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THE APPLICATION OF BASIC NUMBERS TO BESSEL'S AND LEGENDRE'S FUNCTIONS

(SECOND PAPER)

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CONTENTS.

Introduction.

In a previous paper the basic exponential function $E_p(x)$ was used to obtain a generalized form of Neumann's addition theorem for Bessel's function J_0. The generalized functions $J_{[n]}$ belong to the class commonly known as q functions. In this paper the function $E_p(x)$ will be used to obtain various relations between the generalized forms of Bessel's and Legendre's functions. An example of such a relation is

$$E_p(ir\cos\theta) = \{q-4\}!\sum_{n=0}^{\infty} i^n[2n+q-2]\,\mathfrak{P}_n(q,\cos\theta)\,\frac{J_{[n+\frac{1}{2}q-1]}(r)}{r^{\frac{1}{2}q-1}},$$

which reduces when $p = 1$ to

$$e^{ir\cos\theta} = 2^{\frac{1}{2}q-1}\Gamma(\tfrac{1}{2}q-1)\sum_{n=0}^{\infty} i^n(n+\tfrac{1}{2}q-1)P_n(q,\cos\theta)\frac{J_{n+\frac{1}{2}q-1}(r)}{r^{\frac{1}{2}q-1}},$$

a result due to Dr. Hobson ("Bessel Functions," *Proc. London Math. Soc.*, Vol. xxv., p. 59).

A variety of such forms will be obtained, and by means of these theorems the following series of generalized Legendre functions will be formed and its summation effected for all values of n:—

$$\sum_{r=0}^{\infty}(-1)^r\frac{[n]!}{[r]!\,[n-r]!}x^r P_{[n-r]}(x).$$

This series and another of similar type afford remarkable parallels to the binomial form. As in previous papers, $[n]!$ is a convenient expression for $\Gamma_p(n+1)$ and $\{2n\}! = [2]^n\Gamma_{p^2}(n+1)$ while $[n] = (p^n-1)/(p-1)$.

The function

$$E_p(x) = \prod_{m=1}^{\infty}\{1+x(p-1)p^{-2m}\} = 1+\frac{x}{[1]!}+\frac{x^2}{[2]!}+\dots \quad (|p|>1);$$

so

$$E_p\left(\frac{x}{1-p}\right) = \prod_{m=1}^{\infty}(1-xp^{-2m}).$$

This infinite product expression for the basic exponential enables us to connect the theory of the functions $J_{[n]}$ and $P_{[n]}$ with the q theory of theta function products. In previous papers it was found that on inverting the base p in the function $J_{[n]}(x)$ another form of function $p^{n^2}\mathfrak{J}_{[n]}(x)$ was obtained, and that $J_{[n]}\mathfrak{J}_{[m]} = \mathfrak{J}_{[n]}J_{[m]}$ (p. 195, Ser. 2, Vol. 2). In this paper a very simple equation is found to connect the functions, viz.,

$$J_{[n]}(x) = E_p(ix)\,E_p(-ix)\,\mathfrak{J}_{[n]}(x),$$

which leads to many remarkable identities, and opens up a theory for a double J function, denoted $J^p_{m,n}(x)$. The function E_p will, moreover, be used to form a function

$$\mathfrak{J}_{[0]}(2x,\cos\theta) = \sum_{r=0}^{\infty}(-1)^r\frac{(2x)^{2r}}{\{2r\}!\,\{2r\}!}p^{2r^2}\cos^2\theta\cos^2(\theta+\omega)\dots\cos^2(\theta+r\omega-\omega)$$

such that $\quad \mathfrak{J}_{[0]}(2xy,\cos\theta) = J_{[0]}\mathfrak{J}_{[0]}+2\sum_{n=1}^{\infty}(-1)^n J_{[n]}\mathfrak{J}_{[n]},$

in which the arguments of the functions J and $\mathfrak{J}$ on the right side of the equation are $x^2e^{i\theta}$ and $y^2e^{-i\theta}/p$ respectively.

In the section of the paper which deals with theta function products reference is made to the memoirs on q products and series by Prof. L. J.

Rogers ("Certain Infinite Products," *Proc. London Math. Soc.*, Vols. XXIV., XXV.). Various relations are obtained between the coefficients denoted by Prof. Rogers $A_r(\theta)$, $B_r(\theta)$ and the functions denoted $J_{[n]}$ in this and previous papers. As an example,

$$I_{[0]}\left(\frac{x}{p-p^2}\right) = J_{[0]} - p\frac{x}{(p-1)}\mathfrak{J}_{[1]}A_1 + p^4\frac{x^2}{(p-1)(p^2-1)}\mathfrak{J}_{[2]}A_2 - \dots,$$

in which the argument of J is $xe^{-i\theta}/(p-1)$, and the argument of $\mathfrak{J}$ is $xe^{-i\theta}/(p^2-p)$. I is used to denote the generalized Bessel series in which all the terms are positive. A coefficient analogous to Legendre's is expressed in terms of Rogers' coefficients as

$$(1-p)^n P_{[n]}(q, \cos\theta) = \sum_{r=0}^{r=n}(-1)^r \frac{A_{n-r}(\theta)\, B_r(\theta)}{\{2n-2r\}!\,\{2r\}!}\, p^{(4-q)r}.$$

A series of novel form is found to express the theta function product

$$\prod_{m=1}^{\infty}\frac{(1-2xp^{2m}\cos\theta + x^2p^{4m})}{(1+2xp^{2m}\cos\theta+x^2p^{4m})} = \Sigma b_n x^n,$$

$$b_n = (-1)^n \frac{\cos\theta\dots\cos(\theta+n\omega-\omega)}{(p^2-1)(p^4-1)\dots(p^{2n}-1)}$$

$$\times\left\{\sum_{r=0}^{r=n} p^{r^2-rn}\frac{\{2n\}!}{\{2n-2r\}!\,\{2r\}!}\,\frac{\cos\theta\dots\cos(\theta+r\omega-\omega)}{\cos(\theta+n\omega-\omega)\dots\cos(\theta+n\omega-r\omega)}\right\}.$$

The properties of the functions $A_r(\theta)$, $B_r(\theta)$ and their use in the investigation of theta function products is shown in the papers of Prof. L. J. Rogers (*loc. cit.*). Finally, two theorems analogous to Neumann's theorems for the expansion of arbitrary functions in series (1) of Bessel's functions, (2) of Legendre's functions, are given.

$$f(x) = \sum_0^{\infty} a_n J_{[n]}(x),$$

where
$$a_n = \frac{1}{2\pi i}\int_C O_n(t)\, f(t)\, dt,$$

and
$$O_n(t) = \frac{\{2n\}!}{t^{n+1}}\left\{1 - \frac{t^2}{[2][2n-2]} + \dots\right\}.$$

The expression for $f(x)$ in a series of $P_{[n]}(x)$ functions gives

$$a_n = \frac{[2n+1]}{2\pi i}\int_C Q_{[n]}(t)\, f(t)\, dt.$$

I hope that the results given in the paper may be of interest not merely because of their remarkable likeness to well known results in the case of ordinary functions, but because they link the functions J_n, P_n, ...

by means of their q generalizations with the q theory of elliptic functions. With regard to the expansions of arbitrary functions in basic (or q) series, it seems noteworthy that functions not involving p should be capable of expression in terms of functions involving this arbitrary parameter.

§ 1.

It will be well to collect together in one section certain subsidiary results required in subsequent work. The analysis required to establish these results has no novelty of method, and if given at length would be somewhat tedious, so the main points only of the analysis will be given. Some of these results, such as Lemma II., Lemmas IV., V., have a certain interest in themselves apart from the work which is based upon them.

LEMMA I.—

$$1-p^{2-2r}\frac{[2r][2n+q-6]}{[2][2n+q-2r-4]}$$
$$+p^{6-4r}\frac{[2r][2r-2]}{[2][4]}\frac{[2n+q-4][2n+q-10]}{[2n+q-2r-4][2n+q-2r-6]}-\dots,$$

the p prefix of the general term being $p^{s(s+1)-2sr}$ ($s=1, 2, 3, \dots$). The sum of this series to $(r+1)$ terms is zero. In general the series can easily be summed term by term; the sum of s terms is always a factor of the $(s+1)$-th term: for example, the sum of the first three terms is

$$p^{6-4r}\frac{[2r-2][2r-4][2n+q-4][2n+q-6]}{[2][4][2n+q-2r-4][2n+q-2r-6]}.$$

In case r is a positive integer, the sum to $(r+1)$ terms is zero, owing to the appearance of a zero factor in the numerator of the product which represents the sum of the series.

