{\pm\pi i\alpha_2}\Gamma(1-\alpha_1)\Gamma(1-\rho+\alpha_2) \\
&= I_4'\Gamma(\alpha_2)\Gamma(1-\rho+\alpha_2) = I_8'e^{\pm\pi i\alpha_2}\Gamma(\alpha_2)\Gamma(\rho-\alpha_1).
\end{aligned}
$$

In each case the upper or lower sign is taken as $I(x)$ is $\pm$: when x is real we have a point on one of the various systems of cross-cuts by which the integrals $I_1, \ldots, I_{12}$; $I_1', \ldots, I_{12}'$, are limited, and therefore some of the formulæ are illusory.

14. We have now obtained the twenty-four solutions of Kummer's equation and we have found the relations between each of the six sets of four which are substantially the same, and also the relations connecting any three of the six fundamental solutions. Our transformations (I.), ..., (XII.) are all, however, transformations by which substantially the variable is changed into its reciprocal. We now proceed to shew that the contour integrals can be so modified as to give directly the change of x into $1-x$; and, in fact, any other of the six transformations

$$x, \quad 1/x, \quad 1-x, \quad 1/(1-x), \quad x/(x-1), \quad (x-1)/x.$$

And, further, we will obtain directly from the theory of the contour integrals the relations given in § 13.

For this purpose we need a lemma which proves to be of fundamental importance in the theory.

15. LEMMA.—*If α_1, α_2, β_1, β_2 be any complex quantities of finite modulus, certain special cases excepted, and if the contour of the integral be parallel to the imaginary axis with loops if necessary to ensure that*

positive sequences of poles of the subject of integration lie to the right of the contour and negative sequences to the left,

$$-\frac{1}{2\pi\iota}\int \Gamma(\alpha_1+s)\,\Gamma(\alpha_2+s)\,\Gamma(\beta_1-s)\,\Gamma(\beta_2-s)\,ds = \frac{\Gamma(\alpha_1+\beta_1)\,\Gamma(\alpha_1+\beta_2)\,\Gamma(\alpha_2+\beta_1)\,\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_1+\alpha_2+\beta_1+\beta_2)}.$$

It is evident from the asymptotic expansion of the gamma function that, unless α_1, α_2, β_1, β_2 have such relations that the contour cannot be drawn, the integral will exist and have a definite finite value.

When α_1, α_2, β_1, β_2 are such that $R(1-\alpha_1-\alpha_2-\beta_1-\beta_2)>0$, the contour of the integral can be bent round so as to include negative sequences of poles of the subject of integration, and by Cauchy's theorem it will be equal to

$$\sum_{n=0}^{\infty}\frac{(-)^n}{n!}\,\Gamma(\beta_1+\alpha_1+n)\,\Gamma(\alpha_2-\alpha_1-n)\,\Gamma(\alpha_1+\beta_2+n) + \sum_{n=0}^{\infty}\frac{(-)^n}{n!}\,\Gamma(\beta_1+\alpha_2+n)\,\Gamma(\alpha_1-\alpha_2-n)\,\Gamma(\alpha_2+\beta_2+n)$$

$$= \Gamma(\alpha_1+\beta_1)\,\Gamma(\alpha_2-\alpha_1)\,\Gamma(\alpha_1+\beta_2)\,F\{\alpha_1+\beta_1,\ \alpha_1+\beta_2;\ 1+\alpha_1-\alpha_2;\ 1\} + \Gamma(\alpha_2+\beta_1)\,\Gamma(\alpha_1-\alpha_2)\,\Gamma(\alpha_2+\beta_2)\,F\{\alpha_2+\beta_1,\ \alpha_2+\beta_2;\ 1+\alpha_2-\alpha_1;\ 1\}.$$

By Gauss's theorem this is equal to

$$\Gamma(\alpha_1+\beta_1)\,\Gamma(\alpha_2-\alpha_1)\,\Gamma(\alpha_1+\beta_2)\,\frac{\Gamma(1+\alpha_1-\alpha_2)\,\Gamma(1-\alpha_1-\alpha_2-\beta_1-\beta_2)}{\Gamma(1-\alpha_2-\beta_1)\,\Gamma(1-\alpha_2-\beta_2)}$$

$+$ a similar expression obtained by interchanging α_1 and α_2

$$= -\Gamma(1-\alpha_1-\alpha_2-\beta_1-\beta_2)\,\Gamma(\alpha_1+\beta_1)\,\Gamma(\alpha_1+\beta_2)\,\Gamma(\alpha_2+\beta_1)\,\Gamma(\alpha_2+\beta_2)$$

$$\times\frac{1}{\pi\sin\pi(\alpha_1-\alpha_2)}\{\sin\pi(\alpha_2+\beta_1)\sin\pi(\alpha_2+\beta_2)-\sin\pi(\alpha_1+\beta_1)\sin\pi(\alpha_1+\beta_2)\}$$

$$= \Gamma(1-\alpha_1-\alpha_2-\beta_1-\beta_2)\,\Gamma(\alpha_1+\beta_1)\,\Gamma(\alpha_1+\beta_2)\,\Gamma(\alpha_2+\beta_1)\,\Gamma(\alpha_2+\beta_2) \times\frac{\sin\pi(\alpha_1+\alpha_2+\beta_1+\beta_2)}{\pi}$$

$$= \frac{\Gamma(\alpha_1+\beta_1)\,\Gamma(\alpha_1+\beta_2)\,\Gamma(\alpha_2+\beta_1)\,\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_1+\alpha_2+\beta_1+\beta_2)}.$$

We have limited ourselves by the restriction that

$$R(1-\alpha_1-\alpha_2-\beta_1-\beta_2)>0.$$

But the original integral and the final expression are, except for isolated points, analytic functions of α_1, α_2, β_1, and β_2. The theorem is therefore true in general.

16. *We now proceed to shew that, if J_1 denote the integral*

$$-\frac{1}{2\pi\iota}\int_C \Gamma(-s)\,\Gamma(\rho-\alpha_1-\alpha_2-s)\,\Gamma(\alpha_1+s)\,\Gamma(\alpha_2+s)\,(1-x)^s\,ds,$$

wherein $|\arg(1-x)| < 2\pi$, *and the contour C is parallel to the imaginary axis and passes between the sequences of positive and negative poles of the subject of integration, then*

$$J_1 = I_1\,\Gamma(\rho-\alpha_1)\,\Gamma(\rho-\alpha_2),$$

where I_1 is the integral defined in § 5.

I have previously shewn* that, if D be a contour parallel to the imaginary axis with loops leaving the positive sequence of poles 0, 1, 2, ... on the right and the negative sequence s, $s-1$, $s-2$, ... on the left,

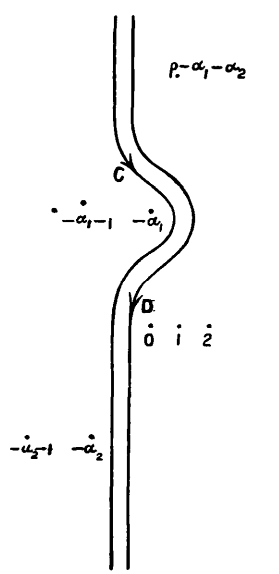
$$\Gamma(-s)\,(1-x)^s = -\frac{1}{2\pi\iota}\int_D \Gamma(\phi-s)\,\Gamma(-\phi)\,(-x)^\phi\,d\phi,$$

wherein $\quad |\arg(-x)| < \pi, \quad |\arg(1-x)| < \pi.$

We therefore have

$$J_1 = -\frac{1}{2\pi\iota}\int_C\left(-\frac{1}{2\pi\iota}\right)\int_D \Gamma(\rho-\alpha_1-\alpha_2-s)\,\Gamma(\alpha_1+s)\,\Gamma(\alpha_2+s)$$
$$\times\,\Gamma(\phi-s)\,\Gamma(-\phi)\,(-x)^\phi\,d\phi.$$

Now we may invert the order of integration, for each integral is absolutely and uniformly convergent. We may conveniently take the contours as in the figure. Hence we have



$$J_1 = -\frac{1}{2\pi\iota}\int_D \Gamma(-\phi)\,(-x)^\phi\,d\phi$$
$$\times\,-\frac{1}{2\pi\iota}\int_C \Gamma(\rho-\alpha_1-\alpha_2-s)\,\Gamma(\phi-s)$$
$$\times\,\Gamma(\alpha_1+s)\,\Gamma(\alpha_2+s)\,ds$$
$$= -\frac{1}{2\pi\iota}\,\Gamma(\rho-\alpha_1)\,\Gamma(\rho-\alpha_2)$$
$$\times\int_D \frac{\Gamma(-\phi)\,\Gamma(\alpha_1+\phi)\,\Gamma(\alpha_2+\phi)}{\Gamma(\rho+\phi)}\,(-x)^\phi\,d\phi$$

(by the lemma of § 15)

$$= \Gamma(\rho-\alpha_1)\,\Gamma(\rho-\alpha_2)\,I_1.$$

In the investigation we have assumed that

$$|\arg(-x)| < \pi$$

which gives the cross-cut necessary to define I_1.

