ON FUNCTION SUM THEOREMS CONNECTED WITH THE SERIES $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$.

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[Received February 12th, 1906.—Read March 8th, 1906.]

1. The transformation of the series

$$x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \ldots,$$

or more generally the integral

$$-\int_0 \log(1-x)\,\frac{dx}{x},$$

where x is a real quantity, has been considered in Bertrand's *Calcule Intégral* § 270 (1870), and connections are there established between this function of x and the same functions of its co-anharmonic ratios $1-x$, $\frac{1}{x}$, $\frac{1}{1-x}$, $\frac{x}{x-1}$, $\frac{x-1}{x}$. It will, however, be more convenient and will lead to conciser results if we take the function

$$-\tfrac{1}{2}\int_0 \left\{ \frac{\log(1-x)}{x} + \frac{\log x}{1-x} \right\} dx, \tag{1}$$

which, if x is real and not greater than unity, may be represented by

$$x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \ldots + \tfrac{1}{2}\log x \log(1-x),$$

the logarithms being taken in their real principal values.

Calling this function Lx, we see immediately that

$$Lx + L(1-x) = L1, \tag{2}$$

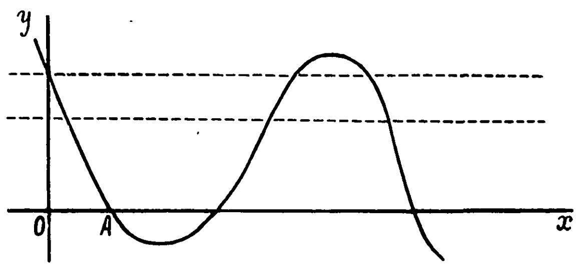
and, since $\log x \log(1-x)$ has zero for its limiting value when $x = 0$, we see that $L1 = \frac{1}{6}\pi^2$.

The equation (2) gives us a connection between *two* L functions whose arguments depend on *one* variable. This relation, we may shew, is a particular case of a linear connection between n^2+1 such functions whose arguments depend on n independent variables.

Let

$$y = 1-p_1x+p_2x^2-\ldots\pm p_nx^n = (1-\mu_1x)(1-\mu_2x)\ldots(1-\mu_nx),$$

where the μ's are all real and positive, represent a curve in rectangular



Cartesian coordinates, and let us suppose for the present that this curve cuts the line $y = 1$ in n real points, including $(0, 1)$. In this case it is evident that the curve will cut the line $y = 1-m$ in n real points, provided $m < 1$; so that we may put

$$m-p_1x+p_2x^2-\ldots\pm p_nx^n = m(1-\lambda_1x)(1-\lambda_2x)\ldots(1-\lambda_nx),$$

where the λ's are also real.

Now consider the sum of the n^2 terms

$$\Sigma\Sigma dL\frac{\mu_s}{\lambda_r}-\Sigma\Sigma dL\frac{\lambda_r}{\mu_s}, \tag{3}$$

where in the positive terms $\mu_s < \lambda_r$, and in the negative terms $\lambda_r < \mu_s$, while r and s have all values from 1 to n. Since

$$2dLx \equiv \log x\, d\log(1-x)-\log(1-x)\, d\log x,$$

we have

$$\begin{aligned}2\Sigma\Sigma dL\frac{\mu_s}{\lambda_r} &= \Sigma\Sigma\left[(\log\mu_s-\log\lambda_r)\, d\left\{\log(\lambda_r-\mu_s)-\log\lambda_r\right\}\right.\\ &\qquad\left.-\left\{\log(\lambda_r-\mu_s)-\log\lambda_r\right\} d(\log\mu_s-\log\lambda_r)\right]\\ &= \Sigma\Sigma\left[(\log\mu_s-\log\lambda_r)\, d\log(\lambda_r-\mu_s)\right.\\ &\qquad-\log(\lambda_r-\mu_s)\, d(\log\mu_s-\log\lambda_r)\\ &\qquad\left.-\log\mu_s\, d\log\lambda_r+\log\lambda_r\, d\log\mu_s\right],\end{aligned}$$

while

$$\begin{aligned}-2\Sigma\Sigma dL\frac{\lambda_r}{\mu} &= -\Sigma\Sigma\left[(\log\lambda_r-\log\mu_s)\, d\log(\mu_s-\lambda_r)\right.\\ &\qquad-\log(\mu_s-\lambda_r)\, d(\log\lambda_r-\log\mu_s)\\ &\qquad\left.-\log\lambda_r\, d\log\mu_s+\log\mu_s\, d\log\lambda_r\right].\end{aligned}$$

Hence twice the whole algebraic sum (3) may be represented by one formula

$$\sum_{s=1}^{n}\sum_{r=1}^{n}\left[(\log\mu_s-\log\lambda_r)\,d\log(\lambda_r\sim\mu_s)-\log(\lambda_r\sim\mu_s)\,d(\log\mu_s-\log\lambda_r)\right.$$
$$\left.-\log\mu_s\,d\log\lambda_r+\log\lambda_r\,d\log\mu_s\right]. \quad (4)$$

Since, identically,

$$u^n-p_1u^{n-1}+\ldots\pm p_n=\quad(u-\mu_1)(u-\mu_2)\ldots(u-\mu_n),$$

and $$mu^n-p_1u^{n-1}+\ldots\pm p_n=m(u-\lambda_1)(u-\lambda_2)\ldots(u-\lambda_n),$$

we have $$(\lambda_r-\mu_1)(\lambda_r-\mu_2)\ldots(\lambda_r-\mu_n)=\lambda_r^n-p_1\lambda_r^{n-1}+\ldots$$
$$=(1-m)\,\lambda_r^n, \quad (5)$$

and $$(\mu_s-\lambda_1)(\mu_s-\lambda_2)\ldots(\mu_s-\lambda_n)=\frac{1}{m}\{m\mu_s^n-p_1\mu_s^{n-1}+\ldots\}$$
$$=\frac{m-1}{m}\mu_s^n;$$

so that $$(\lambda_1\sim\mu_s)(\lambda_2\sim\mu_s)\ldots(\lambda_n\sim\mu_s)=\frac{1-m}{m}\mu_s^n\quad(\text{since } m<1). \quad (6)$$

Now, in (4), the terms

$$\Sigma\Sigma\,\{\log\mu_s\,d\log(\lambda_r\sim\mu_s)-\log(\lambda_r\sim\mu_s)\,d\log\mu_s\}$$

can be reduced by summing first with respect to r, and, using (6), and become

$$\Sigma\left\{\log\mu_s\,d\left(n\log\mu_s+\log\frac{1-m}{m}\right)-\left(n\log\mu_s+\log\frac{1-m}{m}\right)d\log\mu_s\right\},$$

$$=\log p_n\,d\log\frac{1-m}{m}-\log\frac{1-m}{m}\,d\log p_n. \quad (7)$$

Similarly, by summing first with respect to s, and using (5), the corresponding set of terms in (4),

$$\Sigma\Sigma\,\{-\log\lambda_r\,d\log(\lambda_r\sim\mu_s)+\log(\lambda_r\sim\mu_s)\,d\log\lambda_r\},$$

reduce to

$$\Sigma[-\log\lambda_r\,d\{n\log\lambda_r+\log(1-m)\}+\{n\log\lambda_r+\log(1-m)\}\,d\log\lambda_r],$$

$$=-\log\frac{p_n}{m}\,d\log(1-m)+\log(1-m)\,d\log\frac{p_n}{m}. \quad (8)$$

The remaining terms in (4) can be reduced independently in r and s, and we get

$$\Sigma\Sigma\,\{-\log\mu_s\,d\log\lambda_r+\log\lambda_r\,d\log\mu_s\}$$
$$=-\log p_n\,d\log\frac{p_n}{m}+\log\frac{p_n}{m}\,d\log p_n$$
$$=\quad\log p_n\,d\,\log m\,-\log\,m\,d\,\log p_n. \quad (9)$$