LEMMA II.—

$$\frac{x^n}{\{2n\}!}=J_{[n]}+\frac{[2n+4]}{[2]}J_{[n+2]}+\dots$$
$$+p^{s(s-1)}[2n+4s]\frac{[2n+2s-2][2n+2s-4]\dots[2n+2]}{[2][4]\dots[2s]}J_{[n+2s]}+\dots.$$

There is no difficulty, on taking out the terms involving x^{n+2r}, in showing that the sum of the coefficients of x^{n+2r} vanishes, x^n arises only from $J_{[n]}$, and its coefficient is $1/\{2n\}!$ The convergence, &c., of this series is discussed in a note at the end of the paper. A similar series for $\mathfrak{J}$ functions is there given.

LEMMA III.—

$$P_{[n]}(x) = p^{\frac{1}{2}(n^2-n)} \Sigma x^n \frac{[n][n-1]\dots[n-2r+1]}{[2]^2[4]^2\dots[2r]^2} \times \left(1-\frac{p^3}{x^2}\right)\left(1-\frac{p^5}{x^2}\right)\dots\left(1-\frac{p^{2r+1}}{x^2}\right)p^{2r^2+r-2rn}. \qquad (\alpha)$$

It has been shown in a preceding paper, Ser. 2, Vol. 2, p. 215, that

$$P_{[n]}(x) = \Sigma x^n \frac{[n][n-1]\dots[n-2r+1]}{[2]^2[4]^2\dots[2r]^2}\left(p^2-\frac{p^3}{x^2}\right)\left(p^6-\frac{p^5}{x^2}\right)\dots\left(p^{4r-2}-\frac{p^{2r+1}}{x^2}\right). \qquad (\beta)$$

Invert the base p in this theorem; then

$$P_{[n]}(x) \quad \text{becomes} \quad p^{-\frac{1}{2}[n(n-1)]-2n}P_{[n]}(xp^2)$$

and the expression on the right side of (β) becomes

$$x^n \frac{[n]\dots[n-2r+1]}{[2]^2\dots[2r]^2} p^{2r^2+r-2nr}\left(1-\frac{p^{-1}}{x^2}\right)\dots\left(1-\frac{p^{2r-3}}{x^2}\right).$$

On replacing x by xp^{-2}, we have the theorem as stated in (α).

LEMMA IV.—

$$\frac{\mu^n}{[n]!} = \frac{\{q-4\}!}{\{2n+q-4\}!}\left[P_{[n]}(q,\mu)+p^3\frac{[2n+q-6]}{[2]}P_{[n-2]}(q,\mu) + p^6\frac{[2n+q-4][2n+q-10]}{[2][4]}P_{[n-4]}+\dots\right]. \qquad (\gamma)$$

The p prefix of the general term of the series within the large brackets is p^{3r}, and

$$P_{[n]}(q,\mu) = \sum_{r=0}^{\infty}(-1)^r p^{r(r+2)}\frac{\{2n+q-2r-4\}!}{\{q-4\}!\,\{2r\}!\,\{n-2r\}!}\mu^{n-2r}$$

Comparing $P_{[n]}(q,\mu)$ with the function denoted $P_n(p,\mu)$ by Dr. Hobson (see *Proc. London Math. Soc.*, Vol. xxv., p. 57), it is seen that when the base p is unity the function $P_{[n]}(q,\mu)$ reduces to $P_n(p,\mu)$, allowing for the use of q in place of Hobson's parameter p, since p is already appropriated to denote the base of the functions. In series (γ), if we expand the P functions and arrange according to powers of μ, it is plain that μ^n arises only from $P_{[n]}$ and that the term involving it is $\mu^n/[n]!$, and the coefficients of the terms involving μ^{n-2r} form the series

$$(-1)^r p^{r^2-r}\frac{\{2n+q-2r-4\}!}{\{2n+q-4\}!\,\{2r\}!\,[n-2r]!}\left\{1-p^{2-2r}\frac{[r][2n+q-6]}{[2][2n+q-2r-4]}+\dots\right\},$$

which is identically zero by Lemma I., and so the required result (γ) is established.

LEMMA V.—

$$\frac{\mu^n}{[n]!} = \frac{\{q-4\}!}{\{2n+q-4\}!}\left\{\mathfrak{P}_n(q,\mu)+\frac{[2n+q-6]}{[2]}\mathfrak{P}_{n-2}(q,\mu)\right.$$

$$\left.+\frac{[2n+q-4][2n+q-10]}{[2][4]}\mathfrak{P}_{n-4}(q,\mu)+\dots\right\}, \quad (\delta)$$

$$\mathfrak{P}_n(q,\mu) = \sum_{r=0}^{\infty}(-1)^r p^{r(r-1)}\frac{\{2n+q-2r-4\}!}{\{q-4\}!\{2r\}![n-2r]!}\mu^{n-2r}.$$

Series (δ) differs from the analogous series in Lemma IV., in that there are no p prefixes to the terms of the series. This lemma may be established on just the same lines as Lemma IV., and depends on the identity given in Lemma I. We notice that

$$\mathfrak{P}_n(q,\mu) = p^{-\frac{1}{4}n}P_{[n]}(q,\mu p^{\frac{1}{2}}).$$

In case $q = 3$, Lemma IV. reduces to a result given in a previous paper [(74), page 217, Ser. 2, Vol. 2]. The double forms of these theorems afford examples of the double forms of all (q) function series and products. This duality of form is necessary to give convergent series, as the base $|p| > 1$ or < 1.

§ 2.

Form the product

$$E_p(r\cos\theta)\{A_0+A_1r^2+A_2r^4+\dots\}$$

in which A_0, A_1, ... are p functions of θ, viz.,

$$A_n = (-1)^n\frac{(p^3-p^2\cos^2\theta)(p^5-p^6\cos^2\theta)\dots(p^{2n+1}-p^{4n-2}\cos^2\theta)}{[2]^2[4]^2\dots[2n]^2}.$$

The series $\Sigma A_n r^{2n}$ and $E_p(r\cos\theta)$ are absolutely convergent if $|p| > 1$. The coefficient of r^n in the product is therefore

$$\frac{1}{[n]!}\Sigma\cos^n\theta\frac{[n][n-1]\dots[n-2r+1]}{[2]^2[4]^2\dots[2r]^2}\frac{(p^3-p^2\cos^2\theta)\dots(p^{2r+1}-p^{4r-2}\cos^2\theta)}{\cos^{2r}\theta},$$

and this series, by Lemma III., is $\frac{1}{[n]!}P_{[n]}(\cos\theta)$; so that we write

$$E_p(r\cos\theta)B(r,\theta) = \sum_{n=0}^{\infty}\frac{r^n}{[n]!}P_{[n]}(\cos\theta),$$

in which

$$B(r,\theta) = \sum_{n=0}^{\infty}(-1)^n\frac{(p^3-p^2\cos^2\theta)\dots(p^{2n+1}-p^{4n-2}\cos^2\theta)}{[2]^2[4]^2\dots[2n]^2}r^{2n},$$

reducing when $p = 1$ to

$$e^{r\cos\theta} J_0(r \sin\theta) = \Sigma \frac{r^n}{n!} P_n(\cos\theta).$$

In a similar way, forming the product $E_{p^{-1}}(r\cos\theta)\,\mathfrak{B}(r,\theta)$, viz.,

$$E_{p^{-1}}(r\cos\theta)\,\{\mathfrak{A}_0+\mathfrak{A}_1 r^2+\mathfrak{A}_2 r^4+\ldots\},$$

in which $$E_{p^{-1}}(r\cos\theta) = 1+\frac{r\cos\theta}{[1]!}+p\frac{r^2\cos^2\theta}{[2]!}+\ldots,$$

$$\mathfrak{A}_n = (-1)^n \frac{(p^3-\cos^2\theta)(p^5-\cos^2\theta)\ldots(p^{2n+1}-\cos^2\theta)}{[2]^2[4]^2\ldots[2n]^2}\, p^{-2n},$$

we find that the coefficient of r^n in the resulting series is

$$\frac{1}{[n]!}\, p^{\frac{1}{2}[n(n-1)]} \Sigma \cos^n\theta \frac{[n][n-1]\ldots[n-2r+1]}{[2]^2[4]^2\ldots[2r]^2}$$
$$\times \frac{(p^3-\cos^2\theta)\ldots(p^{2r+1}-\cos^2\theta)}{\cos^{2r}\theta}\, p^{2r^2+r-2rn}.$$

This, by (a), Lemma III., is $\frac{1}{[n]!} P_{[n]}(\cos\theta)$.