* *Quarterly Journal of Mathematics*, Vol. XXXVIII., pp. 108–116.

This includes the condition $|\arg(1-x)| < \pi$. We notice that the integral J_1 represents the function over the extended range

$$|\arg(1-x)| < 2\pi.$$

17. Suppose now that $|1-x| < 1$. We may bend round the contour of the integral J_1 so as to include the positive sequences of poles of the subject of integration, and we therefore obtain, by Cauchy's theorem,

$$\frac{\Gamma(a_1)\,\Gamma(a_2)\,\Gamma(\rho-a_1)\,\Gamma(\rho-a_2)}{\Gamma(\rho)}\,F\{a_1, a_2;\ \rho;\ x\}$$

$$= \Gamma(-a_1-a_2)\,\Gamma(a_1)\,\Gamma(a_2)\,F\{a_1, a_2;\ 1+a_1+a_2-\rho;\ 1-x\}$$

$$+\Gamma(\rho-a_1)\,\Gamma(\rho-a_2)\,\Gamma(a_1+a_2-\rho)\,(1-x)^{\rho-a_1-a_2}$$

$$\times F\{\rho-a_1,\ \rho-a_2;\ 1+\rho-a_1-a_2;\ 1-x\};$$

or $F\{a_1, a_2;\ \rho;\ x\}$

$$= \frac{\Gamma(\rho-a_1-a_2)\,\Gamma(\rho)}{\Gamma(\rho-a_1)\,\Gamma(\rho-a_2)}\,F\{a_1, a_2;\ 1+a_1+a_2-\rho;\ 1-x\}$$

$$+\frac{\Gamma(a_1+a_2-\rho)\,\Gamma(\rho)}{\Gamma(a_1)\,\Gamma(a_2)}\,(1-x)^{\rho-a_1-a_2}\,F\{\rho-a_1,\ \rho-a_2;\ 1+\rho-a_1-a_2;\ 1-x\}.$$

This is a direct transformation from argument x to argument $(1-x)$. It is equivalent to our old relation (IX.) of § 12.

Suppose that similarly $|1-x| > 1$ and we bend the contour of the integral J_1 round to the left so as to include the negative sequences of poles of the subject of integration, we obtain

$F\{a_1, a_2;\ \rho;\ x\}$

$$= \frac{\Gamma(a_2-a_1)\,\Gamma(\rho)}{\Gamma(a_2)\,\Gamma(\rho-a_1)}\,(1-x)^{-a_1}\,F\{a_1,\ \rho-a_2;\ 1+a_1-a_2;\ 1/(1-x)\}$$

$$+\frac{\Gamma(a_1-a_2)\,\Gamma(\rho)}{\Gamma(a_1)\,\Gamma(\rho-a_2)}\,(1-x)^{-a_2}\,F\{a_2,\ \rho-a_1;\ 1+a_2-a_1;\ 1/(1-x)\}.$$

This is equivalent to our old relation (I.) of § 12.

The formula thus gives a direct transformation from x to $1/(1-x)$.

18. We can now shew that the integral J_1 gives rise directly to the set of equalities obtained indirectly in § 13 between I_1, I_9, I_1', I_9'. We have already seen that $J_1 = \Gamma(\rho-a_1)\,\Gamma(\rho-a_2)\,I_1$.

Again, if in the integral we write $s-a_1-a_2+\rho$ for s, it becomes

$$J_1 = (1-x)^{\rho-a_1-a_2}$$

$$\times\left(-\frac{1}{2\pi\iota}\right)\int \Gamma(-s)\,\Gamma(a_1+a_2-\rho-s)\,\Gamma(\rho-a_2+s)\,\Gamma(\rho-a_1+s)\,(1-x)^s ds.$$

This is the same integral as the former when a_1 is replaced by $\rho-a_1$ and

a_2 by $\rho-a_2$. The transformation of § 16 therefore gives us

$$J_1 = \Gamma(a_1)\,\Gamma(a_2)\,(1-x)^{\rho-a_1-a_2} \times\left(-\frac{1}{2\pi\iota}\right)\int \frac{\Gamma(\rho-a_1+\phi)\,\Gamma(\rho-a_2+\phi)\,\Gamma(-\phi)}{\Gamma(\rho+\phi)}\,(-x)^{\phi}\,d\phi$$

$$= \Gamma(a_1)\,\Gamma(a_2)\,I_1',$$

Similarly, by an obvious change of the variable, J_1 may be written

$$-\frac{1}{2\pi\iota}(1-x)^{-a_1}\int \Gamma(-s)\,\Gamma(a_2-a_1-s)\,\Gamma(a_1+s)\,\Gamma(\rho-a_2+s)\,(1-x)^{-s}ds$$

$$= \Gamma(a_2)\,\Gamma(\rho-a_1) \times\left(-\frac{1}{2\pi\iota}\right)(1-x)^{-a_1}\int \frac{\Gamma(-\phi)\,\Gamma(a_1+\phi)\,\Gamma(\rho-a_2+\phi)}{\Gamma(\rho+\phi)}\left(\frac{x}{1-x}\right)^{\phi}d\phi$$

$$= \Gamma(a_2)\,\Gamma(\rho-a_1)\,I_9,$$

and, by symmetry, $\qquad J_1 = \Gamma(a_1)\,\Gamma(\rho-a_2)\,I_9'.$

We thus have by direct transformation the first set of equalities of § 13, or, if we prefer so to regard it, the equalities (A) of § 11.

19. We can similarly obtain by direct transformation the other five sets of equalities of § 13, and all possible transformations and relations between three integrals of the system by means of the analogous integrals

$$J_2 = -\frac{1}{2\pi\iota}(-x)^{1-\rho}\int \Gamma(-s)\,\Gamma(\rho-a_1-a_2-s) \times\Gamma(1+a_1-\rho+s)\,\Gamma(1+a_2-\rho+s)\,(1-x)^{s}\,ds,$$

$$J_3 = -\frac{1}{2\pi\iota}\int \Gamma(-s)\,\Gamma(1-\rho-s)\,\Gamma(a_1+s)\,\Gamma(a_2+s)\,x^{s}\,ds,$$

$$J_4 = -\frac{1}{2\pi\iota}(x-1)^{\rho-a_1-a_2}\int \Gamma(-s)\,\Gamma(1-\rho-s)\,\Gamma(\rho-a_1+s)\,\Gamma(\rho-a_2+s)\,x^{s}\,ds.$$

$$J_5 = -\frac{1}{2\pi\iota}(-x)^{-a_1}\int \Gamma(-s)\,\Gamma(\rho-a_1-a_2-s)\,\Gamma(a_1+s) \times\Gamma(1+a_1-\rho+s)\left(\frac{x-1}{x}\right)^{s}ds,$$

$$J_6 = -\frac{1}{2\pi\iota}(-x)^{-a_2}\int \Gamma(-s)\,\Gamma(\rho-a_1-a_2-s)\,\Gamma(a_2+s) \times\Gamma(1+a_2-\rho+s)\left(\frac{x-1}{x}\right)^{s}ds.$$

The general theory of the hypergeometric solutions of Kummer's equation is evidently complete. When the quantities a_1, a_2, ρ or their differences are integers, the integrals are still valid representations of the solutions even though the hypergeometric series degenerate into different forms involving logarithmic terms. The case when $\rho = 1$, $a_1 = a_2 = \frac{1}{2}$ is discussed later: it gives rise to the quarter-periods of the Jacobian elliptic functions.

We now proceed to an analogous development of the Riemann P-functions, where the greater symmetry of the theory shews the elegance of Riemann's generalisation.

PART II.

The Riemann P-Functions.