Collecting the reduced forms (7), (8), (9), we have

$$2\Sigma\Sigma\left(dL\frac{\mu_s}{\lambda_r}-dL\frac{\lambda_r}{\mu_s}\right)$$

$$\begin{aligned} &= \log p_n\{d\log(1-m)-d\log m\}-\{\log(1-m)-\log m\}\,d\log p_n \\ &\quad -(\log p_n-\log m)\,d\log(1-m)+\log(1-m)(d\log p_n-d\log m) \\ &\quad +\log p_n\,d\log m-\log m\,d\log p_n \\ &= \log m\,d\log(1-m)-\log(1-m)\,d\log m \\ &= 2dLm. \end{aligned}$$

Hence, integrating, we have

$$\Sigma\Sigma\left(L\frac{\mu_s}{\lambda_r}-L\frac{\lambda_r}{\mu_s}\right)=C+Lm. \tag{10}$$

To determine the constant C, we may notice that, when m approaches the value unity, the values of the λ's approach equality with the μ's, and we may suppose λ_r to be that λ which is equal to μ_r in the limit. Now μ_s/λ_r is ultimately equal to λ_s/μ_r, and, if r is not equal to s, the functions having these fractions or their reciprocals as arguments will cancel each other. Noticing from the diagram, where $OA=1/\mu_1$, that $\mu_1<\lambda_1$, $\mu_2>\lambda_2$, $\mu_3<\lambda_3$, ..., we see that, if n is even, then the left-hand side of (10) vanishes and $C=-L1$; whereas, if n is odd, the left-hand side is $L1$ and $C=0$.

We have then, for all positive integral values of n, a linear relation consisting of n^2+1 L-functions depending on n independent variables; and it is easy to see that the arguments are all real provided the curve in the diagram cuts the axis of x in n real points, and the line $y=1-m$ is taken sufficiently near to the axis of x to cut every wave of the curve.

When $n=2$, the relation is

$$L\frac{\mu_1}{\lambda_1}+L\frac{\mu_2}{\lambda_1}-L\frac{\lambda_2}{\mu_1}-L\frac{\lambda_2}{\mu_2}=-L1+Lm,$$

where $$m=\frac{\mu_1\mu_2}{\lambda_1\lambda_2}$$

and $$\frac{\lambda_1+\lambda_2}{\lambda_1\lambda_2}=\frac{\mu_1+\mu_2}{\mu_1\mu_2}.$$

Let $$\frac{\lambda_2}{\mu_2}=x,\quad m=y,$$

so that $$\frac{\mu_1}{\lambda_1}=xy;$$

then $$Lx+Ly = L(xy)+L\frac{\mu_2}{\lambda_1}+L1-L\frac{\lambda_2}{\mu_1}$$

$$= L(xy)+L\frac{\mu_2}{\lambda_1}+L\frac{\mu_1-\lambda_2}{\mu_1}, \quad \text{by (1).}$$

But $$\mu_1+\mu_2 = y(\lambda_1+\lambda_2);$$

therefore $$\lambda_1 xy+\mu_2 = y(\lambda_1+\mu_2 x),$$

so that $$\frac{\mu_2}{\lambda_1} = \frac{y(1-x)}{1-xy};$$

while $$\frac{\lambda_2}{\mu_1} = \frac{1}{y}\frac{\mu_2}{\lambda_1} = \frac{1-x}{1-xy},$$

so that $$1-\frac{\lambda_2}{\mu_1} = \frac{x(1-y)}{1-xy}.$$

Thus, finally, we have

$$Lx+Ly = L(xy)+L\left\{\frac{x(1-y)}{1-xy}\right\}+L\left\{\frac{y(1-x)}{1-xy}\right\}. \tag{11}$$

This formula is apparently the simplest that can be obtained in two independent variables. It is remarkable that, although (1) has been made use of in determining it, yet it is not possible to deduce (1) directly from it. We might infer, then, that formulæ included in (10) containing more than two variables may be so reduced that (11) cannot be deduced as a particular case; and, conversely, it is possible that formulæ (10) cannot be deduced by repeated application of (11). Certain facts render this very probable. For instance, by making $y = x$, we have

$$2Lx = L(x^2)+2L\left(\frac{x}{1+x}\right), \tag{12}$$

a relation which is closely connected with the obvious identity

$$2\psi x+2\psi(-x) = \psi(x^2), \tag{13}$$

where $$\psi x = x+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\ldots,$$

when we take note of the connection given in Bertrand's *Calc. Int.*, § 270, (26), between

$$\psi(-x) \quad \text{and} \quad \psi\left(\frac{x}{1+x}\right).$$

However, the equally obvious identity

$$3\psi x+3\psi(x\rho)+3\psi(x\rho^2) = \psi(x^3),$$

where $$1+\rho+\rho^2 = 0,$$

does not appear to be made deducible in any way from (1), even when this formula is made adaptable to imaginary arguments.

2. By transforming the right-hand side of § 1, (11), by means of § 1, (1), we get

$$Lx+Ly+L(1-xy)+L\left(\frac{1-y}{1-xy}\right)+L\left(\frac{1-x}{1-xy}\right)=3L1.$$

These five arguments taken in the order

$$x, \quad y, \quad \frac{1-x}{1-xy}, \quad 1-xy, \quad \frac{1-y}{1-xy}$$

form a cyclic group in which any constituent is the same function of the two preceding it.

If a and b are the sides of a right-angled spherical triangle, these arguments may be written

$$\cos^2 a, \quad \cos^2 b, \quad \sin^2 A, \quad \sin^2 c, \quad \sin^2 B,$$

the squared cosines of Napier's circular parts.

This property shews that, if we apply § 1, (1) to the function of any consecutive three in the cyclic arrangement of the arguments, we again get § 1, (11) in another form, *e.g.*, transferring Ly, $L\left(\frac{1-x}{1-xy}\right)$, and $L(1-xy)$, we get

$$Lx+L\left(\frac{1-y}{1-xy}\right)=L\left\{\frac{x(1-y)}{1-xy}\right\}+L(xy)+L(1-y),$$

which represents § 1, (11), where y is replaced by $(1-x)/(1-xy)$.

The equation § 1, (11) may be written in another interesting form as follows.