We write the theorems

$$\left.\begin{aligned} E_p(r\cos\theta)\, B(r,\theta) &= \Sigma \frac{r^n}{[n]!} P_{[n]}(\cos\theta), \\ E_{p^{-1}}(r\cos\theta)\, \mathfrak{B}(r,\theta) &= \Sigma \frac{r^n}{[n]!} P_{[n]}(\cos\theta). \end{aligned}\right\} \qquad (1)$$

Since $E_p(r\cos\theta)\, E_{1/p}(-r\cos\theta) = 1$ and the B functions involve only even powers of r, it follows that

$$B(r,\theta)\,\mathfrak{B}(r,\theta)$$
$$= \left\{1+\frac{r^2}{[2]!}P_{[2]}+\frac{r^4}{[4]!}P_{[4]}+\ldots\right\}^2 - \left\{rP_{[1]}+\frac{r^3}{[3]!}P_{[3]}+\ldots\right\}^2$$

(cf. *Proc. London Math. Soc.*, Vol. xxv., p. 66).

Again, since for the E function $E_p(a)\, E_{p^{-1}}(-a) = 1$, we can write

$$E_{p^{-1}}(-r\cos\theta) \sum_{n=0}^{\infty} \frac{r^n}{[n]!} P_{[n]}(\cos\theta) = B(r,\theta). \qquad (2)$$

This gives us a new form of $B(r,\theta)$ expressed in a series of ascending powers of r. The coefficient of r^n in the expression on the left side of (2) is

$$\frac{1}{[n]!}\left[P_{[n]}-\frac{[n]}{[1]}\cos\theta P_{[n-1]}+p\frac{[n][n-1]}{[2]!}\cos^2\theta P_{[n-2]}\right.$$
$$\left.-\dots(-1)^n p^{\frac{1}{2}[n(n-1)]}\cos^n\theta P_{[0]}\right]. \quad (3)$$

The coefficients of the series within the large brackets follow the binomial form

$$(1-x)_n = 1-\frac{[n]}{[1]}x+p\frac{[n][n-1]}{[2]!}x^2-\dots$$

(*Proc. London Math. Soc.*, Ser. 2, Vol. 2, p. 193).

Now $B(r, \theta)$ is an even function of r; so that when n is an odd integer it necessarily follows that the expression (3) is identically zero. In case n is an even integer we obtain, by equating coefficients of r^{2n} and replacing $\cos\theta$ by x,

$$\frac{1}{[2n]!}\left[P_{[2n]}-\frac{[2n]}{[1]}xP_{[2n-1]}+p\frac{[2n][2n-1]}{[2]!}x^2P_{[2n-2]}-\dots\right]$$
$$=\frac{(p^2x^2-p^3)(p^6x^2-p^5)\dots(p^{4n-2}x^2-p^{2n+1})}{[2]^2[4]^2\dots[2n]^2}. \quad (4)$$

Similarly, from the second expression of (1) we obtain that, if n be odd,

$$P_{[n]}-\frac{[n]}{[1]}xP_{[n-1]}+\frac{[n][n-1]}{[2]!}x^2P_{[n-2]}-\dots = 0,$$

and for $2n$ (even integer)

$$P_{[2n]}-\frac{[2n]}{[1]}xP_{[2n-1]}+\frac{[2n][2n-1]}{[2]!}x^2P_{[2n-2]}-\dots$$
$$=[2n]!\,\frac{(x^2-p^3)(x^2-p^5)\dots(x^2-p^{2n+1})}{[2]^2[4^2]\dots[2n]^2}. \quad (5)$$

These theorems are easily verified for integral values of n, and suggest interesting extensions in case n be not integral.

In (5), if we take out the terms involving x^{2n-2r} from the left side, the coefficient of x^{2n-2r} is found to be

$$p^{r(r+2)}\frac{[4n-2r]!}{\{2r\}!\,\{4n-2r\}!\,[2n-2r]!}$$
$$\times\left\{1-\frac{[2n]}{[1]}\frac{[2n-2r]}{[4n-2r-1]}+\frac{[2n][2n-1]}{[2]!}\frac{[2n-2r][2n-2r-1]}{[4n-2r-1][4n-2r-3]}-\dots\right\}.$$

The sum of the Heinean series within the large brackets is easily found to be for all values of n

$$\frac{[2n]!\,\{2n-2r-1\}!}{\{2n\}!\,\{4n-2r-1\}!},$$

and the coefficient of x^{2n-2r} may be written

$$p^{r(r+2)}\frac{[2n]!}{\{2n\}!^2}\frac{\{2n\}!}{\{2n-2r\}!\,\{2r\}!}. \tag{6}$$

Now
$$x^{2n}\mathop{\mathrm{L}}_{\kappa=\infty}\frac{(x^2-p^3)(x^2-p^5)\dots(x^2-p^{2\kappa+1})}{(x^2-p^{2n+3})(x^2-p^{2n+5})\dots(x^2-p^{2n+2\kappa+1})}.$$

may be expanded without difficulty (cf. *Proc. London Math. Soc.*, Ser. 2, Vol. 1, p. 73) as

$$\sum_{r=0}^{r=\infty}\frac{[2n][2n-2]\dots[2n-2r+2]}{[2][4]\dots[2r]}p^{r(r+2)}x^{2n-2r}.$$

On comparing this with (6), viz., the coefficient of x^{2n-2r} obtained from $P_{[2n]}-\frac{[2n]}{[1]}xP_{[2n-1]}+\dots$, we are justified in writing generally for all values of $2n$, except positive odd integral values,

$$\sum_0^\infty(-1)^r\frac{[2n]!}{[r]!\,[2n-r]!}x^rP_{[2n-r]}(x)$$
$$=x^{2n}\frac{[2n]!}{\{2n\}!^2}\mathop{\mathrm{L}}_{\kappa=\infty}\frac{(x^2-p^3)(x^2-p^5)\dots(x^2-p^{2\kappa+1})}{(x^2-p^{2n+3})(x^2-p^{2n+5})\dots(x^2-p^{2n+2\kappa+1})}\quad(|p|<1). \tag{7}$$

A similar theorem may be obtained as a general expression of

$$\sum_{r=0}^\infty(-1)^rp^{r(r-1)}\frac{[2n]!}{[r]!\,[2n-r]!}x^rP_{[2n-r]}(x)$$
$$=\frac{[2n]!}{\{2n\}!}(p^2x^2-p^3)\dots(p^{4n-2}x^2-p^{2n+1}). \tag{8}$$

A result which is easily derived from (1) is

$$\left\{1-\frac{2(1+p)}{[2]!}r^2\cos^2\theta+\frac{2(1+p)(1+p^2)(1+p^3)}{[4]!}r^4\cos^4\theta-\dots\right\}B(ir,\theta)\,\mathfrak{B}(ir,\theta)$$
$$=1+\sum_{n=1}^\infty(-1)^n\left\{P_{[2n]}+\frac{[2n]}{[1]}P_{[2n-1]}P_{[1]}+\frac{[2n][2n-1]}{[2]!}P_{[2n-2]}P_{[2]}+\dots+P_{[2n]}\right\}\frac{r^{2n}}{[2n]!}, \tag{9}$$

reducing, in case $p=1$, to

$$\cos(2r\cos\theta)\,J_0^2(r\sin\theta)=1+\sum_{n=1}^\infty(-1)^n\left\{2P_{2n}+2\frac{2n}{1}P_{2n-1}P_1+\dots\right\}\frac{r^{2n}}{2n!}.$$

§ 3.