20. The differential equation for Riemann's P-function is, in the form due to Papperitz,*

$$\frac{d^2y}{dx^2}+\left\{\frac{1-\alpha-\alpha'}{x-a}+\frac{1-\beta-\beta'}{x-b}+\frac{1-\gamma-\gamma'}{x-c}\right\}\frac{dy}{dx}$$
$$+\left\{\frac{\alpha\alpha'(a-b)(a-c)}{x-a}+\frac{\beta\beta'(b-c)(b-a)}{x-b}+\frac{\gamma\gamma'(c-a)(c-b)}{x-c}\right\}$$
$$\times\frac{y}{(x-a)(x-b)(x-c)}=0,$$

where $$\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=1.$$

Put $$z=\frac{x-a}{x-b}\frac{c-b}{c-a}\quad\text{and}\quad y=z^{\alpha}(1-z)^{\gamma}w,$$

and we obtain

$$\frac{d^2w}{dz^2}+\left(\frac{1+\alpha-\alpha'}{z}+\frac{1+\gamma-\gamma'}{z-1}\right)\frac{dw}{dz}+\{\beta\beta'+(\alpha+\gamma)(1-\alpha'-\gamma')\}\frac{w}{z(z-1)}=0.$$

Take now
$$a_1=\alpha+\beta+\gamma,$$
$$a_2=\alpha+\beta'+\gamma,$$
$$\rho=1+\alpha-\alpha',$$

so that $$a_1a_2=\beta\beta'+(\alpha+\gamma)(1-\alpha'-\gamma'),$$

since $$\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=1,$$

and we have $$\frac{d^2w}{dz^2}+\frac{\rho-(1+a_1+a_2)z}{z(1-z)}\frac{dw}{dz}-\frac{a_1a_2w}{z(1-z)}=0.$$

This is the ordinary equation of the hypergeometric function, and we have seen that a solution is

$$w=-\frac{1}{2\pi\iota}\int\frac{\Gamma(a_1+s)\,\Gamma(a_2+s)}{\Gamma(\rho+s)}\Gamma(-s)(-z)^s\,ds,$$

when $|\arg(-z)|<\pi$.

* Papperitz, *Mathematische Annalen*, T. xxv., p. 213.

Hence a solution of Papperitz's equation is

$$-\frac{1}{2\pi\iota}\left(\frac{x-c}{x-a}\frac{b-a}{b-c}\right)^{\gamma}\int\frac{\Gamma(a+\gamma-s)\,\Gamma(\beta+s)\,\Gamma(\beta'+s)}{\Gamma(1-a'-\gamma+s)}\left(-\frac{x-a}{c-a}\frac{c-b}{x-b}\right)^{s}ds,$$

when $\left|\arg -\frac{x-a}{c-a}\frac{c-b}{x-b}\right| < \pi$.

21. Leaving for the present the question of the precise determination of the functions $\left(\frac{x-c}{x-a}\frac{b-a}{b-c}\right)^{\gamma}$ and $\left(-\frac{x-a}{c-a}\frac{c-b}{x-b}\right)^{s}$ we can, by making all possible symmetrical interchanges in this solution, obtain the twenty-four integrals which satisfy Papperitz's equation. These we now proceed to indicate, denoting them by K with such suffixes and accents as correspond in order to the twenty-four solutions (A), ..., (F) (§ 11) of Kummer's equation when $a = 0$, $b = \infty$, $c = 1$, and when a_1, a_2, ρ are connected with a, β, γ, a', β', γ' by the relations of the previous paragraph, coupled with $a = 0$, $\gamma = 0$. Thus we have $a = 0$, $a' = 1-\rho$, $\beta = a_1$, $\beta' = a_2$, $\gamma = 0$, $\gamma' = \rho-a_1-a_2$. We thus write down the integrals in groups of four, each of which will be subsequently proved to be substantially the same functions of x. And in brackets we indicate the corresponding solution I of the more special case.

$$\left.\begin{aligned}
K_1 &= -\frac{1}{2\pi\iota}\left(\frac{x-c}{x-a}\frac{b-a}{b-c}\right)^{\gamma}\int\frac{\Gamma(a+\gamma-s)}{\Gamma(1-a'-\gamma+s)}\Gamma(\beta+s)\,\Gamma(\beta'+s)\\
&\qquad\times\left(-\frac{x-a}{c-a}\frac{c-b}{x-b}\right)^{s}ds\ (I_1)\\
K_2 &= -\frac{1}{2\pi\iota}\left(\frac{x-c}{x-a}\frac{b-a}{b-c}\right)^{\gamma'}\int\frac{\Gamma(a+\gamma'-s)}{\Gamma(1-a'-\gamma'+s)}\Gamma(\beta+s)\,\Gamma(\beta'+s)\\
&\qquad\times\left(-\frac{x-a}{c-a}\frac{c-b}{x-b}\right)^{s}ds\ (I_1')\\
K_3 &= -\frac{1}{2\pi\iota}\left(\frac{x-b}{x-a}\frac{c-a}{c-b}\right)^{\beta}\int\frac{\Gamma(a+\beta-s)}{\Gamma(1-a'-\beta+s)}\Gamma(\gamma+s)\,\Gamma(\gamma'+s)\\
&\qquad\times\left(-\frac{x-a}{b-a}\frac{b-c}{x-c}\right)^{s}ds\ (I_9)\\
K_4 &= -\frac{1}{2\pi\iota}\left(\frac{x-b}{x-a}\frac{c-a}{c-b}\right)^{\beta'}\int\frac{\Gamma(a+\beta'-s)}{\Gamma(1-a'-\beta'+s)}\Gamma(\gamma+s)\,\Gamma(\gamma'+s)\\
&\qquad\times\left(-\frac{x-a}{b-a}\frac{b-c}{x-c}\right)^{s}ds\ (I_9')
\end{aligned}\right\}.\quad (A)$$

If we interchange a and a' in this set we get in order K_5, K_6, K_7, K_8, which correspond respectively to I_2, I'_2, I_{10}, I'_{10} and form a set (B) of four integrals.

Interchange in (A) γ and α, γ' and α', b and c, and we get a set (C) of four integrals K_9, K_{10}, K_{11}, K_{12} which correspond to I_5, I'_5, I_{11}, I'_{11} respectively.

Interchange γ and γ' in the set (C) and we get the set (D) of four integrals K_{13}, K_{14}, K_{15}, K_{16} which correspond to I_6, I'_6, I_{12}, I'_{12}.

Interchange in (A) a and b, α and β, α' and β', and we get the set (E) of four integrals K_{17}, K_{18}, K_{19}, K_{20} which correspond to I_3, I'_3, I_7, I'_7.

Finally, interchanging β and β' in (E) we get the set (F) of four integrals K_{21}, K_{22}, K_{23}, K_{24} which correspond to I_4, I'_4, I_8, I'_8.

22. We have now to give an accurate definition of the many-valued functions which occur in the previous integrals $K_1, \ldots, K_{24}$.

Represent the three quantities a, b, c by points A, B, C by the usual Argand diagram, and, as in Riemann's memoir, let us assume that these points are so placed that, when we go round the circle through A, B, and C in a positive counter-clockwise direction, we pass from A to C to B.

We now define $\{(x-c)/(x-a)\}^\gamma$ by a cross-cut along the arc AC. (By this we mean the arc which excludes the point B: the other arc will be denoted by ABC.) When x lies within the circle, arg $\{(x-c)/(x-a)\}$ is taken to lie between $\pi+B$, its value just inside the arc AC, and B, its value on the arc ABC. When x lies outside the circle, arg $\{(x-c)/(x-a)\}$ is taken to lie between B, its value on the arc ABC, and $B-\pi$, its value just outside the arc AC.

The argument of $\dfrac{b-c}{b-a}$ is thus B. Hence $\dfrac{x-c}{b-c}\dfrac{b-a}{x-a}$ has a cross-cut along AC. Its argument ranges from π just within AC to zero on ABC, and to $-\pi$ just outside AC.

Similarly $\dfrac{x-a}{c-a}\dfrac{c-b}{x-b}$ has a cross-cut along BA, and its argument ranges from π just within BA to zero on BCA and to $-\pi$ just outside BA. And $\dfrac{x-b}{a-b}\dfrac{a-c}{x-c}$ has a cross-cut along CB, and its argument ranges from π just within CB to zero on CAB and to $-\pi$ just outside CB.

Further, in the preceding integrals we always take such values of $\left(-\dfrac{x-a}{c-a}\dfrac{c-b}{x-b}\right)^s$ and similar terms as have arguments less than π. Hence

$$\arg\left(-\frac{x-a}{c-a}\frac{c-b}{x-b}\right) = \arg\frac{x-a}{c-a}\frac{c-b}{x-b} \begin{matrix} -\pi \\ +\pi \end{matrix} \quad \text{as } x \text{ lies } \begin{matrix}\text{within} \\ \text{without}\end{matrix} \text{ the circle.}$$

Hence $-\dfrac{x-a}{c-a}\dfrac{c-b}{x-b}$ has a cross-cut along the arc ACB, and the value of its argument ranges from $-\pi$ just within ACB to zero on AB, and to π just outside ACB. The argument of the reciprocal of this expression is minus the argument of the expression. Similar definitions apply to the other similar terms which intervene in the integrals defined in the previous paragraph.