By direct application of the formula, we get

$$L(1-x)+L\left(\frac{1}{1+m}\right)=L\left(\frac{1-x}{1+m}\right)+L\left\{\frac{m(1-x)}{m+x}\right\}+L\left(\frac{x}{m+x}\right).$$

By § 1, (1), this may be written

$$Lx+L\left(\frac{x}{x+m}\right)+L\left\{\frac{(1-x)m}{x+m}\right\}+L\left(\frac{1-x}{1+m}\right)=L1+L\left(\frac{1}{1+m}\right).$$

Now, let the function $Lx+L\left(\frac{x}{x+m}\right)$ be called Mx, where we may call x the argument and m the parameter. Then

$$L\left\{\frac{(1-x)m}{x+m}\right\}+L\left(\frac{1-x}{1+m}\right)=M\left\{\frac{(1-x)m}{x+m}\right\},$$

and $$L1+L\left(\frac{1}{1+m}\right) = M1.$$

Hence $$Mx+M\left\{\frac{(1-x)m}{x+m}\right\} = M1. \tag{1}$$

When $m = \infty$, this relation reduces to § 1, (1).

When $m = 1$, we have

$$f(x)+f\left(\frac{1-x}{1+x}\right) = f(1),$$

where $$f(x) = Lx+L\left(\frac{x}{1+x}\right),$$

a result which is virtually given by Bertrand in connection with the series

$$x+\frac{x^3}{3^2}+\frac{x^5}{5^2}+\ldots \quad \text{(see § 273)}.$$

Again, we have

$$Lx-L(xy)+L1-L\left\{\frac{x(1-y)}{1-xy}\right\}-L\left\{\frac{y(1-x)}{1-xy}\right\} = L1-Ly,$$

i.e., $$Lx-L(xy)+L\left(\frac{1-x}{1-xy}\right)-L\left(y\frac{1-x}{1-xy}\right) = L1-Ly. \tag{2}$$

Looking upon y as a parameter, and writing Yx for $Lx-Lxy$, we get

$$Yx+Y\left(\frac{1-x}{1-xy}\right) = Y1. \tag{3}$$

The equations (1) and (3) may be looked upon as a kind of generalization of the formula § 1, (2), where, if $x = \sin^2\theta$, $1-x = \cos^2\theta$, we get a type of relation of the form $F(\theta)+F(\frac{1}{2}\pi-\theta) = F(\frac{1}{2}\pi)$. In (3), if $x = \operatorname{sn}^2 u$, $y = k^2$, the relation is of the form $F(u)+F(K-u) = F(K)$. It is possible that the terms in § 1, (10) may be so grouped that by looking upon one variable as a modulus the identity may represent a relation between functions of the $(n-1)$ remaining variables, each associated with this modulus; but it is not easy to see how such a grouping may be effected.

3. The value of $L(x+yi)$ when the argument is complex will depend upon the path of integration, but will be unambiguous provided the path does not cross the axis of real quantities at any point whose abscissa >1 or <0. This value will be called the principal value of $L(x+yi)$.

We may easily see then that the formula § 1, (10) can be employed in the case of imaginary arguments, provided we make all the μ's real and positive. For, provided we make the line $y = 1-m$ lie close enough to the axis of x, we have a case in which all the λ's are real, even if the waves of the curve do not reach $y = 1$. When m is made to diminish so that two of the intersections are lost, then two of the λ's will become imaginary and will remain imaginary up to the final value of m, viz., unity. The corresponding arguments will therefore never again cross the axis of real quantities, and the corresponding L-functions will have their principal values. The same remark applies to all imaginary arguments that may occur.

If, however, the μ's are imaginary, the same process of reasoning fails, as may be seen in the following simple example.

In the formula $$2Lz = 2z^2+2L\left(\frac{z}{1+z}\right),$$

let $z = \rho$, where $$\rho^2+\rho+1 = 0,$$

so that apparently $$2L\rho = L\rho^2+2L(-\rho^2).$$

Now, if $z = e^{\theta i}$, the value of Lz may be taken as

$$z+\frac{z^2}{2^2}+\frac{z^3}{3^2}+\ldots+\tfrac{1}{2}\log z\log(1-z),$$

the principal values of the logarithms being taken. Thus

$$L(e^{\theta i}) = \cos\theta+\frac{\cos 2\theta}{2^2}+\ldots+i\left(\sin\theta+\frac{\sin 2\theta}{2^2}+\ldots\right)$$
$$-\tfrac{1}{2}\theta i\left\{\cos\theta+\tfrac{1}{2}\cos 2\theta+\ldots+i(\sin\theta+\tfrac{1}{2}\sin 2\theta+\ldots)\right\}.$$

The real part of this is

$$\cos\theta+\frac{\cos 2\theta}{2^2}+\frac{\cos 3\theta}{3^2}+\ldots+\tfrac{1}{2}\theta\left(\sin\theta+\frac{\sin 2\theta}{2}-\ldots\right)$$
$$= \frac{\pi^2}{6}-\frac{\pi}{2}\theta+\frac{\theta^2}{4}+\tfrac{1}{2}\theta\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$$
$$= \frac{\pi^2}{6}-\frac{\pi\theta}{4} \quad (\text{provided } \theta > 0 < \pi),$$

and we get the same value for the real part of $L(e^{-\theta i})$, if $\theta > 0 < \pi$. Thus the real parts of the principal values of $L\rho$ and $L\rho^2$ are zero, while the real part of $L(-\rho^2)$ is $\frac{1}{12}\pi^2$; so that the relation cannot be true. It will be easily seen, however, that, if we make z move from (0, 0) to (0, 1) and thence along the circle $r = 1$ from $\theta = 0$ to $\theta = \frac{2}{3}\pi$, then z^2 will move across the axis of real quantities on the negative side of the origin, and the value of $L\rho^2$ will not be the same as if the point $z = \rho^2$ had been reached by making $\theta = -\frac{2}{3}\pi$. In fact it differs by $\pi i\log(1-z)$, where

$z = \rho^2$, corresponding to a circuit about 0 of $\frac{\log z}{1-z}$, *i.e.*, $-\frac{1}{2}\frac{1}{1-z}\int\frac{dz}{z}$ in the integrand, *i.e.*, $-\frac{\pi i}{1-z}$. The real part of $\pi i \log(1-\rho^2)$ is now $-\frac{1}{6}\pi^2$, which cancels with the real part of $2L(-\rho^2)$ in the formula.

It is seen then that even in the simplest formula depending on § 1, (11) imaginary values of the arguments may lead to what at first sight seem anomalous results. It may be observed also that in this formula, whenever the μ's are real, the λ's must be real; so that imaginary arguments always imply imaginary values of the μ's.

The actual point at which the reasoning in § 1 fails for imaginary arguments lies in taking logarithms of each side of equations § 1, (5), (6), but it is easy to see that the logarithms differ by some multiples of $2\pi i$. Where such corrections have to be made, it will be seen that the reduced form for § 1, (4) will contain further terms consisting of $2\pi i$ multiplied by differentials of logarithms; so that § 1, (10) must be corrected by the addition of terms consisting of $2\pi i$ multiplied by logarithms of known arguments, and an integral equation is thereby always obtained. General rules for determining such terms and the constant of integration would be very difficult to determine; but, on the other hand, the system of arguments of the L-functions would be simplified, as there is nothing to prevent taking complex arguments with norm greater than unity, and so in the L-function sum we may start by taking all terms positive and keeping the μ's uniformly in all the numerators of the arguments.