Writing (Lemma V.)

$$I_{[n+\frac{1}{2}q-1]}(r) = \frac{r^{n+\frac{1}{2}q-1}}{\{2n+q-2\}!}\left\{1+\frac{r^2}{[2]\,[2n+q]}+\ldots\right\},$$

$$\mathfrak{P}_n(q,\mu) = \frac{\{2n+q-4\}!}{[n]!\,\{q-4\}!}\left\{\mu^n - \frac{[n][n-1]}{[2][2n+q-4]}\mu^{n-2}\right.$$

$$\left. + p^2\frac{[n][n-1][n-2][n-3]}{[2][4][2n+q-4][2n+q-6]}\mu^{n-4}+\ldots\right\},$$

consider the series

$$\{q-4\}!\sum_{n=0}^{\infty}\frac{[2n+q-2]}{r^{\frac{1}{2}q-1}}I_{[n+\frac{1}{2}q-1]}(r)\,\mathfrak{P}_n(q,\mu).$$

The coefficient of r^n in this series is

$$\frac{\{q-4\}!}{\{2n+q-4\}!}\left\{\mathfrak{P}_n(q,\mu)+\frac{[2n+q-6]}{[2]}\mathfrak{P}_{n-2}(q,\mu),+\ldots\right\},$$

and this, by Lemma V., is equal to $\mu^n/[n]!$; so that we are able to write, subject to the absolute convergence of the series,

$$\{q-4\}!\sum_{n=0}^{\infty}\frac{[2n+q-2]}{r^{\frac{1}{2}q-1}}I_{[n+\frac{1}{2}q-1]}(r)\,\mathfrak{P}_n(q,\cos\theta)$$

$$= 1+\frac{r\cos\theta}{[1]!}+\frac{r^2\cos^2\theta}{[2]!}+\ldots = E_p(r\cos\theta). \quad (10)$$

Replacing r by ir,

$$\{q-4\}!\sum_{n=0}^{\infty}i^n\frac{[2n+q-2]}{r^{\frac{1}{2}q-1}}J_{[n+\frac{1}{2}q-1]}(r)\,\mathfrak{P}_n(q,\cos\theta) = E_p(ir\cos\theta). \quad (11)$$

By means of Lemma IV. we may show in the same way that

$$\{q-4\}!\sum_{n=0}^{\infty}i^n p^{-\frac{1}{2}n}\frac{[2n+q-2]}{r^{\frac{1}{2}q-1}}J_{[n+\frac{1}{2}q-1]}(r)\,P_{[n]}(q,\cos\theta) = E_p(irp^{-\frac{1}{2}}\cos\theta) \quad (12)$$

and again from these theorems, by inversion of the base p,

$$\{q-4\}!\sum_{n=0}^{\infty}i^n\frac{[2n+q-2]}{r^{\frac{1}{2}q-1}}p^{2q+\frac{1}{2}[n(n-1)]}\mathfrak{J}_{[n+\frac{1}{2}q-1]}(r)\,\mathfrak{P}'_n(q,\cos\theta) = E_{p^{-1}}(ir\cos\theta),$$
$$(13)$$

$$\{q-4\}!\sum_{n=0}^{\infty}i^n p^{\frac{1}{2}n}\frac{[2n+q-2]}{r^{\frac{1}{2}q-1}}p^{\frac{1}{2}[n(n-1)]}\mathfrak{J}_{[n+\frac{1}{2}q-1]}(r)\,P'_{[n]}(q,\cos\theta)$$

$$= E_{p^{-1}}(irp^{\frac{1}{2}}\cos\theta), \quad (14)$$

in which $P_{[n]}$ and $\mathfrak{P}_n$ are defined as in Lemmas IV. and V., while we define

$$\mathfrak{P}'_n(q, \mu) = \sum_{r=0}^{\infty} (-1)^r p^{r(r+q-2)} \frac{\{2n+q-2r-4\}!}{\{q-4\}!\,\{2r\}!\,[n-2r]!} \mu^{n-2r},$$

$$P'_{[n]}(q, \mu) = \sum_{r=0}^{\infty} (-1)^r p^{r(r+q-5)} \frac{\{2n+q-2r-4\}!}{\{q-4\}!\,\{2r\}!\,[n-2r]!} \mu^{n-2r}.$$

The four forms (11), (12), (13), (14) reduce to Dr. Hobson's result if the base p be made unity. When the parameter $q = 3$ the functions P reduce to the basic generalizations of Legendre's function.

§ 4.

It can be shown without difficulty that

$$\prod_{m=1}^{\infty} \{1+x(p-1)p^{-2m}\} = 1+\frac{x}{[2]}+\frac{x^2}{[2][4]}+\ldots = E_{p^2}\left(\frac{x}{[2]}\right).$$

In a previous paper it has been shown (p. 198, Ser. 2, Vol. 2) that

$$E_{p^2}\left(\frac{\lambda t}{[2]}\right) E_{1/p^2}\left(-\frac{\lambda t^{-1}}{[2]}\right) = 1+\frac{\lambda}{[2]}(t-t^{-1})+\frac{\lambda^2}{[2][4]}(t-t^{-1})(t-p^2t^{-1})+\ldots.$$

Replacing t by $e^{i\theta}$, p by $e^{i\omega}$, we can write

$$\frac{E_{p^2}\left(\frac{\lambda t}{[2]}\right)}{E_{p^2}\left(\frac{\lambda}{[2]t}\right)} = 1+\frac{i\lambda}{[2]}\,2\sin\theta+\frac{i^2\lambda^2}{[2][4]}\,4p\sin\theta\sin(\theta+\omega)+\ldots$$
$$+\frac{i^n\lambda^n}{\{2n\}!}p^{\frac{1}{2}[n(n-1)]}2^n\sin\theta\sin(\theta+\omega)\ldots\sin(\theta+n\omega-\omega). \quad (15a)$$

Similarly

$$\frac{E_{p^2}\left(-\frac{\lambda t^{-1}}{[2]}\right)}{E_{p^2}\left(-\frac{\lambda t}{[2]}\right)} = 1+\frac{i\lambda}{[2]}\,2\sin\theta+\frac{i^2\lambda^2}{[2][4]}\,4p\sin\theta\sin(\theta-\omega)+\ldots$$
$$+\frac{i^n\lambda^n}{\{2n\}!}p^{\frac{1}{2}[n(n-1)]}2^n\sin\theta\sin(\theta-\omega)\ldots\sin(\theta-n\omega+\omega). \quad (15\beta)$$

Take the product of these series; then, by result (8), p. 195, Ser. 2, Vol. 2, we obtain

$$\frac{\sum_{-\infty}^{+\infty} t^n J_{[n]}(\lambda)}{\sum_{-\infty}^{+\infty} t^{-n} J_{[n]}(\lambda)} = 1+c_1\lambda+c_2\lambda^2+\ldots,$$

in which

$$c_n = \frac{2^n i^n}{\{2n\}!} p^{\frac{1}{2}[n(n-1)]} \sin\theta \sin(\theta+\omega) \ldots \sin(\theta+n\omega-\omega)$$

$$\times \left\{ \sum_{r=0}^{n} p^{r^2-rn} \frac{\{2n\}!}{\{2r\}!\,\{2n-2r\}!} \frac{\sin\theta \ldots \sin(\theta-r\omega+\omega)}{\sin(\theta+n\omega-\omega)\ldots\sin(\theta+n\omega-r\omega)} \right\}. \quad (15\gamma)$$

The product of the E functions, besides being expressed in series of $J_{[n]}$ functions, may be expressed in theta function product form: for

$$E_{p^2}\left(\frac{\lambda t}{[2]}\right) = \prod_{m=1}^{\infty} \{1+\lambda(p-1)tp^{-2m}\}.$$

On replacing λ by $ix/(1-p)$ we obtain

$$E_{p^2}\left(\frac{ixt}{1-p^2}\right) = \prod_{m=1}^{\infty} \{1-ixtp^{-2m}\};$$

so that the product of (15α) and (15β) gives us, if $|p| > 1$,

$$\left.\begin{aligned} &\prod_{m=1}^{\infty} \frac{(1-2xp^{-2m}\sin\theta+x^2p^{-4m})}{(1+2xp^{-2m}\sin\theta+x^2p^{-4m})} = \sum_{n=0}^{\infty} a_n x^n, \\ &a_n = (-1)^n \frac{\sin\theta\ldots\sin(\theta+n\omega-\omega)}{(p^2-1)(p^4-1)\ldots(p^{2n}-1)} \left\{ \sum_{r=0}^{n} p^{r^2-rn} \frac{\{2n\}!}{\{2r\}!\,\{2n-2r\}!} \right. \\ &\qquad\qquad \left. \times \frac{\sin\theta\ldots\sin(\theta+\omega-r\omega)}{\sin(\theta+n\omega-\omega)\ldots\sin(\theta+n\omega-r\omega)} \right\} p^{\frac{1}{2}(n^2-n)}. \end{aligned}\right\} \quad (16)$$