We see that, with such definitions,

$$\arg\frac{x-a}{x-b}+\arg\frac{x-b}{x-c}+\arg\frac{x-c}{x-a}=\begin{matrix}2\pi \text{ within} \\ 0 \text{ without}\end{matrix}\Big\}\text{ the circle;}$$

and also $$\left(\frac{x-a}{c-a}\frac{c-b}{x-b}\right)^{a}\left(\frac{x-b}{a-b}\frac{a-c}{x-c}\right)^{a}\left(-\frac{x-c}{b-c}\frac{b-a}{x-a}\right)^{a}=1,$$

whether x be within or without the circle, for the three terms in order have cross-cuts along AB, BC, and ABC, and these cross-cuts neutralise one another.

For brevity we shall write

$$u=\frac{x-b}{a-b}\frac{a-c}{x-c},\quad v=\frac{x-c}{b-c}\frac{b-a}{x-a},\quad w=\frac{x-a}{c-a}\frac{c-b}{x-b}.$$

The Functions P_a, $P_{a'}$, ..., $P_{\gamma'}$.

23. By § 21 it is evident that there is a solution of Papperitz' equation which near $x=a$ admits an expansion of the form

$$w^{a}\{1+C_1w+C_2w^2+\ldots\}.$$

This solution when $|\arg w|<\pi$ we denote by P_a. Thus P_a is defined with respect to a cross-cut along the arc AB: on the inside of this arc $\arg w=\pi$, on the outside it is $-\pi$, and on the arc ACB it is zero.

From the theory of the differential equation we know that equally P_a must admit near $x=a$ an expansion in powers of

$$1/v=\frac{x-a}{b-a}\frac{b-c}{x-c},$$

with index a at $x=a$. Now v^{a} has a cross-cut along AC and $\arg v$ ranges from π just within AC to zero on ABC, and then to π just outside AC. Hence, if $|\arg(-1/v)|<\pi$, $(-1/v)^{a}$ has a cross-cut along ABC, and $\arg(-1/v)$ ranges from π just within ABC to zero on AC, and then to $-\pi$ just outside ABC. Hence the arguments of w^{a} and $(-1/v)^{a}$ have the same range, and near $x = a$ they have the same cross-cut. Also

$$\frac{(-1/v)^{\alpha}}{w^{\alpha}} = \left(\frac{c-a}{c-x}\frac{x-b}{a-b}\right)^{\alpha},$$

and this ratio is unity at $x = a$. Therefore P_{α} may be equally expressed in the form

$$(-1/v)^{\alpha}\{1+D_1/v+D_2/v^2+\dots\},$$

wherein $|\arg(-1/v)| < \pi$. By writing α' for α we derive $P_{\alpha'}$ from P_{α}.

Similarly P_{β} near $x = b$ has a cross-cut along the arc BC and may be expressed in either of the forms

$$u^{\beta}\{1+E_1u+E_2u^2+\dots\}, \text{ wherein } |\arg u| < \pi;$$

or $$(-1/w)^{\beta}\{1+F_1/w+F_2/w^2+\dots\}, \text{ wherein } |\arg(-1/w)| < \pi.$$

And P_{γ} has a cross-cut near $x = c$ along the arc CA, and may be expressed in either of the forms

$$v^{\gamma}\{1+G_1v+G_2v^2+\dots\}, \text{ wherein } |\arg v| < \pi;$$

or $$(-1/u)^{\gamma}\{1+H_1/u+H_2/u^2+\dots\}, \text{ wherein } |\arg(-1/u)| < \pi.$$

24. We proceed now to shew that

$$P_{\alpha} = \left(\frac{x-c}{a-c}\frac{a-b}{x-b}\right)^{\gamma}\left(\frac{x-a}{c-a}\frac{c-b}{x-b}\right)^{\alpha}$$
$$\times F\left\{\alpha+\beta+\gamma,\ \alpha+\beta'+\gamma;\ 1-\alpha'+\alpha;\ \frac{x-a}{c-a}\frac{c-b}{x-b}\right\},$$

$$K_1 = e^{\mp\pi\iota\alpha}\frac{\Gamma(\alpha+\beta+\gamma)\,\Gamma(\alpha+\beta'+\gamma)}{\Gamma(1-\alpha'+\alpha)}P_{\alpha},$$

the upper or lower sign being taken as x lies within or without the circle. We have

$$K_1 = -\frac{1}{2\pi\iota}v^{\gamma}\int\frac{\Gamma(\alpha+\gamma-s)}{\Gamma(1-\alpha'-\gamma+s)}\Gamma(\beta+s)\,\Gamma(\beta'+s)(-w)^{s}\,ds,$$

with the previous specification of v^{γ} and $(-w)^{s}$.

When $|w| < 1$, we may bend round the contour so as to include the positive sequence of poles of the subject of integration, and we have, by Cauchy's theorem,

$$K_1 = v^{\gamma}(-w)^{\alpha+\gamma}\frac{\Gamma(\alpha+\beta+\gamma)\,\Gamma(\alpha+\beta'+\gamma)}{\Gamma(1-\alpha'+\alpha)}$$
$$\times F\{\alpha+\beta+\gamma,\ \alpha+\beta'+\gamma;\ 1-\alpha'+\alpha;\ w\}.$$

Now, by § 22, $v^{\gamma}(-w)^{\gamma} = (1/u)^{\gamma}$, and $(-w)^{\alpha} = w^{\alpha}e^{\mp\pi\iota\alpha}$.

Hence

$$K_1 = (1/u)^{\gamma} w^{\alpha} e^{\mp\pi\iota\alpha} \frac{\Gamma(\alpha+\beta+\gamma)\,\Gamma(\alpha+\beta'+\gamma)}{\Gamma(1-\alpha'+\alpha)}$$
$$\times F\{\alpha+\beta+\gamma,\ \alpha+\beta'+\gamma;\ 1-\alpha'+\alpha;\ w\}.$$

Now, at $x = a$, $(1/u)^{\gamma}$ approaches the value unity.

Hence, by the definition of § 23,

$$K_1 = e^{\mp\pi\iota\alpha} \frac{\Gamma(\alpha+\beta+\gamma)\,\Gamma(\alpha+\beta'+\gamma)}{\Gamma(1-\alpha'+\alpha)} P_{\alpha}.$$

We thus have the equalities given.

25. If we treat the integral K_2 in the same way, we get

$$K_2 = e^{\mp\pi\iota\alpha} \frac{\Gamma(\alpha+\beta+\gamma')\,\Gamma(\alpha+\beta'+\gamma')}{\Gamma(1-\alpha'+\alpha)} P_{\alpha},$$

$$P_{\alpha} = (1/u)^{\gamma'} w^{\alpha} F\{\alpha+\beta+\gamma',\ \alpha+\beta'+\gamma';\ 1-\alpha'+\alpha;\ w\}.$$

Similarly
$$K_3 = \frac{\Gamma(\alpha+\beta+\gamma)\,\Gamma(\alpha+\beta+\gamma')}{\Gamma(1-\alpha'+\alpha)} P_{\alpha},$$

$$P_{\alpha} = u^{\beta}(-1/v)^{\alpha} F\{\alpha+\beta+\gamma,\ \alpha+\beta+\gamma';\ 1-\alpha'+\alpha;\ 1/v\}.$$

Finally,
$$K_4 = \frac{\Gamma(\alpha+\beta'+\gamma)\,\Gamma(\alpha+\beta'+\gamma')}{\Gamma(1-\alpha'+\alpha)} P_{\alpha},$$

$$P_{\alpha} = u^{\beta'}(-1/v)^{\alpha} F\{\alpha+\beta'+\gamma,\ \alpha+\beta'+\gamma';\ 1-\alpha'+\alpha;\ 1/v\}.$$

We have thus deduced from the theory of the differential equation the equalities

$$\begin{aligned} K_1 e^{\pm\pi\iota\alpha}\Gamma(\alpha+\beta+\gamma')\,\Gamma(\alpha+\beta'+\gamma') &= K_2 e^{\pm\pi\iota\alpha}\Gamma(\alpha+\beta+\gamma)\,\Gamma(\alpha+\beta'+\gamma) \\ &= K_3\Gamma(\alpha+\beta'+\gamma)\,\Gamma(\alpha+\beta'+\gamma') \\ &= K_4\Gamma(\alpha+\beta+\gamma)\,\Gamma(\alpha+\beta+\gamma'). \end{aligned}$$

These equalities shew that the group of integrals (A) of § 21 are substantially equivalent to P_{α}. Similarly (B), (C), (D), (E), (F) respectively are equivalent to $P_{\alpha'}$, P_{γ}, $P_{\gamma'}$, P_{β}, $P_{\beta'}$.