4. It has been shewn in § 1 that a function exists such that, for any arguments x, y which lie between 0 and 1,

$$f(x)+f(y)-f(xy) = f(u)+f(v), \tag{1}$$

where
$$u = \frac{x(1-y)}{1-xy} \quad \text{and} \quad v = \frac{y(1-x)}{1-xy}.$$

To find the differential equation which must be satisfied by any such functions we must operate on both sides by $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}$, so as to annihilate the left-hand side.

Now
$$x(1-x)\frac{\partial u}{\partial x} = \frac{1-y}{(1-xy)^2}x(1-x) = u(1-u);$$

therefore
$$x(1-x)\frac{\partial}{\partial x}f(u) = u(1-u)\frac{d}{du}f(u) = F(u) \text{ say.}$$

But
$$\frac{\partial u}{\partial y} = -\frac{x(1-x)}{(1-xy)^2};$$

therefore
$$x(1-x)\frac{\partial^2}{\partial x\,\partial y} f(u) = -\frac{x(1-x)}{(1-xy)^2}\frac{dFu}{du},$$

i.e.,
$$\frac{\partial^2}{\partial x\,\partial y} f(u) = -\frac{1}{(1-xy)^2}\frac{dFu}{du}.$$

The operation $x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ leaves a function of xy unchanged, while
$$\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) u = \frac{x}{1-xy};$$

therefore
$$\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)\frac{\partial^2}{\partial x\,\partial y} f(u) = -\frac{x}{(1-xy)^3}\frac{d^2}{du^2}F(u)$$
$$= -\frac{u(1-u)}{(1-x)(1-y)(1-xy)}\frac{d^2}{du^2}F(u).$$

By symmetry, the same operation on $f(v)$ gives
$$\frac{v(1-v)}{(1-x)(1-y)(1-xy)}\frac{d^2}{dv^2}F(v),$$

whence
$$u(1-u)\frac{\partial^2}{\partial u^2}u(1-u)\frac{d}{du}f(u) = v(1-v)\frac{d^2}{dv^2}v(1-v)\frac{d}{dv}f(v),$$

where u and v may be considered as independent. Hence
$$u(1-u)\frac{d^2}{du^2}u(1-u)\frac{d}{du}f(u) = \text{a constant.} \tag{2}$$

The complete solution of this equation is easily arrived at by straightforward integration and is of the form
$$A_1 Lu + A_2 \log u + A_3 \log(1-u) + A_3;$$

so that it is possible that a more general form than Lu for $f(u)$ in (1) may satisfy that relation. Such may be shewn to be the case, provided no demand is made that the functions of the several arguments should be identical. In fact, if we write $2L(x, a)$ as denoting
$$Lx - \tfrac{1}{2}\log a \log(1-x) + \tfrac{1}{2}\log(1-a)\log x - La, \tag{3}$$

we shall have identically

$$L(x, a)+L(y, b)$$
$$= L(xy, ab)+L\left(x\frac{1-y}{1-xy},\ a\frac{1-b}{1-ab}\right)+L\left(y\frac{1-x}{1-xy},\ b\frac{1-a}{1-ab}\right); \quad (4)$$

for the L-functions obviously cancel out, and the logarithmic terms may be easily shown to vanish. For such an identity to hold it is clear that any multiple of the logarithmic terms might have been assumed in defining $L(x, a)$, but for reasons hereafter obvious the form above given will be most important. The cancelling of these logarithmic products may easily be proved as follows, in the most general case of § 1, (10).

Let $a_1, a_2, \ldots, a_n, \beta_1, \beta_2, \ldots, \beta_n, \kappa$ be a set of quantities related in the manner given in § 1, viz.,

$$\kappa(1-\beta_1 x)(1-\beta_2 x)\ldots(1-\beta_n x) = \kappa-1+(1-a_1 x)(1-a_2 x)\ldots(1-a_n x),$$

and let us suppose that in the equation § 1, (3), viz.,

$$\pm\Sigma\Sigma\left\{\log\frac{\mu_s}{\lambda_r}\, d\log\left(1-\frac{\mu_s}{\lambda_r}\right)-\log\left(1-\frac{\mu_s}{\lambda_r}\right) d\log\frac{\mu_s}{\lambda_r}\right\},$$

we take the operation d to denote changes of the letter μ into β, λ into a, and m into κ. Then the double sum will reduce as before to $\log m\, d\log(1-m)-\log(1-m)\, d\log m$, which now denotes

$$\log m\log(1-\kappa)-\log(1-m)\log\kappa,$$

and
$$\Sigma\Sigma\pm L\left(\frac{\mu_s}{\lambda_r},\ \frac{\beta_s}{a_r}\right) = L(m, \kappa), \quad (5)$$

the constant C disappearing.

5. When the argument z of a L function is complex, we must define $2Lz$ as the value of the integral of

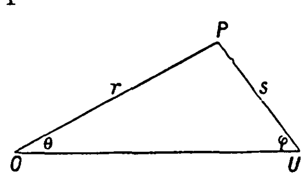
$$-\log(1-z)\, d\log z+\log z\, d\log(1-z),$$

taken along a path which starts from the origin, and where $\log(1-z)$ and $\log z$ have those values which correspond to integration along the same path. The principal value of Lz will correspond to any path connecting 0 and z for which $\log(1-z)$ and $\log z$ retain their principal values, *i.e.*, one in which z is never real unless it is a positive proper fraction.

If we take

$$z = re^{\theta i} \quad\text{and}\quad 1-z = se^{-\phi i},$$

$$\log z = \log r+\theta i \quad\text{and}\quad \log(1-z) = \log s-\phi i,$$



then the principal value will correspond to the case in which θ and ϕ both lie between π and $-\pi$, r and s being of course always positive.

We have now

$$2dLz = -(\log s-\phi i)\, d(\log r+\theta i)+(\log r+\theta i)\, d(\log s-\phi i)$$

$$= -\log s\, d\log r+\log r\, d\log s-\phi\, d\theta+\theta\, d\phi$$

$$+i(\theta\, d\log s+\phi\, d\log r-\log s\, d\theta-\log r\, d\phi). \qquad (1)$$

The real part of Lz will be written $R(\theta, \phi)$, while the coefficient of i in $2dLz$ is easily seen to be

$$\theta d\{\log\sin\theta-\log\sin(\theta+\phi)\}+\phi d\{\log\sin\phi-\log\sin(\theta+\phi)\}$$

$$-\{\log\sin\theta-\log\sin(\theta+\phi)\}\, d\theta+\{\log\sin\phi-\log\sin(\theta+\phi)\}\, d\phi$$

$$=(\theta\cot\theta-\log\sin\theta)\, d\theta+(\phi\cot\phi-\log\sin\phi)\, d\phi$$

$$-\{(\theta+\phi)\cot(\theta+\phi)-\log\sin(\theta+\phi)\}\, d(\theta+\phi).$$

Since the path of integration for Lz will always be taken as starting from 0, we see that initially $\phi=0$; so that the coefficient of i in $L(re^{\theta i})$ is of the form

$$f(\theta)+f(\phi)-f(\theta+\phi), \qquad (2)$$

where

$$f(\theta)=\tfrac{1}{2}\int_0 (\theta\cot\theta-\log\sin\theta)\, d\theta$$

$$=\int_0 \frac{\theta}{\tan\theta}\, d\theta-\tfrac{1}{2}\theta\log\sin\theta.$$

The symbol $T\theta$ will be used to denote the function $\int_0 \frac{\theta}{\tan\theta}\, d\theta$.