Similarly,

$$\left.\begin{aligned} &\prod_{m=1}^{\infty} \frac{(1-2xp^{-2m}\cos\theta+p^{-4m}x^2)}{(1+2xp^{-2m}\cos\theta+p^{-4m}x^2)} = \sum_{n=0}^{\infty} b_n x^n, \\ &b_n = (-1)^n \frac{\cos\theta\ldots\cos(\theta+n\omega-\omega)}{(p^2-1)\ldots(p^{2n}-1)} \left\{ \sum p^{r^2-rn} \frac{\{2n\}!}{\{2r\}!\,\{2n-2r\}!} \right. \\ &\qquad\qquad \left. \times \frac{\cos\theta\ldots\cos(\theta+\omega-r\omega)}{\cos(\theta+n\omega-\omega)\ldots\cos(\theta+n\omega-r\omega)} \right\} p^{\frac{1}{2}(n^2-n)}. \end{aligned}\right\} \quad (17)$$

Since

$$E_{p^2}\left(\frac{\lambda\kappa t}{[2]}\right) E_{1/p^2}\left(\frac{\lambda\kappa}{[2]\,t}\right) = 1 + \frac{\lambda\kappa}{[2]}\, 2\cos\theta + p\,\frac{\lambda^2\kappa^2}{[2][4]}\, 2^2 \cos\theta\cos(\theta+\omega) + \ldots$$

$$+ p^{\frac{1}{2}[n(n-1)]} \frac{\lambda^n\kappa^n}{\{2n\}!}\, 2^n \cos\theta\ldots\cos[\theta+(n-1)\omega] + \ldots \quad (18)$$

and

$$E_{p^2}\left(-\frac{\lambda\kappa^{-1}t}{[2]}\right)E_{1/p^2}\left(-\frac{\lambda\kappa^{-1}}{[2]t}\right)$$
$$=1-\frac{\lambda\kappa^{-1}}{[2]}2\cos\theta+p\frac{\lambda^2\kappa^{-2}}{[2][4]}2^2\cos\theta\cos(\theta+\omega)+\dots, \quad (19)$$

also we know that

$$E_{p^2}\left(\frac{\lambda\kappa t}{[2]}\right)E_{p^2}\left(-\frac{\lambda\kappa^{-1}t}{[2]}\right)=J_{[0]}(\lambda t)+(\kappa-\kappa^{-1})J_{[1]}(\lambda t)+(\kappa^2+\kappa^{-2})J_{[2]}(\lambda t)+\dots, \quad (20)$$

$$E_{1/p^2}\left(\frac{\lambda\kappa}{[2]t}\right)E_{1/p^2}\left(-\frac{\lambda\kappa^{-1}}{[2]t}\right)$$
$$=\mathfrak{J}_{[0]}\left(\frac{\lambda}{pt}\right)+p(\kappa-\kappa^{-1})\mathfrak{J}_{[1]}\left(\frac{\lambda}{pt}\right)+p^4(\kappa^2+\kappa^{-2})\mathfrak{J}_{[2]}\left(\frac{\lambda}{pt}\right)+\dots. \quad (21)$$

On taking the product of (20) and (21) we find

$$\left\{J_{[0]}(\lambda e^{i\theta})+\sum_{n=1}^{\infty}[\kappa^n+(-\kappa)^n]J_{[n]}(\lambda e^{i\theta})\right\}$$
$$\times\left\{\mathfrak{J}_{[0]}\left(\frac{\lambda}{p}e^{-i\theta}\right)+\sum_{n=1}^{\infty}p^{n^2}[\kappa^n+(-\kappa)^n]\mathfrak{J}_{[n]}\left(\frac{\lambda}{p}e^{-i\theta}\right)\right\}$$
$$=\mathfrak{J}_{[0]}(2\lambda,\cos\theta)+\Sigma p^{n^2}[\kappa^n+(-\kappa)^{-n}]\mathfrak{J}_{[n]}(2\lambda,\cos\theta), \quad (22)$$

in which $J_{[n]}(\lambda e^{i\theta})$ and $\mathfrak{J}_{[n]}(\lambda e^{-i\theta}/p)$ are the generalizations of Bessel's function defined p. 195, Ser. 2, Vol. 2, but

$$\mathfrak{J}_{[n]}(2\lambda,\cos\theta)=\Sigma(-1)^n\frac{(2\lambda)^{n+2r}}{\{2r\}!\{2n+2r\}!}p^{2r(n+r)}\cos\theta\cos(\theta+\omega)\dots$$
$$\times\cos[\theta+(n+r-1)\omega]\cos\theta\cos(\theta+\omega)\dots\cos[\theta+(r-1)\omega]. \quad (23)$$

In case $p=1$, this function reduces immediately to $J_n(2\lambda\cos\theta)$.

By equating coefficients in (22) we obtain interesting forms of quasi-addition theorems. For example, from the terms independent of κ we find

$$J_{[0]}\mathfrak{J}_{[0]}+2\Sigma(-1)^n p^{n^2}J_{[n]}\mathfrak{J}_{[n]}=\mathfrak{J}_{[0]}(2\lambda,\cos\theta),$$

in which

$$\mathfrak{J}_{[0]}(2\lambda,\cos\theta)=\sum_{r=1}^{\infty}(-1)^r\frac{(2\lambda)^{2r}}{\{2r\}!\{2r\}!}p^{2r^2}\cos^2\theta\cos^2(\theta+\omega)\dots\cos^2[\theta+(r-1)\omega], \quad (24)$$

the arguments of the functions J and $\mathfrak{J}$ on the left side of (24) being $\lambda e^{i\theta}$ and $\lambda e^{-i\theta}/p$ respectively. By similar reasoning we may show that more generally

$$\mathfrak{J}_{[0]}(2xy,\cos\theta)=J_{[0]}(x^2e^{i\theta})\mathfrak{J}_{[0]}(y^2p^{-1}e^{-i\theta})+2\Sigma(-1)^n p^{n^2}J_{[n]}\mathfrak{J}_{[n]}, \quad (24a)$$

and other theorems of similar kind may be obtained by equating coefficients of κ^n in (22).

§ 5.

The expansion $$\frac{1}{(1-2x\cos\theta+x^2)^{\frac{1}{2}q-1}} = \sum_{n=0}^{\infty} x^n P_n(q, \cos\theta) \tag{25}$$

is well known (cf. *Proc. London Math. Soc.*, Vol. xxv., p. 57), and the properties of the coefficients $P_n(q, \cos\theta)$ have been investigated by Dr. Hobson. It may be of interest to form the basic coefficient analogous to $P_n(q, \cos\theta)$ and connect it by means of the E function with other generalizations of P_n and J_n. Take the basic analogue of the binomial theorem

$$\prod_{r=1}^{\infty} \frac{(1-p^r x)}{(1-p^{n+r}x)} = 1 - p\frac{[n]}{[1]}x + p^3\frac{[n][n-1]}{[2]!}x^2 - \dots,$$

and let us apply this to obtain the analogue of (25). Changing p into p^2, and making $n = 1-\frac{1}{2}q$, we find

$$\prod_{r=1}^{\infty} \frac{(1-p^{2r}xe^{i\theta})}{(1-p^{2-q+2r}xe^{i\theta})} = 1 - p^2\frac{[2-q]}{[2]}xe^{i\theta} + p^6\frac{[2-q][-q]}{[2][4]}x^2e^{2i\theta} - \dots. \tag{26}$$

In the usual way, by taking the product of two such series, we find

$$\prod_{r=1}^{\infty} \frac{(1-2p^{2r}x\cos\theta+x^2p^{4r})}{(1-2p^{2-q+2r}x\cos\theta+x^2p^{4-2q+4r})} = 1+\Sigma(-1)^n x^n p^{(4-q)n} P_{[n]}(q, \cos\theta), \tag{27}$$

in which

$$P_{[n]}(q, \cos\theta) = \frac{[q-2][q]\dots[q+2n-4]}{\{2n\}!}$$
$$\times\left\{2\cos n\theta + \frac{[q-2][2n]}{[2][q+2n-4]}2\cos(n-2)\theta + \dots\right\}. \tag{28}$$