26. We can verify the relations just found by direct transformation. Apply the result of § 16 to the function

$$K_1 = (1/u)^\gamma w^a e^{\mp\pi\iota a} \frac{\Gamma(a+\beta+\gamma)\,\Gamma(a+\beta'+\gamma)}{\Gamma(1-a'+a)}$$
$$\times F\{a+\beta+\gamma,\ a+\beta'+\gamma;\ 1-a'+a;\ w\}.$$

We find

$$K_1(1/w)^a e^{\pm\pi\iota a}\,\Gamma(a+\beta'+\gamma')\,\Gamma(a+\beta+\gamma')$$
$$= -\frac{1}{2\pi\iota} u^a \int \Gamma(a+\gamma+s)\,\Gamma(a+\gamma'+s)\,\Gamma(\beta-s)\,\Gamma(\beta'-s)\,u^s\,ds,$$

the direction of integration of the contour being downwards.

Interchanging β and γ, β' and γ', b and c, we find

$$K_1(1/w)^a e^{\pm\pi\iota a}\,\Gamma(a+\beta'+\gamma')\,\Gamma(a+\beta+\gamma')$$
$$= K_3 u^a(-v)^a\,\Gamma(a+\beta'+\gamma')\,\Gamma(a+\beta'+\gamma),$$

so that
$$K_1 e^{\pm\pi\iota a}\Gamma(a+\beta+\gamma') = K_3\Gamma(a+\beta'+\gamma),$$

since
$$w^a u^a(-v)^a = 1 \quad (\S\ 22).$$

The other relations can be obtained in like manner.

We note that incidentally we have shewn that

$$P_a \frac{\Gamma(a+\beta+\gamma)\,\Gamma(a+\beta'+\gamma)\,\Gamma(a+\beta+\gamma')\,\Gamma(a+\beta'+\gamma')}{\Gamma(1-a'+a)}$$
$$= -\frac{1}{2\pi\iota} w^a \int \Gamma(\gamma-s)\,\Gamma(\gamma'-s)\,\Gamma(a+\beta+s)\,\Gamma(a+\beta'+s)\,u^{-s}\,ds,$$

wherein $|\arg u| < \pi$. This and analogous expressions constitute the best definitions of the fundamental solutions P_a, $P_{a'}$, P_β, $P_{\beta'}$, P_γ, $P_{\gamma'}$.

27. We know from the theory of linear differential equations that a linear relation connects any three independent solutions. Adopting Riemann's notation we must therefore have relations of the form*

$$P_a = a_\beta P_\beta + a_{\beta'} P_{\beta'},$$

where a_β and $a_{\beta'}$ are constants.

* Such relations are of course part of Riemann's definition. He did not define his P-functions as solutions of Papperitz's equation.

We may easily establish this relation directly, and shew that

$$a_\beta = \frac{\Gamma(1-a'+a)\,\Gamma(\beta'-\beta)}{\Gamma(a+\beta'+\gamma)\,\Gamma(a+\beta'+\gamma')}\, e^{\pm\pi\iota a},$$

$$a_{\beta'} = \frac{\Gamma(1-a'+a)\,\Gamma(\beta-\beta')}{\Gamma(a+\beta+\gamma)\,\Gamma(a+\beta+\gamma')}\, e^{\pm\pi\iota a},$$

the upper or lower sign being, as usual, taken as x lies within or without the circle.

We have

$$P_a e^{\mp\pi\iota a} \frac{\Gamma(a+\beta+\gamma)\,\Gamma(a+\beta'+\gamma)}{\Gamma(1-a'+a)}$$

$$= K_1 = -\frac{1}{2\pi\iota}\, v^\gamma \int \frac{\Gamma(a+\gamma-s)}{\Gamma(1-a'-\gamma+s)}\, \Gamma(\beta+s)\,\Gamma(\beta'+s)(-w)^s\, ds.$$

If $|w| > 1$ we may bend round the contour so as to include the negative sequences of poles of the subject of integration, and we have by Cauchy's theorem

$$K_1 = v^\gamma(-w)^{-\beta} \frac{\Gamma(a+\beta+\gamma)\,\Gamma(\beta'-\beta)}{\Gamma(1-a'-\gamma-\beta)}$$

$$\times F\{a+\beta+\gamma,\ a'+\gamma+\beta;\ 1-\beta'+\beta;\ 1/w\}$$

+ a similar expression obtained by interchanging β and β'.

Hence, from § 23, since

$$a+a'+\beta+\beta'+\gamma+\gamma' = 1,$$

$$P_a e^{\mp\pi\iota a} \frac{\Gamma(a+\beta+\gamma)\,\Gamma(a+\beta'+\gamma)}{\Gamma(1-a'+a)}$$

$$= \frac{\Gamma(a+\beta+\gamma)\,\Gamma(\beta'-\beta)}{\Gamma(a+\beta'+\gamma')}\, P_\beta + \frac{\Gamma(a+\beta'+\gamma)\,\Gamma(\beta-\beta')}{\Gamma(a+\beta+\gamma')}\, P_{\beta'}.$$

We thus have the given values of a_β and $a_{\beta'}$.

Similarly we have the relation

$$P_a = a_\gamma P_\gamma + a_{\gamma'} P_{\gamma'},$$

wherein

$$a_\gamma = \frac{\Gamma(1-a'+a)\,\Gamma(\gamma'-\gamma)}{\Gamma(a+\beta+\gamma')\,\Gamma(a+\beta'+\gamma')}\, e^{\mp\pi\iota\gamma}$$

$$a_{\gamma'} = \frac{\Gamma(1-a'+a)\,\Gamma(\gamma-\gamma')}{\Gamma(a+\beta+\gamma)\,\Gamma(a+\beta'+\gamma)}\, e^{\mp\pi\iota\gamma'}.$$

These values of a_γ, $a_{\gamma'}$ are not symmetrical with those just obtained for a_β, $a_{\beta'}$, and this may be expected *a priori*, since P_γ, $P_{\gamma'}$ have a cross-cut along CA, while P_β, $P_{\beta'}$ have a cross-cut along BC.

The new result is easily obtained. We have

$$P_a \frac{\Gamma(a+\beta+\gamma)\,\Gamma(a+\beta+\gamma')}{\Gamma(1-a'+a)}$$

$$= K_3 = -\frac{1}{2\pi\iota} w^{-\beta} \int \frac{\Gamma(a+\beta-s)}{\Gamma(1-a'-\beta+s)} \Gamma(\gamma+s)\,\Gamma(\gamma'+s)\,(-1/v)^s\,ds$$

$$= w^{-\beta}(-1/v)^{-\gamma} \frac{\Gamma(a+\beta+\gamma)\,\Gamma(\gamma'-\gamma)}{\Gamma(a+\beta'+\gamma')}$$

$$\times F\{a+\beta+\gamma,\ a'+\beta+\gamma;\ 1-\gamma'+\gamma;\ v\}$$

+a similar expression obtained by interchanging γ and γ'

$$= \frac{\Gamma(a+\beta+\gamma)\,\Gamma(\gamma'-\gamma)}{\Gamma(a+\beta'+\gamma')} P_\gamma e^{\mp\pi\iota\gamma} + \frac{\Gamma(a+\beta+\gamma')\,\Gamma(\gamma-\gamma')}{\Gamma(a+\beta'+\gamma)} e^{\mp\pi\iota\gamma'} P_{\gamma'}.$$

We thus have the given values of a_γ, $a_{\gamma'}$.

We can now write down by cyclical interchange the values of β_γ, $\beta_{\gamma'}$, … .

28. The preceding results verify Riemann's manuscript* relations between the ratios of the coefficients, and we can immediately obtain the relations given by him in his memoir.†

We have

$$\frac{a_\beta}{a'_\beta} = \frac{\Gamma(1-a'+a)\,\Gamma(\beta'-\beta)}{\Gamma(a+\beta'+\gamma)\,\Gamma(a+\beta'+\gamma')} e^{\pm\pi\iota a} \Big/ \frac{\Gamma(1-a+a')\,\Gamma(\beta'-\beta)}{\Gamma(a'+\beta'+\gamma)\,\Gamma(a'+\beta'+\gamma')} e^{\pm\pi\iota a'}.$$

Therefore $$\frac{a_\beta \sin\pi(a+\beta+\gamma')\,e^{\mp\pi\iota a}}{a'_\beta \sin\pi(a'+\beta+\gamma')\,e^{\mp\pi\iota a'}}$$

$$= \frac{\Gamma(1-a'+a)\,\Gamma(a'+\beta'+\gamma')\,\Gamma(a'+\beta+\gamma')}{\Gamma(1-a+a')\,\Gamma(a+\beta+\gamma')\,\Gamma(a+\beta'+\gamma')} = \frac{a_\gamma}{a'_\gamma}.$$

Hence $$\frac{a_\gamma}{a'_\gamma} = \frac{a_\beta \sin(a+\beta+\gamma')\pi\, e^{\mp a\pi\iota}}{a'_\beta \sin(a'+\beta+\gamma')\pi\, e^{\mp a'\pi\iota}} = \frac{a_{\beta'} \sin(a+\beta'+\gamma')\pi\, e^{\mp a\pi\iota}}{a'_{\beta'} \sin(a'+\beta'+\gamma')\pi\, e^{\mp a'\pi\iota}}.$$

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* Note (1), p. 84, Riemann, *Œuvres Mathématiques* (Paris, 1898), or *Mathematische Werke* (1892), p. 86. These relations are found in several places among Riemann's manuscripts, but they are not in the first German edition of his collected works.