The formula § 1, (10) will lead, by consideration of imaginary terms, to sum-formulæ connecting T-functions in conjunction with such functions as $\theta\log\sin\theta$, but, by virtue of the extended form § 4, (5), it is not difficult to see that these latter functions will vanish identically for

$$2L(re^{\theta i}, re^{-\theta i})$$

$$= L(re^{\theta i})-L(re^{-\theta i})-\tfrac{1}{2}(\log r-\theta i)(\log s-\phi i)+\tfrac{1}{2}(\log s+\phi i)(\log r+\phi i)$$

$$= L(re^{\theta i})-L(re^{-\theta i})+i(\theta\log s+\phi\log r)$$

$$= 2i\{T\theta+T\phi-T(\theta+\phi)\}.$$

If, then, in § 4, (5) all pairs of arguments such as μ_s/λ_r and β_s/α_r, m and κ are conjugate complexes, we shall obtain a relation in which a sum of $3(n^2+1)$ T functions, involving n θ's and n ϕ's, is equated either to zero or (see the end of § 3) to a sum of logarithms. The case derived from § 1, (11) will be considered below *ab initio* in § 8, where it will be seen that the T sum will be equated to zero; and it is probable in all cases that the relation will be free from logarithmic terms.

If $r < 1$, we may write

$$\begin{aligned} d\{T\theta+T\phi-T(\theta+\phi)\} &= \theta\, d\log s+\phi\, d\log r \\ &= d(\theta\log s)-\log s\, d\theta+\phi\, d\log r \\ &= d(\theta\log s)-\tfrac{1}{2}\log(1-2r\cos\theta+r^2)\,d\theta \\ &\qquad +\frac{1}{r}\tan^{-1}\frac{r\sin\theta}{1-r\cos\theta}\,dr \\ &= d(\theta\log s)+\left(r\cos\theta+\frac{r^2}{2}\cos 2\theta+\dots\right)d\theta \\ &\qquad +\left(\sin\theta+\frac{r^2}{2}\sin 2\theta+\dots\right)dr \\ &= d(\theta\log s)+d\left(r\sin\theta+\frac{r^2}{2^2}\sin 2\theta+\frac{r^3}{3^2}\sin 3\theta+\dots\right). \end{aligned}$$

Integrating from 0, where $s = 1$ and $r = 0$, we have

$$\begin{aligned} T\theta+T\phi-T(\theta+\phi) &= r\sin\theta+\frac{r^2}{2^2}\sin 2\theta+\frac{r^3}{3^2}\sin 3\theta+\dots \\ &\qquad +\tfrac{1}{2}\theta\log(1-2r\cos\theta+r^2). \end{aligned} \tag{3}$$

If we treat r as constant, this right-hand side may be written in the form $\int_0 \frac{r\theta\sin\theta}{1-2r\cos\theta+r^2}\,d\theta$, but this integral will not represent the left-hand side when $r > 1$. For the latter implies a path of integration from 0, and, if r is treated as a constant, we must first take the path along the axis of x as far as $(r, 0)$ before proceeding along the circle $x^2+y^2 = r^2$. But, if $r > 1$, this first path would pass through U and pass along the axis beyond it, which is not permissible.

However, if $r < 1$, the ϕ-coordinate of $1/re^{\theta i}$ is easily seen to be $\pi-\theta-\phi$; so that the expression $T\theta+T\phi-T(\theta+\phi)$ becomes

$$T\theta+T(\pi-\theta-\phi)-T(\pi-\phi).$$

Now $\quad dT\theta+dT(\pi-\theta) = \left\{\frac{\theta}{\tan\theta}-\frac{\pi-\theta}{\tan(\pi-\theta)}\right\}d\theta = \frac{\pi}{\tan\theta}\,d\theta;$

therefore $\quad T\theta+T(\pi-\theta) = C+\pi\log\sin\theta,$

where $\quad C = 2T(\tfrac{1}{2}\pi).$

But

$$T\tfrac{1}{2}\pi = \int_0^{\frac{1}{2}\pi}\theta\cot\theta\,d\theta = [\theta\log\sin\theta]_0^{\frac{1}{2}\pi}-\int_0^{\frac{1}{2}\pi}\log\sin\theta\,d\theta = -\int_0^{\frac{1}{2}\pi}\log\sin\theta\,d\theta,$$

which is known to be equal to $\frac{1}{2}\pi\log 2$.

Hence $$T\theta+T(\pi-\theta) = \pi\log(2\sin\theta), \tag{4}$$
and
$$T\theta+T(\pi-\phi-\theta)-T(\pi-\phi) = T\theta+T\phi-T(\phi+\theta)-\pi\log\frac{\sin\phi}{\sin(\phi+\theta)}$$
$$= T\theta+T\phi-T(\phi+\theta)-\pi\log r. \tag{5}$$

If, in (3), $r=1$, then $\phi=\frac{1}{2}(\pi-\theta)$.

But by differentiating it may be easily verified that
$$T\theta+T\{\tfrac{1}{2}(\pi-\theta)\}-T\{\tfrac{1}{2}(\pi+\theta)\} = 2T(\tfrac{1}{2}\theta);$$
so that, changing θ into 2θ, we have
$$T\theta = \theta\log(2\sin\theta)+\tfrac{1}{2}\left(\sin 2\theta+\frac{1}{2^2}\sin 4\theta+\frac{1}{3^2}\sin 6\theta+\dots\right). \tag{6}$$
Hence also
$$T\theta+T\phi-T(\theta+\phi) = \theta\log s+\phi\log r$$
$$+2\left\{\sin\theta\sin\phi\sin(\theta+\phi)+\frac{1}{2^2}\sin 2\theta\sin 2\phi\sin 2(\theta+\phi)+\dots\right\}. \tag{7}$$

In analogy to the above transformation, we may consider the difference of any two L functions in the form $L(\rho e^{u})-L(\rho e^{-u})$, where u is positive and $\rho e^{u}<1$.

If $1-\rho e^{u}=\sigma e^{-v}$ and $1-\rho e^{-u}=\sigma e^{v}$, we have $\rho=\sinh v/\sinh(u+v)$ and $\sigma=\sinh u/\sinh(u+v)$; and
$$2d[L(\rho e^{u})-L(\rho e^{-u})] = -(\log\sigma-v)\,d(\log\rho+u)+(\log\rho+u)\,d(\log\sigma-v)$$
$$+(\log\sigma+v)\,d(\log\rho-u)-(\log\rho-u)\,d(\log\sigma+v)$$
$$= 2(v\,d\log\rho+u\,d\log\sigma-\log\sigma\,du-\log\rho\,dv);$$
therefore $$L(\rho e^{u})-L(\rho e^{-u}) = f(u)+f(v)-f(u+v),$$
where $$f(u) = \int_0 (u\coth u-\log\sinh u)\,du = 2\int_0 \frac{u}{\tanh u}\,du-u\log\sinh u. \tag{8}$$

If in § 2, (2) we put $x=\rho e^{u}$, $y=e^{-2u}$, we get
$$L(\rho e^{u})-L(\rho e^{-u}) = L1-Le^{-2u}-Le^{-2v}+Le^{-2u-2v},$$
which may be easily identified with (8).