The infinite products

$$\prod_{r=0}^{\infty} \frac{1}{(1-2xp^m\cos\theta+x^2p^{2m})} = 1+\Sigma\frac{A_r(\theta)}{(1-p^r)!}x^r,$$

$$\prod_{r=0}^{\infty} (1+2xp^{m+1}\cos\theta+x^2p^{2m+2}) = 1+\Sigma\frac{B_r(\theta)}{(1-p^r)!}x^r$$

have been discussed by Prof. L. J. Rogers (*Proc. London Math. Soc.*, Vols. xxiv., xxv.). In the notation of Rogers' papers we have

$$\left\{1+\Sigma\frac{A_r(\theta, p^2)}{\{2r\}!}\left(\frac{xp^{4-q}}{1-p}\right)^r\right\}\left\{1+\Sigma\frac{B_r(\theta, p^2)}{\{2r\}!}\left(\frac{x}{1-p}\right)^r\right\}$$
$$= 1+\Sigma(-1)^n x^n p^{(4-q)n} P_{[n]}(q, \cos\theta). \tag{29}$$

Equating coefficients of x^n, we see that

$$p^{(4-n)r}P_{[n]}(q, \cos\theta) = \frac{1}{(1-p)^n}\sum_{r=0}^{n}(-1)^r\frac{A_r B_{n-r}}{\{2r\}!\{2n-2r\}!}p^{(4-q)r}. \tag{30}$$

The discussion of this for general values of n would be interesting, but is not attempted here.

The properties of the coefficients A_r, B_r are discussed by Prof. L. J. Rogers (*loc. cit.*). Some interesting relations may be found between the coefficients $J_{[n]}$ and A_n. We have

$$\prod_{m=1}^{\infty}\{1+2\kappa x p^{-2m}\cos\theta+\kappa^2 x^2 p^{-4m}\}^{-1}$$

$$=\left\{1+\frac{\kappa x e^{i\theta}}{p-1}+\frac{\kappa^2 x^2 e^{2i\theta}}{(p-1)(p^2-1)}+\dots\right\}\left\{1+\frac{\kappa x e^{-i\theta}}{p-1}+\dots\right\}$$

$$=1+\frac{\kappa x}{(p-1)}A_1(\theta)+\frac{\kappa^2x^2}{(p-1)(p^2-1)}A_2(\theta)+\dots, \tag{31}$$

$$A_r(\theta)=2\cos r\theta+2\frac{[r]}{[1]}\cos(r-2)\theta+2\frac{[r][r-1]}{[2]!}\cos(r-4)\theta+\dots.$$

Consider the product $$E_{p^2}\left(\frac{\kappa\lambda}{[2]t}\right)E_{1/p^2}\left(-\frac{\lambda}{[2]\kappa t}\right). \tag{32}$$

Since $E_p(a)E_{p^{-1}}(-a)=1$, we can write (32) as

$$\frac{E_{p^2}\left(\frac{\kappa\lambda t}{[2]}\right)E_{p^2}\left(\frac{\kappa\lambda}{[2]t}\right)}{E_{p^2}\left(\frac{\kappa\lambda t}{[2]}\right)E_{p^2}\left(\frac{\lambda}{[2]\kappa t}\right)}=\prod_{m=1}^{\infty}\frac{\{1+\kappa\lambda t(p-1)p^{-2m}\}\{1+\kappa\lambda t^{-1}(p-1)p^{-2m}\}}{\{1+\kappa\lambda t(p-1)p^{-2m}\}\{1+\kappa^{-1}\lambda t^{-1}(p-1)p^{-2m}\}}. \tag{33}$$

Replace λ by $x/(p-1)$, κ by $e^{i\phi}$, t by $e^{i\theta}$; then the infinite product (33) is

$$\prod_{m=1}^{\infty}\frac{(1+2\kappa x p^{-2m}\cos\theta+\kappa^2x^2p^{-4m})}{(1+2xp^{-2m}\cos(\theta+\phi)+x^2p^{-4m})}, \tag{34}$$

while (32) becomes $$\sum_{r=0}^{\infty}\kappa^r J_{[r]}+\sum_{r=1}^{\infty}(-1)^r p^{r^2}\kappa^{-r}\mathfrak{J}_{[r]},$$

the arguments of J and $\mathfrak{J}$ being respectively $xe^{-i\theta}/(p-1)$ and $xe^{-i\theta}/(p^2-p)$ [cf. (22), p. 201, Ser. 2, Vol. 2]. Hence

$$\prod_{m=1}^{\infty}\frac{1}{(1+2xp^{-2m}\cos(\theta+\phi)+x^2p^{-4m})}$$

$$=\left\{1+\sum_{r=1}^{\infty}(-1)^r\frac{\kappa^r x^r}{(1-p^r)!}A_r(\theta)\right\}\left\{\sum_0^{\infty}\kappa^r J_{[r]}+\sum_1^{\infty}(-1)^r\kappa^{-r}p^{r^2}\mathfrak{J}_{[r]}\right\}. \tag{35}$$

The infinite product on the left may be expressed in a series of generalized Bessel functions as

$$I_{[0]}\left(\frac{x}{p-p^2}\right)+\sum_{r=1}^{\infty}2p^{r^2}\cos(r\theta+r\phi)I_{[n]}\left(\frac{x}{p-p^2}\right), \tag{36}$$

in which $$I_{[n]}(a) = \sum_{r=0}^{\infty} \frac{a^{n+2r}}{\{2r\}!\,\{2n+2r\}!}\, p^{2r(n+r)},$$

because in E function form the product is

$$\frac{1}{E_{p^2}\left(\frac{\kappa x t}{(p-1)[2]}\right) E_{p^2}\left(\frac{x\kappa^{-1}t^{-1}}{(p-1)[2]}\right)} = E_{p^{-2}}\left(-\frac{\kappa x t}{(p-1)[2]}\right) E_{p^{-2}}\left(-\frac{x\kappa^{-1}t^{-1}}{(p-1)[2]}\right)$$

$$= I_{[0]} + \sum_{n=1}^{\infty} (-1)^n p^{n^2}(\kappa^n t^n + \kappa^{-n} t^{-n})\, I_{[n]}.$$

Equating coefficients in (35) and (36), we find

$$I_{[0]}\left(\frac{x}{p-p^2}\right) = J_{[0]} - p\,\frac{x}{(p-1)}\,\mathfrak{J}_{[1]} A_1 + p^4\,\frac{x^2}{(p-1)(p^2-1)}\,\mathfrak{J}_{[2]} A_2 - \dots, \tag{37}$$

$$p^{n^2} e^{-ni\theta} I_{[n]} = p^n \mathfrak{J}_{[n]} - p^{(n+1)^2}\,\frac{x}{(p-1)}\,\mathfrak{J}_{[1]} A_1 + \dots, \tag{38}$$

in which the argument of I is $x/(p-p^2)$, of $\mathfrak{J}$ is $xe^{-i\theta}/(p^2-p)$, of J is $xe^{-i\theta}/(p-1)$.

§ 6.

Expansion of an Arbitrary Function in Series of Functions $J_{[n]}$ or $\mathfrak{J}_{[n]}$.