† *Loc. cit.*, p. 70.

This is equivalent to Riemann's result, for he takes the case when $a=0$, $b=\infty$, $c=1$, and $I(x)$ is positive, which corresponds to the case when x lies within the circle ABC in our more general investigation.

29. It is an obvious investigation to try to obtain for P_α an expansion near $x=a$ in ascending powers of $x-a$, our previous expansions having been in ascending powers of

$$w=\frac{x-a}{c-a}\frac{c-b}{x-b} \quad \text{or} \quad \frac{1}{v}=\frac{x-a}{b-a}\frac{b-c}{x-c}.$$

This investigation leads to hypergeometric functions of two variables of the type first introduced into analysis by Appell,* and is, in fact, equivalent to a theorem given by him.

We will briefly indicate a proof that when $|x-a|<|b-a|$ or $|c-a|$,

$$P_\alpha\left(\frac{a-b}{x-a}\frac{c-a}{c-b}\right)^\alpha\left(\frac{a-b}{x-b}\right)^\beta\left(\frac{a-c}{x-c}\right)^\gamma\frac{\Gamma(\alpha+\beta+\gamma')\,\Gamma(\alpha+\beta+\gamma)\,\Gamma(\alpha+\beta'+\gamma)}{\Gamma(1-\alpha'+\alpha)}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(m+\alpha+\beta'+\gamma)\,\Gamma(n+\alpha+\beta+\gamma')\,\Gamma(m+n+\alpha+\beta+\gamma)}{m!\,n!\,\Gamma(1-\alpha'+\alpha+m+n)}\times\left(\frac{x-a}{c-a}\right)^m\left(\frac{x-a}{b-a}\right)^n,$$

We take for simplicity the case when x does not lie within or on the sides of the smaller angle whose vertex is A and whose sides are AB and AC produced indefinitely in the directions AB and AC respectively. In this case, as may be readily verified by reference to a figure,

$$u^{-s}=\left(\frac{x-c}{c-a}\frac{b-a}{x-b}\right)^s=\left(1-\frac{x-a}{c-a}\right)^{s-\gamma}\left(1-\frac{x-a}{b-a}\right)^{-s-\alpha-\beta}\left(\frac{x-c}{a-c}\right)^\gamma\left(\frac{x-b}{a-b}\right)^{\alpha+\beta},$$

where $$\left|\arg\frac{x-c}{a-c}\right|<\pi \quad\text{and}\quad \left|\arg\frac{x-b}{a-b}\right|<\pi.$$

Also $$\frac{x-c}{a-c}=1+\frac{x-a}{a-c},\qquad \frac{x-b}{a-b}=1+\frac{x-a}{a-b},$$

where $$\left|\arg\frac{x-a}{a-c}\right|<\pi,\quad\text{and}\quad \left|\arg\frac{x-a}{a-b}\right|<\pi.$$

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* Appell, *Liouville* (1882), Sér. 3, T. VIII., pp. 173-216. Previous notes had appeared in the *Comptes Rendus*.

Therefore, by § 26 and the theorem quoted in § 16,

$$P_a \frac{\Gamma(\alpha+\beta+\gamma)\,\Gamma(\alpha+\beta'+\gamma)\,\Gamma(\alpha+\beta+\gamma')\,\Gamma(\alpha+\beta'+\gamma')}{\Gamma(1-a'+a)} w^{-a} \times \left(\frac{x-c}{a-c}\right)^{-\gamma}\left(\frac{x-b}{a-b}\right)^{-a-\beta}$$

$$= -\frac{1}{2\pi\iota} \int \Gamma(\gamma-s)\,\Gamma(\gamma'-s)\,\Gamma(a+\beta+s)\,\Gamma(a+\beta'+s) \times \left(1-\frac{x-a}{c-a}\right)^{s-\gamma}\left(1-\frac{x-a}{b-a}\right)^{-s-a-\beta} ds$$

$$= \left(-\frac{1}{2\pi\iota}\right)^3 \int ds \int d\phi \int d\psi\, \Gamma(\gamma'-s)\,\Gamma(\gamma+\phi-s)\,\Gamma(-\phi)\,\Gamma(\psi+a+\beta+s) \times \Gamma(-\psi)\,\Gamma(a+\beta'+s)\left(\frac{x-a}{a-c}\right)^{\phi}\left(\frac{x-a}{a-b}\right)^{\psi}.$$

The order of integration may be inverted, and we get, by § 15,

$$\Gamma(a+\beta'+\gamma')\left(-\frac{1}{2\pi\iota}\right)^2 \iint \Gamma(-\phi)\,\Gamma(-\psi)\,\Gamma(\psi+a+\beta+\gamma')\,\Gamma(\phi+a+\beta'+\gamma) \times \frac{\Gamma(\phi+\psi+a+\beta+\gamma)}{\Gamma(\phi+\psi+1-a'+a)}\left(\frac{x-a}{a-c}\right)^{\phi}\left(\frac{x-a}{a-b}\right)^{\psi} d\phi\, d\psi$$

$$= \Gamma(a+\beta'+\gamma') \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \times \frac{\Gamma(m+a+\beta'+\gamma)\,\Gamma(n+a+\beta+\gamma')\,\Gamma(m+n+a+\beta+\gamma)}{m!\,n!\,\Gamma(1-a'+a+m+n)}\left(\frac{x-a}{c-a}\right)^{m}\left(\frac{x-a}{b-a}\right)^{n}.$$

We thus have the given result if that value of $\left(\frac{a-b}{x-a}\frac{c-a}{c-b}\right)^{a}$ is taken which is equal to $w^{-a}\left(\frac{a-b}{x-b}\right)^{a}$, where w has a cross-cut along the arc AB, and $|\arg w| < \pi$. P_a is thus defined with respect to a cross-cut along the arc AB, and the result is therefore, by § 23, valid for all values of x for which the series is convergent.

Appell would denote the double series, with unity for its first term, by

$$F_1\left\{a+\beta+\gamma;\ a+\beta'+\gamma,\ a+\beta+\gamma';\ 1-a'+a;\ \frac{x-a}{c-a},\ \frac{x-a}{b-a}\right\}.$$

And the preceding result is equivalent to his theorem that

$$F\{a_1, a_2;\ \rho;\ \xi+\eta\} = (1-\eta)^{-a_1} F_1\left\{a_1;\ a_2,\ \rho-a_2;\ \rho;\ \frac{\xi}{1-\eta};\ \frac{-\eta}{1-\eta}\right\}.$$

The theory of the transformation of Appell's series can be developed entirely by the contour integrals introduced in the present paragraph.

PART III.

The Jacobian Elliptic Integrals K and K′ as Functions of k^2.

30. It has been stated that the contour integrals introduced into the preceding theory are valid even when degenerate cases of the hypergeometric series arise which involve a logarithmic term. As an example we will take the important case when $a_1 = a_2 = \frac{1}{2}$, $\rho = 1$, when Kummer's equation becomes the differential equation for Jacobi's elliptic integrals K and K' considered as functions of $x = k^2$.

The equation has been often considered, among others by Fuchs* and Tannery.† But the theory is invariably complicated, and most writers have, explicitly or implicitly, confined themselves to the case when $I(x)$ is positive. Tannery's investigation, which is given in Forsyth's treatise,‡ is cumbrous, and the investigation given by Schlesinger§ is difficult to follow, and not altogether accurate. The fundamental relation

$$K(x) \mp \iota K'(x) = x^{-\frac{1}{2}} K(1/x),$$

is, in fact, difficult to obtain from the Jacobian elliptic integrals. There appears to be an error in Forsyth's formula,‖ and in his corresponding substitution for $\iota K'/K$ corresponding to $x = \infty$. The complete investigation, whether $I(x)$ is $\pm$, is given in § 36 of the present paper.

31. The differential equation of the quarter-periods of the Jacobian elliptic functions is

$$x(1-x)\frac{d^2y}{dx^2} + (1-2x)\frac{dy}{dx} - \tfrac{1}{4}y = 0.$$

It corresponds to Kummer's equation with $a_1 = a_2 = \frac{1}{2}$, $\rho = 1$.

From the general theory, or by direct substitution, we readily see that a solution is

$$-\frac{1}{2\pi\iota}\int \{\Gamma(-s)\,\Gamma(\tfrac{1}{2}+s)\}^2 (1-x)^s\, ds.$$

The integral is convergent if $|\arg(1-x)| < 2\pi$, if we assume as usual that the contour of integration is parallel to the imaginary axis with loops

* Fuchs, *Crelle*, T. LXXI. (1870), pp. 121–127.