6. We may now establish the formula (*v.* Bertrand's *Calc. Int.*, § 271) connecting Lz with $L\{z/(z-1)\}$ and Lz with $L(1/z)$ where z is complex.

The four bipolar coordinates of $z/(z-1)$ corresponding to r, s, θ, ϕ are r/s, $1/s$, $\theta+\phi-\pi$, $-\pi$; therefore

$$2dL\left(\frac{z}{z-1}\right) = \log s\, d\log\frac{r}{s} - \log\frac{r}{s}\, d\log s + \phi d(\theta+\phi) - (\theta+\phi)d\phi + \pi\, d\phi$$
$$+ i\left\{-\phi\left(\frac{dr}{s} - \frac{ds}{s}\right) - (\theta+\phi-\pi)\frac{ds}{s} + \log s\, d(\theta+\phi) + \log\frac{r}{s}\, d\phi\right\}.$$

Hence $$2dLz + 2dL\left(\frac{z}{z-1}\right) = i\pi\frac{ds}{s} + \pi\, d\phi$$

by § 5, (1), and, integrating from $s = 1$, $\phi = 0$,

$$Lz + L\left(\frac{z}{z-1}\right) = \tfrac{1}{2}\pi i \log s + \tfrac{1}{2}\pi\phi. \tag{1}$$

Again, the four bipolar coordinates $r^{-1}e^{-\theta i}$ are $1/r$, s/r, $-\theta$, $\theta+\phi-\pi$; so that

$$2dL\left(\frac{1}{r}e^{\theta i}\right) = \log\frac{s}{r}\, d\log r - \log r\, d\log\frac{s}{r} \mp (\pi-\theta-\phi)d\theta - \theta d(\theta+\phi)$$
$$+ i\left\{-\theta d\log\frac{s}{r} + (\pi-\theta-\phi)\, d\log r + \log\frac{s}{r}\, d\theta + \log r\, d(\theta+\phi)\right\};$$

therefore $\qquad 2dL(re^{\theta i}) + 2dL(r^{-1}e^{-\theta i}) = -\pi\, d\theta + \pi i\, d\log r$

by § 5, (1), and $\qquad Lz + L(1/z) = C - \tfrac{1}{2}\pi\theta + \tfrac{1}{2}\pi i \log r$

when $z = 1$, $r = 1$, and $\theta = 0$; so that $C - 2L1 = \frac{1}{3}\pi^2$ and

$$Lz + L(1/z) = 2L1 - \tfrac{1}{2}\pi\theta + \tfrac{1}{2}\pi i \log r \tag{2}$$

(cf. Bertrand, *Calc. Int.*, § 272).

7. We have seen in § 2 that, if z and ω are complexes, we are not justified in assuming that

$$Lz + L\omega = L(z\omega) + L\left(z\frac{1-\omega}{1-z\omega}\right) + L\left(\omega\frac{1-z}{1-z\omega}\right),$$

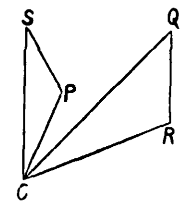
but that the formula would have to be corrected by the introduction of logarithmic terms.

It is not easy to give a concise geometric meaning to the related complexes in this equation, but, by § 6, (1), we see that there can be a relation derived from the above connecting

$$Lz, \quad L\omega, \quad L\left\{\frac{z(1-\omega)}{z-1}\right\}, \quad L\left\{\frac{\omega(1-z)}{\omega-1}\right\}, \quad \text{and} \quad L(z\omega).$$

These complexes are connected with one another and with O, U in a manner which is worthy of notice.

If P, Q, R, S have complexes z_1, z_2, z_3, z_4, it is possible to find uniquely a point C (say, z_0) such that the triangle CPS, by rotation about C and enlargement (or diminution), may take the position CRQ; that is, there is a unique *centre of similitude* of SP and QR, which, moreover, is the centre of similitude of SQ and PR. The necessary and sufficient condition is that



$$(z_1-z_0)(z_2-z_0) = (z_3-z_0)(z_4-z_0),$$

i.e.,
$$z_0 = \frac{z_3 z_4 - z_1 z_2}{z_3+z_4-z_1-z_2}.$$

Let $\quad z_1 = z, \quad z_2 = \omega, \quad z_3 = \dfrac{z(1-\omega)}{z-1}, \quad z_4 = \dfrac{\omega(1-z)}{\omega-1}.$

Then the vectors
$$OP.OQ = OR.OS;$$
so that O is the centre of similitude of PS and RQ or of QS and RP. Now the vectors drawn from U are

$$1-z, \quad 1-\omega, \quad \frac{1-z\omega}{1-z}, \quad \frac{1-z\omega}{1-\omega};$$

so that
$$UP.UR = UQ.US,$$
and U is the centre of similitude of PQ and SR or of PS and QR.

Lastly, if V is the point whose complex is $z\omega$, the vectors drawn to V from P, Q, R, S are

$$z(1-\omega), \quad \omega(1-z), \quad \frac{z(1-\omega z)}{1-z}, \quad \frac{\omega(1-\omega z)}{1-\omega};$$

so that
$$VP.VS = VQ.VR$$
and V is the centre of similitude of PR and QS or of PQ and RS. Hence O, U, V are the three centres of similitude of pairs of sides of the quadrangle $PQRS$.

It may be observed that the *shape* of $PQRS$ is quite general.

8. In establishing a function-sum theorem for the T-function of § 5 the most convenient method will be based on the formula in § 2 where the five arguments are expressed in the cyclic form

$$z, \quad \omega, \quad \frac{1-z}{1-z\omega}, \quad 1-z\omega, \quad \frac{1-\omega}{1-z\omega},$$

after replacing x and y by z and ω.

If (θ_1, ϕ_1), (θ_2, ϕ_2), ..., (θ_5, ϕ_5) are the bipolar angular coordinates of the points represented by these five arguments, we see immediately that

$$\phi_4 = -\theta_1-\theta_2,$$

and hence, in virtue of the cyclic property, the five points are

$$(\theta_1, -\theta_3-\theta_4), \quad (\theta_2, -\theta_4-\theta_5), \quad (\theta_3, -\theta_5-\theta_1), \quad \ldots.$$

There is, however, one relation connecting the five θ's, which will necessarily be cyclic. Taking $z_1, z_2, \ldots$ as representing the five complexes and $r_1, s_1, r_2, s_2, \ldots$ as their bivectorial coordinates, we have $z_3z_4 = 1-z$; so that $r_3r_4 = s_1$. But in all cases

$$r = \sin\phi/\sin(\phi+\theta) \quad \text{and} \quad s = \sin\theta/\sin(\phi+\theta);$$

so that $\sin\phi_3 \sin\phi_4 \sin(\phi_1+\theta_1) = \sin(\phi_3+\theta_3)\sin(\phi_4+\theta_4)\sin\theta_1$,

i.e., by substituting for the ϕ's

$$\sin(\theta_1+\theta_5)\sin(\theta_1+\theta_2)\sin(\theta_3+\theta_4-\theta_1) + \sin(\theta_1+\theta_5-\theta_3)\sin(\theta_1+\theta_2-\theta_4)\sin\theta_1 = 0. \quad (2)$$