If $|t| > |x|$,

$$\frac{1}{t-x} = \frac{1}{t} + \frac{x}{t^2} + \frac{x^2}{t^3} + \dots.$$

Now express each power of x in a series of $J_{[n]}$ functions by Lemma II. as

$$\frac{x^n}{t^{n+1}} = \frac{\{2n\}!}{t^{n+1}}\left\{ J_{[n]} + \frac{[2n+4]}{[2]} J_{[n+2]} + p^2\,\frac{[2n+8][2n+2]}{[2][4]} J_{[n+4]} + \dots \right\}.$$

Arrange the series according to the order of the J functions. Terms involving $J_{[n]}$ will arise from the series which represent

$$\frac{x^n}{t^{n+1}},\quad \frac{x^{n-2}}{t^{n-1}},\quad \frac{x^{n-4}}{t^{n-3}},\quad \dots,$$

and these terms will give

$$J_{[n]}(x)\,\frac{\{2n\}!}{t^{n+1}}\left\{ 1 - \frac{t^2}{[2n-2][2]} + p^2\,\frac{t^4}{[2n-2][2n-4][2][4]} - \dots \right\}. \tag{39}$$

If $$O_{[n]}(t) = \frac{\{2n\}!}{t^{n+1}}\left\{ 1 + \frac{t^2}{[2][2n-2]} + p^2\,\frac{t^4}{[2][4][2n-2][2n-4]} + \dots \right\}$$

in which the p prefix of the general term is $p^{r(r-1)}$, we may write

$$\frac{1}{t-x} = O_{[0]}(t)J_{[0]}(x) + \sum_{n=1}^{\infty} O_{[n]}(t)J_{[n]}(x) \tag{40}$$

(cf. Heine, *Kugelfunctionen*, Vol. I., p. 249). Thus

$$\begin{aligned} f(x) &= \frac{1}{2\pi i}\int_C \frac{f(t)}{t-x}\,dt \\ &= \frac{1}{2\pi i}\int_C f(t)\left\{O_{[0]}(t)J_{[0]}(x) + \sum_{n=1}^{\infty} O_{[n]}(t)J_{[n]}(x)\right\}dt \\ &= a_0 J_{[0]}(x) + a_1 J_{[1]}(x) + a_2 J_{[2]}(x) + \ldots, \\ a_n &= \frac{1}{2\pi i}\int_C O_{[n]}(t) f(t)\,dt. \end{aligned} \tag{41}$$

In a similar way by means of (77), p. 218, Ser. 2, Vol. 2,

$$f(x) = \sum_{n=0}^{\infty} a_n P_{[n]}(x),$$

$$a_n = \frac{[2n+1]}{2\pi i}\int_C f(t) Q_{[n]}(t)\,dt \quad (|\,t\,| > |\,x\,| > 1). \tag{42}$$

§ 7.

In connection with the theory of the function $E_p(x)$, we may show that

$$J_{[n]}(x) = E_p(ix)E_p(-ix)\mathfrak{J}_{[n]}(x),$$

and derive thence many interesting results. It is easily shown that

$$\begin{aligned} E_{p^{-1}}(-x)\,I_{[n]}(x) = \frac{x^n}{\{2n\}!}\Big\{1 - x + \frac{[2n+3]}{[2][2n+2]}x^2 - \frac{[2n+5]}{[3]!\,[2n+2]}x^3 \\ + \frac{[2n+5][2n+7]}{[4]!\,[2n+2][2n+4]}x^4 - \ldots\Big\}, \end{aligned} \tag{43}$$

$$I_{[n]}(x) = i^{-n}J_{[n]}(ix).$$

Inverting the base p, we obtain

$$E_p(-x)\,\mathrm{I}_{[n]}(x) = \frac{x^n}{\{2n\}!}\left\{1 - x + \frac{[2n+3]}{[2][2n+2]}x^2 - \ldots\right\}, \tag{43a}$$

in which $\qquad \mathrm{I}_{[n]}(x) = i^{-n}\mathfrak{J}_{[n]}(ix),$

because $\frac{x^n}{\{2n\}!}$ becomes $p^{n^2}\frac{x^n}{\{2n\}}$, $I_{[n]}(x)$ becomes $p^{n^2}\mathrm{I}_{[n]}(x)$, and the series within the large brackets of (43) is unchanged by the inversion of p.

Hence
$$E_{p-1}(-x)\, I_{[n]}(x) = \dot{E}_p(-x)\, \mathrm{I}_{[n]}(x), \tag{44}$$

and it easily follows that

$$E_{p-1}(-ix)\, J_{[n]}(x) = E_p(-ix)\, \mathfrak{J}_{[n]}(x), \tag{45}$$

also
$$J_{[n]}(x) = E_p(ix)\, E_p(-ix)\, \mathfrak{J}_{[n]}(x)\,; \tag{46}$$

so that the extension of the series for Lommel's product $J_m(x)\, J_{[n]}(x)$ (*Proc. R.S.*, Vol. LXXIV., p. 67) may be written

$$\begin{aligned} &E_p(ix)\, E_p(-ix)\, \mathfrak{J}_{[m]}\, \mathfrak{J}_{[n]} \\ &\quad = E_{p-1}(ix)\, E_{p-1}(-ix)\, J_{[m]}\, J_{[n]} \\ &\quad = \sum_{r=0}^{\infty} (-1)^r \frac{\{2m+2n+4r\}!}{\{2m+2n+2r\}!\, \{2m+2r\}!\, \{2n+2r\}!\, \{2r\}!}\, x^{m+n+2r}. \end{aligned} \tag{47}$$

It follows also that

$$J_{[m]}\, J_{[n]} = E_p^2(ix)\, E_p^2(-ix)\, \mathfrak{J}_{[m]}\, \mathfrak{J}_{[n]}.$$

At this point we see that interesting equations may be formed giving relations between the functions $J_{[n]}$ and analogous functions which may be conveniently termed multiple Bessel functions. The notation for such a function will be

$$J^p_{m,\,n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2n+2m+4r}}{\{2m+2n+2r\}!\, \{2m+2r\}!\, \{2n+2r\}!\, \{2r\}!}. \tag{48}$$

In the denominator of the general term we have the same expression as appears in the denominator of the general term of (47), which is a generalization of a series due to Lommel.

When m and n are integers the function may be derived from the function $J_{[n]}$ in the same way that $J_{[n]}$ was derived from E (cf. p. 196, Ser. 2, Vol. 2).

Consider the product $\quad J_{[n]}(xt)\, J_{[n]}(xt^{-1})$.

We may arrange in a series of ascending and descending powers of t as

$$J^p_{n,\,0}(x) + \sum_{m=1}^{\infty} (-1)^m (t^{2m} + t^{-2m})\, J^p_{n,\,m}(x). \tag{49}$$

Consider the equations

$$E_{p^2}\left(\frac{xt}{p+1}\right) E_{p^2}\left(-\frac{xt^{-1}}{p+1}\right) = J_{[0]}(x) + \sum_{n=1} (t^n + t^{-n})\, J_{[n]}(x), \tag{50}$$

$$E_{p^{-2}}\left(\frac{xt}{p+1}\right) E_{p^{-2}}\left(-\frac{xt^{-1}}{p+1}\right) = \mathfrak{J}_{[0]}\left(\frac{x}{p}\right) + \Sigma p^{n^2}(t^n + t^{-n})\, \mathfrak{J}_{[n]}\left(\frac{x}{p}\right). \tag{50a}$$

Since $$E_p \mathfrak{J}_{[n]} = E_{p^{-1}} J_{[n]} \tag{51}$$

[cf. (45)], we can write the quotient of the two expressions on the left sides of (50) and (50a) as

$$\frac{J^{p^2}_{[n]}\left(\frac{ixt}{p+1}\right) J^{p^2}_{[n]}\left(-\frac{ixt^{-1}}{p+1}\right)}{\mathfrak{J}^{p^2}_{[n]}\left(\frac{ixt}{p+1}\right) \mathfrak{J}^{p^2}_{[n]}\left(-\frac{ixt^{-1}}{p+1}\right)}, \tag{52}$$

the index p^2 being necessary to indicate that the base of the functions is p^2. This quotient may be expressed in series of $J_{m,n}$ functions by means of result (49), viz., as

$$\frac{J^{p^2}_{n,0}\left(\frac{ix}{[2]}\right) + \sum_{m=1}^{\infty} (t^{2m}+t^{-2m})(-1)^m J_{n,m}\left(\frac{ix}{[2]}\right)}{\mathfrak{J}^{p^2}_{n,0}\left(\frac{ix}{[2]}\right) + \sum_{m=1}^{\infty} (t^{2m}+t^{-2m})(-1)^m \mathfrak{J}_{n,m}\left(\frac{ix}{[2]}\right)}. \tag{53}$$

$\mathfrak{J}_{n,m}$ denotes the function derived from the product $\mathfrak{J}_{[n]}(xt)\,\mathfrak{J}_{[n]}(xt^{-1})$ in the same way that $J_{n,m}$ was derived from $J_{[n]}$.