† Tannery, *Annales de l'École Normale Supérieure*, Sér. 2, T. VIII. (1879), pp. 169–194.

‡ Forsyth, *Theory of Differential Equations*, Part III., Vol. IV. (1902), pp. 129–135.

§ Schlesinger, *Linearen Differential Gleichungen*, Bd. II., pp. 476–484.

‖ Forsyth, *Theory of Functions* (2nd edition, 1900), p. 731.

if necessary to ensure that the positive sequence of poles of the subject of integration lies to the right, and the negative sequence to the left of the contour. We uniquely prescribe the integral as a function of x by the condition $|\arg(1-x)| < \pi$, and then we say that it defines $2\pi K(x)$. *Thus $K(x)$ has a cross-cut along the real axis from $+1$ to $+\infty$.*

Since the equation is unaltered when we write $1-x$ for x, we see that similarly

$$-\frac{1}{2\pi\iota}\int\{\Gamma(-s)\,\Gamma(\tfrac{1}{2}+s)\}^2 x^s\,ds$$

is a solution. When $|\arg x| < \pi$, we denote this solution by $2\pi K'(x)$. *Thus $K'(x)$ has a cross-cut along the real axis from $-\infty$ to 0.*

32. These definitions at once lead to a number of relations between K and K' of arguments belonging to the set x, $1/x$, $1-x$, $1/(1-x)$, $x/(x-1)$, $(x-1)/x$.

We have at once

$$\left.\begin{aligned} K(x) &= K'(1-x), &&\text{if } |\arg(1-x)| < \pi \\ K(1-x) &= K'(x), &&\text{if } |\arg x| < \pi \end{aligned}\right\}. \tag{A}$$

Again, in the integral defining $K(x)$ write $-s-\frac{1}{2}$ for s. Then the direction of the contour as always being downwards, we have

$$2\pi K(x) = (1-x)^{-\frac{1}{2}}\left(-\frac{1}{2\pi\iota}\right)\int\{\Gamma(\tfrac{1}{2}+s)\,\Gamma(-s)\}^2\left(1-\frac{x}{x-1}\right)^s ds$$

or $$K(x) = (1-x)^{-\frac{1}{2}}K\left(\frac{x}{x-1}\right), \quad \text{if } |\arg(1-x)| < \pi$$

therefore

$$K(1-x) = x^{-\frac{1}{2}}K\left(\frac{x-1}{x}\right), \quad \text{if } |\arg x| < \pi. \tag{B}$$

Applying the same process to the integral which defines $K'(x)$, we have

$$\left.\begin{aligned} K'(x) &= x^{-\frac{1}{2}}K'(1/x), &&\text{if } |\arg x| < \pi \\ K'(1-x) &= (1-x)^{-\frac{1}{2}}K'\left(\frac{1}{1-x}\right), &&\text{if } |\arg(1-x)| < \pi \end{aligned}\right\}. \tag{C}$$

From (A), we have

$$\left.\begin{aligned} K(1/x) &= K'\left(\frac{x-1}{x}\right), &&\text{if } \left|\arg\frac{x-1}{x}\right| < \pi \\ K\left(\frac{x-1}{x}\right) &= K'(1/x), &&\text{if } |\arg x| < \pi \end{aligned}\right\}. \tag{D}$$

Finally, from (B) and (C) by changing x into $1/x$,

$$\left.\begin{aligned} K(1/x) &= (1-1/x)^{-\frac{1}{2}} K\left(\frac{1}{1-x}\right), \quad \text{if} \quad |\arg(1-1/x)| < \pi \\ K'\left(\frac{x-1}{x}\right) &= (1-1/x)^{-\frac{1}{2}} K'\left(\frac{x}{x-1}\right), \quad \text{if} \quad |\arg(1-1/x)| < \pi \end{aligned}\right\}. \tag{E}$$

These relations may be summed up in the equalities

$$Y_1 = K(x) = K'(1-x) = (1-x)^{-\frac{1}{2}} K\left(\frac{x}{x-1}\right) = (1-x)^{-\frac{1}{2}} K'\left(\frac{1}{1-x}\right),$$

$$Y_3 = x^{-\frac{1}{2}} K(1/x) = x^{-\frac{1}{2}} K'\left(\frac{x-1}{x}\right) = (x-1)^{-\frac{1}{2}} K\left(\frac{1}{1-x}\right)$$

$$= (x-1)^{-\frac{1}{2}} K'\left(\frac{x}{x-1}\right),$$

and $Y_2 = K(1-x) = K'(x) = x^{-\frac{1}{2}} K'\left(\frac{1}{x}\right) = x^{-\frac{1}{2}} K\left(\frac{x-1}{x}\right)$,

though when written in this form we have no indication of the system of cross-cuts by which our equalities are limited. Such limitations must be those which have just been specified.

33. The three functions Y_1, Y_2, Y_3 are evidently solutions of the differential equation. They include all functions that can be obtained from $K(x)$, $K'(x)$ by the fundamental homographic transformations. It remains to find the linear relation connecting these three quantities, and explicit expansions for $K(x)$, $K'(x)$, when $|x| < 1$.

In the first place, by § 16,

$$2\pi K(x) = -\frac{1}{2\pi\iota}\int \{\Gamma(-s)\,\Gamma(\tfrac{1}{2}+s)\}^2 (1-x)^s\, ds$$

$$= -\frac{\pi}{2\pi\iota}\int \frac{\{\Gamma(\frac{1}{2}+s)\}^2\,\Gamma(-s)}{\Gamma(1+s)} (-x)^s\, ds,$$

when $|\arg(-x)| < \pi$.

From the last integral we see, by applying Cauchy's theorem in the usual manner, that, when $|x| < 1$,

$$2K(x) = \sum_{n=0}^{\infty} \frac{\{\Gamma(\frac{1}{2}+n)\}^2}{\{n!\}^2} x^n$$

or

$$K(x) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left\{\frac{1.3\ldots(2n-1)}{2.4\ldots 2n}\right\}^2 x^n.$$

This is the explicit expansion for $K(x)$.

34. Again,

$$2\pi K'(x) = -\frac{1}{2\pi\iota}\int \{\Gamma(-s)\,\Gamma(\tfrac{1}{2}+s)\}^2\, x^s\, ds,$$

and when $|x| < 1$, we may take the contour to include the positive sequence of poles $0, 1, 2, \ldots, \infty$.

Now, when $s = n+\epsilon$, and ϵ is small,

$$\Gamma(-s) = \frac{\Gamma(1-\epsilon)}{(-\epsilon)(-\epsilon-1)\ldots(-\epsilon-n)}$$

$$= \frac{(-)^{n-1}}{n!\,\epsilon}\left\{1-\epsilon\left[\psi(1)+\frac{1}{1}+\ldots+\frac{1}{n}\right]+\ldots\right\}.$$

Hence the residue of $-\{\Gamma(-s)\,\Gamma(\tfrac{1}{2}+s)\}^2 x^s$ at $s = n+\epsilon$ is the coefficient of $1/\epsilon$ in the expansion in ascending powers of ϵ of

$$-\frac{1}{\epsilon^2 (n!)^2}\left\{1-2\epsilon\left[\psi(1)+\frac{1}{1}+\ldots+\frac{1}{n}\right]+\ldots\right\}$$

$$\times\{\Gamma(\tfrac{1}{2}+n)\}^2\{1+2\epsilon\psi(\tfrac{1}{2}+n)+\ldots\}\, x^{n+\epsilon}$$

$$= -\left\{\frac{\Gamma(\frac{1}{2}+n)}{\Gamma(n+1)}\right\}^2 x^n \left\{\log x + 2\psi(n+\tfrac{1}{2}) - 2\left[\psi(1)+\frac{1}{1}+\ldots+\frac{1}{n}\right]\right\}.$$

Now $$\psi(x+1) = \frac{1}{x}+\psi(x).$$

Also $$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\,\Gamma(x)\,\Gamma(x+\tfrac{1}{2}),$$

so that $$2\psi(2x) = 2\log 2 + \psi(x) + \psi(x+\tfrac{1}{2});$$

and therefore $$\psi(1) - \psi(\tfrac{1}{2}) = 2\log 2.$$

Hence $$2\psi(n+\tfrac{1}{2}) - 2\left[\psi(1)+\frac{1}{1}+\ldots+\frac{1}{n}\right]$$

$$= 2\left[\psi(\tfrac{1}{2})-\psi(1)+\frac{2}{2n-1}+\frac{2}{2n-3}+\ldots+\frac{2}{1}-\frac{1}{1}-\frac{1}{2}-\ldots-\frac{1}{n}\right]$$

$$= 4\left[-\log 2+\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{2n-1}-\frac{1}{2n}\right].$$

Hence, when $|x| < 1$ and $|\arg x| < \pi$,

$$2\pi K'(x) = -\sum_{n=0}^{\infty}\left\{\frac{\Gamma(\frac{1}{2}+n)}{\Gamma(n+1)}\right\}^2 x^n \left\{\log x - 4\log 2 + 4\left(\frac{1}{1}-\frac{1}{2}+\ldots-\frac{1}{2n}\right)\right\},$$

or $K'(x) = -\frac{1}{2} \sum_{n=0}^{\infty} \left\{ \frac{1.3 \ldots (2n-1)}{2.4 \ldots 2n} \right\}^2 x^n$

$$\times \left[\log x - 4 \log 2 + 4 \left(\frac{1}{1} - \frac{1}{2} + \ldots - \frac{1}{2n} \right) \right].$$

This is the explicit expansion for $K'(x)$.