This is the relation required, but does not appear here in the requisite cyclic form. To attain this end we may write $\alpha_1, \alpha_2, \ldots$ for $e^{2\theta_1 i}, e^{2\theta_2 i}, \ldots$ and put $z_1 = r_1 e^{\theta_1 i}$ in the form

$$\sin(\theta_3+\theta_4)e^{\theta_1 i}/\sin(\theta_3+\theta_4-\theta_1) = \alpha_1(\alpha_3\alpha_4-1)/(\alpha_3\alpha_4-\alpha_1).$$

With similar transformations for $z_2, z_3, \ldots$, we get

$$1-z_1z_2-z_4 = -\frac{\alpha_1\alpha_2\alpha_4}{d_1d_2d_4}S^*, \quad (3)$$

where $$d_1 = \alpha_3\alpha_4-\alpha_1, \ \ldots, \ \&c.,$$

and $$S = \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 - \alpha_1\alpha_2\alpha_3 - \alpha_2\alpha_3\alpha_4 - \alpha_3\alpha_4\alpha_5 - \alpha_4\alpha_5\alpha_1 - \alpha_5\alpha_1\alpha_2 + \alpha_4\alpha_5 + \alpha_5\alpha_1 + \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_4 - 1, \quad (4)$$

i.e., $$\sin(\theta_1+\theta_2+\theta_3+\theta_4+\theta_5) = \sin(\theta_1+\theta_2+\theta_3-\theta_4-\theta_5) + \text{four other terms cyclically derived.} \quad (5)$$

We may now consider the function sum relation connecting the fifteen T functions corresponding to these five complexes.

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* It is interesting to note that, if the θ's are independent, so that $S \not\equiv 0$, the solution of $\alpha_1, \alpha_2, \ldots$ from the equations $z_1 = \alpha_1(\alpha_3\alpha_4-1)/(\alpha_3\alpha_4-\alpha_1), \ldots$ may be effected. Writing u_1 for $1-z_1-z_3z_4, \ldots$ and v_1 for $1-z_5-z_1-z_2+z_5z_1z_2, \ldots$, we get

$$u_1 = -\alpha_1\alpha_3\alpha_4 S/d_1d_3d_4 \quad \text{and} \quad v_1 = \alpha_2\alpha_3\alpha_4\alpha_5 S/d_5d_1d_2.$$

From these two results we get $\alpha_1u_1v_1/v_2v_5 =$ cyclic analogues and $d_1u_1v_1u_3u_4/v_2v_5 =$ cyclic analogues, whence, by the relation $\alpha_1z_5d_5-\alpha_5d_1 = \alpha_3\alpha_5d_4$, we get $\alpha_1 = -v_2v_5/u_1v_1, \ldots$, thereby obtaining the solution of the α's in terms of the z's. It is remarkable also that $(1-\alpha_1)u_1v_1 = u_1v_1+v_2v_5 = (1-z_1)T$, where T is the cyclic expression

$$2-z_1-z_2-z_3-z_4-z_5+z_1z_2z_3+z_2z_3z_4+z_3z_4z_5+z_4z_5z_1+z_5z_1z_2+z_1z_2z_3-z_1z_2z_3z_4z_5.$$

If $T = 0$, $\alpha_1 = 1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5$; so that the θ's are all zero and the z's all real.

In the differential

$$d\{T\theta_1+T\phi_1-T(\theta_1+\phi_1)+\ldots\},$$
$$= d\{T\theta_1-T(\theta_3+\theta_4)+T(\theta_3+\theta_4-\theta_1)+\ldots\},$$
$$= \theta_1\cot\theta_1\,d\theta_1-(\theta_3+\theta_4)\cot(\theta_3+\theta_4)\,d(\theta_3+\theta_4)$$
$$+(\theta_3+\theta_4-\theta_1)\cot(\theta_3+\theta_4-\theta_1)\,d(\theta_3+\theta_4-\theta_1)+\ldots$$

the coefficient of θ_1 is

$$\cot\theta_1\,d\theta_1-\cot(\theta_1+\theta_2)\,d(\theta_1+\theta_2)-\cot(\theta_1+\theta_5)\,d(\theta_1+\theta_5)$$
$$-\cot(\theta_3+\theta_4-\theta_1)\,d(\theta_3+\theta_4-\theta_1)+\cot(\theta_1+\theta_2-\theta_4)\,d(\theta_1+\theta_2-\theta_4)$$
$$+\cot(\theta_1+\theta_5-\theta_3)\,d(\theta_1+\theta_5-\theta_3),$$
$$= d\log\frac{\sin\theta_1\sin(\theta_1+\theta_2+\theta_4)\sin(\theta_1+\theta_5-\theta_3)}{\sin(\theta_1+\theta_2)\sin(\theta_1+\theta_5)\sin(\theta_3+\theta_4-\theta_1)},$$
$$= 0 \text{ by (2)}.$$

Similarly, the coefficients of θ_2, θ_3, θ_4, θ_5 are zero, and the differential of the sum vanishes. Integrating then from the origin where all the θ's are zero, we see that the sum of the fifteen T functions is zero.

This relation between the θ's, as is also (4), is cyclic, but not symmetrical. If, however, we substitute the ϕ's in the relations, they will both become symmetrical, as may be seen as follows.

Since $$\phi_1 = -\theta_3-\theta_4,\ \ldots,$$

we have $\sigma = -(\theta_1+\theta_2+\theta_3+\theta_4+\theta_5) = \tfrac{1}{2}(\phi_1+\phi_2+\phi_3+\phi_4+\phi_5)$;

so that (4) becomes

$$\sin\sigma = \sin(\sigma-2\phi_1)+\sin(\sigma-2\phi_2)+\sin(\sigma-2\phi_3)+\sin(\sigma-2\phi_4)$$
$$+\sin(\sigma-2\phi_5), \qquad (6)$$

while, since

$$\theta_1 = -\sigma+\phi_2+\phi_5 \quad\text{and}\quad \theta_1+\phi_1 = -\sigma+\phi_1+\phi_2+\phi_5 = \sigma-\phi_3-\phi_4,$$

we have
$$T\phi_1-T(\sigma-\phi_2-\phi_5)-T(\sigma-\phi_3-\phi_4)$$
$$+T\phi_2-T(\sigma-\phi_3-\phi_1)-T(\sigma-\phi_4-\phi_5)$$
$$+\ldots = 0, \qquad (7)$$

which is symmetric in the five angles ϕ_1, ϕ_2, ϕ_3, ϕ_4, ϕ_5.

Referring the series from § 5, (6) for $T\theta$, we see that the above formula holds good, even if we understand by $T\theta$ the series

$$\sin 2\theta + \frac{1}{2^2}\sin 4\theta + \frac{1}{3^2}\sin 6\theta + \ldots,$$

since it is obvious that the logarithmic sum vanishes identically.

If $\phi_5 = 0$, the equation (7) becomes nugatory; for in this case

$$\sin(\sigma - \phi_1 - \phi_2)\cos(\phi_1 - \phi_2) + \sin(\sigma - \phi_3 - \phi_4)\cos(\phi_3 - \phi_4) = 0,$$

i.e., $\sin\tfrac{1}{2}(\phi_1 + \phi_2 - \phi_3 - \phi_4)\sin\tfrac{1}{2}(\phi_1 - \phi_2 + \phi_3 - \phi_4)\sin\tfrac{1}{2}(\phi_1 - \phi_2 - \phi_3 + \phi_4) = 0$;

so that the sum of two of the remaining angles equals the sum of the other two. This makes the function sum in (7) vanish identically.

Moreover, it is impossible that four of the ϕ's should be equal and real. For, if $\phi_1 = \phi_2 = \phi_3 = \phi_4$, then

$$\sin\sigma - \sin(\sigma - 2\phi_5) = 4\sin(\sigma - 2\phi_1) = 4\sin\tfrac{1}{2}\phi_5,$$

$$\sin\phi_5\cos(\sigma - \phi_5) = 2\sin\tfrac{1}{2}\phi_5;$$

therefore $$\cos\tfrac{1}{2}\phi_5\cos(\sigma - \phi_5) = 1, \tag{8}$$

which is impossible, unless ϕ_5 is imaginary.

Let $\phi_4 = \phi_5$, and call each of these ϕ. Then, if $2s = \phi_1 + \phi_2 + \phi_3$, so that $2\sigma = 2s + 2\phi$, we have

$$\sin(s+\phi) = \sin(s + \phi - 2\phi_1) + \sin(s + \phi - 2\phi_2) + \sin(s + \phi - 2\phi_3) + 2\sin(s - \phi),$$

whence $$\tan\phi = \frac{\sin(s - 2\phi_1) + \sin(s - 2\phi_2) + \sin(s - 2\phi_3) + \sin s}{3\cos s - \cos(s - 2\phi_1) - \cos(s - 2\phi_2) - \cos(s - 2\phi_3)}.$$

If $\phi_1 = \phi_2 = \phi_3 = \omega$, we have

$$\tan\phi = \frac{\sin\tfrac{3}{2}\omega - 3\sin\tfrac{1}{2}\omega}{3\cos\tfrac{3}{2}\omega - 3\cos\tfrac{1}{2}\omega} = \tfrac{1}{3}\tan\tfrac{1}{2}\omega,$$

while $$3T\omega + 2T\phi - 3T(\phi - \tfrac{1}{2}\omega) - T(\tfrac{3}{2}\omega - \phi) - 6T(\tfrac{1}{2}\omega) = 0.$$

It is possible therefore that, if the ϕ's are all real, three of them should be equal.

Again, let $\phi_2 = \phi_3 = \phi$, say; $\phi_4 = \phi_5 = \omega$; and $\phi_1 = \psi$, so that $\sigma = \psi + \phi + \omega$. Now (2) may be written

$$\sin(\sigma - \phi_1 - \phi_2)\sin(\sigma - \phi_1 - \phi_2)\sin(\sigma - \phi_2 - \phi_5) + \sin\phi_3\sin\phi_4\sin(\sigma - \phi_3 - \phi_4) = 0,$$

which now becomes

$$\sin\psi\sin(\phi - \psi)\sin(\omega - \psi) + \sin\phi\sin\omega\sin\psi = 0.$$

Rejecting $\sin\psi = 0$, we have

$$\sin(\phi-\psi)\sin(\psi-\omega) = \sin\phi\sin\omega,$$

while the T function relation becomes

$$2T\phi+2T\omega+T(2\psi)-T(\phi-\omega+\psi)-T(\omega-\phi+\psi)$$
$$-4T\psi-2T(\phi-\psi)-2T(\omega-\psi) = 0.$$

It may be observed that

$$4T\psi-T(2\psi) = \int_0 4\psi\left(\frac{1}{\tan\psi}-\frac{1}{\tan 2\psi}\right)d\psi = \int_0 \frac{4\psi\,d\psi}{\sin 2\psi}$$
$$= 4\psi\log(2\sin\psi)-2\psi\log(2\sin 2\psi)$$
$$+2\left(\sin\psi+\frac{1}{2^2}\sin 2\psi+\ldots\right)$$
$$-\tfrac{1}{2}\left(\sin 2\psi+\frac{1}{2^2}\sin 4\psi+\ldots\right)$$
$$= 2\psi\log\tan\psi+2\left(\sin\psi+\frac{1}{3^2}\sin 3\psi+\frac{1}{5^2}\sin 5\psi+\ldots\right);$$

so that

$$\int_0 \frac{\omega}{\sin\omega}\,d\omega = \omega\log\tan\frac{\omega}{2}+2\left(\sin\frac{\omega}{2}+\frac{1}{3^2}\sin\frac{3\omega}{2}+\frac{1}{5^2}\sin\frac{5\omega}{2}+\ldots\right).$$

Returning to the case in which $\phi_1 = \phi_2 = \phi_3 = \phi_4$, we have seen in (8) that

$$\cos\tfrac{1}{2}\phi_5\cos(2\phi_1-\tfrac{1}{2}\phi_5) = 1,$$

where we will suppose that $\tfrac{1}{2}\phi_5 = ui$ and $2\phi_1-\tfrac{1}{2}\phi_5 = \theta$; so that

$$\cosh u = \sec\theta.$$

The T equation becomes

$$4T\phi_1+T\phi_5-6T(\tfrac{1}{2}\phi_5)-4T(\phi_1-\tfrac{1}{2}\phi_5) = 0,$$

i.e.,
$$4\left[T\left(\frac{\theta+ui}{2}\right)-T\left(\frac{\theta-ui}{2}\right)\right] = -T(2ui)+6T(ui),$$

or, if we had put $\tfrac{1}{2}\phi_5 = \theta$, $2\phi_1-\tfrac{1}{2}\phi_5 = ui$, so that again $\cosh u = \sec\theta$, we should have

$$4\left[T\left(\frac{\theta+ui}{2}\right)+T\left(\frac{\theta-ui}{2}\right)\right] = -T(2\theta)+6T\theta.$$

10. It is shewn by Bertrand (*Calc. Int.*, § 271) that for certain values of the argument x the values of ψx may be determined. The notation of the present memoir considerably simplifies the results, which may be obtained as follows.

In § 1, (2) let $x = \frac{1}{2}$; then

$$L\tfrac{1}{2} = \tfrac{1}{2}L1 = \tfrac{1}{12}\pi^2.$$

In § 1, (12) let $x^2 = 2-x$, so that $x = \frac{1}{2}(\sqrt{5}-1)$; then

$$\frac{x}{1+x} = x^2 \quad \text{and} \quad 2Lx = 3L(x^2),$$

while, by § 1, (2), $$Lx + L(x^2) = L1.$$

Hence $$Lx = L\left(\frac{\sqrt{5}-1}{2}\right) = \tfrac{3}{5}L1 = \frac{\pi^2}{10},$$

while $$L(x^2) = L\left(\frac{3-\sqrt{5}}{2}\right) = \tfrac{2}{5}L1 = \frac{\pi^2}{15}.$$

Apart from these cases, it does not seem possible to obtain a special value of Lx for any real or complex argument.