35. *We can now shew that**

$$Y_1 = Y_3 \pm \iota Y_2,$$

the upper or lower sign being taken as $I(x)$ is positive or negative.

When $|x| > 1$, we may take the contour of the integral

$$2K(x) = -\frac{1}{2\pi\iota} \int \frac{\{\Gamma(\frac{1}{2}+s)\}^2 \Gamma(-s)}{\Gamma(1+s)} (-x)^s \, ds$$

to include the sequence of negative poles of the subject of integration.

Now the residue of

$$\frac{\{\Gamma(\frac{1}{2}+s)\}^2 \Gamma(-s)}{\Gamma(1+s)} (-x)^s$$

at $$s = -n - \tfrac{1}{2} + \epsilon$$

is the coefficient of $1/\epsilon$ in the expansion in ascending powers of ϵ of

$$\frac{\sin \pi(n+\frac{1}{2}) \{\Gamma(n+\frac{1}{2})\}^2}{\pi\epsilon^2 \{\Gamma(n+1)\}^2} [1 - 2\epsilon \{\psi(n+\tfrac{1}{2}) - \psi(n+1)\} + \ldots] (-x)^{-n-\frac{1}{2}+\epsilon};$$

and is therefore

$$\frac{(-)^n}{\pi} \left\{ \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \right\}^2 (-x)^{-n-\frac{1}{2}} \left[\log(-x) + 4 \log 2 - 4 \left(\frac{1}{1} - \frac{1}{2} + \ldots - \frac{1}{2n} \right) \right].$$

Therefore

$$2K(x) = (-x)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \left\{ \frac{1.3 \ldots (2n-1)}{2.4 \ldots 2n} \right\}^2 \frac{1}{x^n}$$

$$\times \left\{ -\log \frac{1}{x} \mp \pi\iota + 4 \log 2 - 4 \left(\frac{1}{1} - \frac{1}{2} - \ldots - \frac{1}{2n} \right) \right\}$$

$$= \mp \pi\iota (-x)^{-\frac{1}{2}} \frac{2}{\pi} K\left(\frac{1}{x}\right) + (-x)^{-\frac{1}{2}} 2K'\left(\frac{1}{x}\right),$$

or $$K(x) = \mp \iota (-x)^{-\frac{1}{2}} K(1/x) + (-x)^{-\frac{1}{2}} K'(1/x),$$

when $|\arg(-x)| < \pi$, the upper or lower sign being taken as $I(x)$ is

* Cf. Tannery et Molk, *Fonctions Elliptiques*, T. III., p. 205.

positive or negative. Now

$$(-x)^{-\frac{1}{2}} = \pm \iota x^{-\frac{1}{2}}.$$

Hence
$$K(x) = x^{-\frac{1}{2}} K(1/x) \pm \iota x^{-\frac{1}{2}} K'(1/x),$$

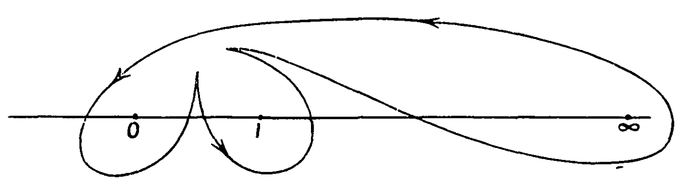
or
$$Y_1 = Y_3 \pm \iota Y_2.$$

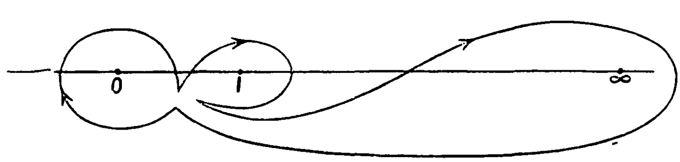
36. From the preceding we may at once obtain the fundamental substitutions which lead to the theory of elliptic modular functions.

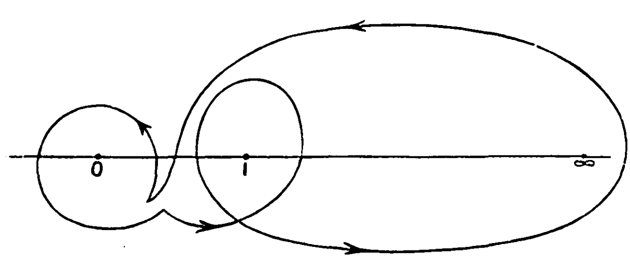
First we recall that near $x = 0$ $K(x)$ is uniform and $K'(x)$ has a cross-cut from $-\infty$ to 0; near $x = 1$ $K(x)$ has a cross-cut from $+1$ to $+\infty$, and $K'(x)$ is uniform, and near $x = \infty$ $K(x)$ has a cross-cut from $+\infty$ to $+1$, $K'(x)$ has a cross-cut from $-\infty$ to 0.

We wish to obtain for the three points such substitutions that the product of all three will be equivalent to a circuit round a point of no singularity—that is to say, to unity.

The first figure represents a possible combination of circuits when $I(x)$ is positive, the second figure when $I(x)$ is negative. The third figure shews that when $I(x)$ is negative, the positive circuits possible in the first case, when $I(x)$ is positive, are no longer available.







We therefore define a *possible* circuit as one which is positive or negative as $I(x)$ is positive or negative.

And now after a possible circuit round the origin

$$K(x) \quad \text{becomes} \quad K(x),$$

$$K'(x) \quad ,, \quad K'(x) \mp 2\iota K(x),$$

the upper or lower sign being taken as $I(x)$ is positive or negative.

Putting $x = 1-\xi$, we see that a possible circuit round $x = 1$ is equivalent to the reverse description of a possible circuit round $\xi = 0$. Therefore after a possible circuit round $x = 1$,

$$K(x) \quad \text{becomes} \quad K(x) \mp 2\iota K'(x),$$

$$K'(x) \quad ,, \quad K'(x).$$

Putting $x = 1/t$, we see that a possible circuit round $x = \infty$ is equivalent to the reverse description of a possible circuit round $t = 0$. Hence, after a possible circuit round $x = \infty$,

$$K(x) = t^{\frac{1}{2}}K(t) \pm \iota t^{\frac{1}{2}}K'(t)$$

becomes $-t^{\frac{1}{2}}K(t) \mp \iota t^{\frac{1}{2}}\{K'(t) \mp 2\iota K(t)\}$

$$= -3t^{\frac{1}{2}}K(t) \mp \iota t^{\frac{1}{2}}K'(t) = -3K(x) \pm 2\iota K'(x),$$

and $$K'(x) = t^{\frac{1}{2}}K'(t)$$

becomes $$-t^{\frac{1}{2}}\{K'(t) \mp 2\iota K(t)\} = K'(x) \pm 2\iota K(x).$$

Hence, corresponding to possible circuits round 0, 1, ∞, the corresponding substitutions for $K(x)$, $\pm\iota K'(x)$ are

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix},$$

agreeing with Schlesinger.*

If $w = \pm\dfrac{\iota K'(x)}{K(x)}$ and $S_a(w)$ be the substitution corresponding to a possible circuit for w round the point a,

$$S_0(w) = w+2,$$

$$S_1(w) = \frac{w}{1-2w},$$

$$S_\infty(w) = \frac{w-2}{2w-3}.$$

* Schlesinger, *Linearen Differentialgleichungen*, Bd. II. (2), p. 46.

37. We may readily verify that the result of a possible circuit round 0, 1, ∞ in succession is a unit substitution.

For, if in $S_0 S_1$, S_0 operates after S_1 so that $S_0 S_1$ represents a circuit round 0 and 1 successively,

$$S_0 S_1 (w) = \frac{-3w+2}{1-2w},$$

$$S_0 S_1 S_\infty (w) = S_0 (w-2) = w.$$

The theory of elliptic modular functions can now be developed in the usual way.