Equating the above expression to the quotient of the two series which form the right sides of equations (50) and (50a), we obtain from cross multiplication, equating constant terms (independent of t),

$$J_{[0]}(x)\,\mathfrak{J}^{p^2}_{n,0}\left(\frac{ix}{[2]}\right) + 2\sum_{m=1}^{\infty}(-1)^m J_{[2m]}\,\mathfrak{J}^{p^2}_{n,m}$$
$$= \mathfrak{J}_{[0]} J^{p^2}_{n,} + 2\sum_{m=1}^{\infty}(-)^m p^{2m^2}\,\mathfrak{J}_{[2m]} J^{p^2}_{n,m}. \tag{54}$$

I do not propose here to enter any further into the theory of multiple J functions; but I consider that the simple relations

$$\left.\begin{aligned} J_{[n]}(x)\,\mathfrak{J}_{[m]}(x) &= \mathfrak{J}_{[m]}(x)\,J_{[n]}(x) \\ \frac{J_{[n]}(x)}{\mathfrak{J}_{[n]}(x)} &= \frac{E_p(ix)}{E_{p^{-1}}(ix)} \\ E_p(x)\,E_{p^{-1}}(y) &= 1 + \frac{(x+y)}{[1]} + \frac{(x+y)(x+py)}{[2]!} + \dots \end{aligned}\right\} \tag{55}$$

will afford a considerable field for investigation, in forms connecting theta function products with the basic or q forms of Bessel's and Legendre's functions.

Taking the product of (43) and (43a),

$$E_p(-x)\,E_{p^{-1}}(-x)\,I_{[n]}(x)\,I_{[m]}(x)$$

$$=\frac{x^{m+n}}{\{2m\}!\,\{2n\}!}\left\{1-x+\frac{[2m+3]}{[2][2m+2]}x^2-\dots\right\}\left\{1-x+\frac{[2n+3]}{[2][2n+2]}x^2-\dots\right\}. \quad (56)$$

The left side of this may be written

$$\left\{1-\frac{2x}{[1]}+\frac{2(1+p)}{[2]!}x^2-\frac{2(1+p)(1+p^2)}{[3]!}x^3+\dots\right\}$$

$$\times\sum_{r=0}^{\infty}\frac{\{2n+2m+4r\}!}{\{2m+2n+2r\}!\,\{2m+2r\}!\,\{2n+2r\}!\,\{2r\}!}x^{m+n+2r}.$$

Also, since $\qquad E_p(x)E_{p^{-1}}(x)=\dfrac{1}{E_{p^{-1}}(-x)\,E_p(-x)},$

$\sum_{r=0}^{\infty}\dfrac{\{2n+2m+4r\}!}{\{2m+2n+2r\}!\,\{2m+2r\}!\,\{2n+2r\}!\,\{2r\}!}x^{m+n+2r}$ may be expressed as the product of three series, viz.,

$$\frac{x^{m+n}}{\{2m\}!\,\{2n\}!}\left\{1-x+\frac{[2m+3]}{[2][2m+2]}x^2-\dots\right\}\left\{1-x+\frac{[2n+3]}{[2][2n+2]}x^2-\dots\right\}$$

$$\times\left\{1+\frac{2x}{[1]}+\frac{2(1+p)}{[2]!}x^2+\frac{2(1+p)(1+p^2)}{[3]!}x^3+\dots\right\}. \quad (57)$$

By equating coefficients and by substituting ix for x, various identities may be deduced from these series: for example, taking

$$2C(x)=E_p(ix)+E_p(-ix)=1-\frac{x^2}{[2]!}+\frac{x^4}{[4]!}-\dots,$$

$$2iS(x)=E_p(ix)-E_p(-ix)=x-\frac{x^3}{[3]!}+\dots,$$

$$J_{[n]}(x)=\frac{x^nC(x)}{(2)_n\Gamma_p(n+1)}\left\{1-\frac{[2n+3]}{[2n+2]}\frac{x^2}{[2]!}+\dots\right\}$$

$$+\frac{x^nS(x)}{(2)_n\Gamma_p(n+1)}\left\{x-\frac{[2n+5]}{[2n+2]}\frac{x^3}{[3]!}+\dots\right\}. \quad (58)$$

This theorem reduces, when $p=1$, to a well known theorem in Bessel's function theory. I hope that the results given may be interesting not merely because of their remarkable likeness to well known results in the case of ordinary functions J_n, P_n, ..., but because the theory links the functions J_n, P_n, ..., by means of their basic or q generalizations, with the q theory of elliptic functions.

Addition to the preceding paper.

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EXPANSIONS OF FUNCTIONS IN SERIES OF BASIC BESSEL COEFFICIENTS

We may show that, if $f(x)$ be a function capable of expansion in an absolutely convergent series

$$\sum_{0}^{\infty} a_s x^s,$$

then, on replacing each power of x by its expression in an infinite series of $J_{[n]}$ functions, we obtain an infinite series $\sum_{0}^{\infty} b_s J_{[s]}$, and that the absolute convergence of this series depends only on the absolute convergence of $\Sigma a_s x^s$ (cf. Gray and Mathews, *Treatise on Bessel Functions*, pp. 19–20, 29–30; Neumann, *Leipzig Berichte*, 1869). A similar generalization of Neumann's result

$$\sum_{0}^{\infty} a_s x^s = \sum_{0}^{\infty} (b_{2s} J_{[s]}^2 + b_{2s+1} J_s J_{s+1})$$

will be given.

Firstly, we may establish without difficulty the three following identities :—

$$\begin{aligned} \frac{x^n}{\{2n\}!} &= J_{[n]} + \frac{[2n+4]}{[2]} J_{[n+2]} + \dots \\ &+ p^{s(s-1)} [2n+4s] \frac{[2n+2s-2][2n+2s-4] \dots [2n+2]}{[2][4] \dots [2s]} J_{[n+2s]} + \dots \\ &= \Sigma c_s J_{[n+2s]} \end{aligned} \qquad (a)$$

[cf. Gray and Mathews, *Bessel Functions*, (47), p. 19],

$$\begin{aligned} \frac{x^{2n}}{\{2n\}!\,\{2n\}!} &= J_{[n]} \mathfrak{J}_{[n]} + \frac{[4n+4]}{[2]} J_{[n+1]} \mathfrak{J}_{[n+1]} + \dots \\ &+ p^{s(s-1)} [4n+4s] \frac{[4n+2s-2][4n+2s-4] \dots [4n+2]}{[2][4] \dots [2s]} J_{[n+s]} \mathfrak{J}_{[n+s]} + \dots, \end{aligned} \qquad (\beta)$$

$$\begin{aligned} \frac{x^{2n-1}}{\{2n\}!\,\{2n-2\}!} &= J_{[n-1]} \mathfrak{J}_{[n]} + \frac{[4n+2]}{[2]} J_{[n]} \mathfrak{J}_{[n+1]} + \dots \\ &+ p^{s(s-1)} [4n][4n+4s-2] \frac{[4n+2][4n+4] \dots [4n+2s-4]}{[2][4] \dots [2s]} \\ &\times J_{[n+s-1]} \mathfrak{J}_{[n+s]} + \dots \end{aligned} \qquad (\gamma)$$

[cf. Gray and Mathews, *Bessel Functions*, (75), (76), p. 29].

In series (a), if we take out from the J series in each term the part involving x^{n+2r}, we find that the coefficient of x^{n+2r} is a series of $r+1$ terms, which reduces identically to zero by a simple property of Heine's series. Moreover, it may be shown that, if S_h denote the sum of the first h terms of series (a),

$$\{2n\}!\,S_h = x^n + \frac{\{2n+2h\}!}{\{2n+4h+2\}!\,\{2h+2\}!}\,x^{n+2h+2} \times\left[1-\frac{x^2}{[2h+4][2n+2h+4][2]}+\dots\right].$$

The series within the large brackets on the right is absolutely convergent for all values of x if $p>1$, and for limited values of x if $p=1$ or $p<1$. In case $p>1$ or $p=1$ the right side may be made as near to x^n as we please by taking h large enough. The series $\sum_{s=h+1}^{\infty} c_s J_{[n+2s]}$ becomes infinitesimal by taking h infinitely large. The relation

$$x^n = \{2n\}!\sum_{0}^{\infty} c_s J_{[n+2s]}$$

holds then for all finite values of x if $p \geqslant 1$.

In series (a), if we invert the base p, we find (since $\{2n\}!$ becomes $p^{-n^2}\{2n\}!$ and $J_{[n]}$ becomes $p^{n^2}\mathfrak{J}_{[n]}$) that

$$\frac{x^n}{\{2n\}!} = \mathfrak{J}_{[n]} + p^{2n+2}\frac{[2n+4]}{[2]}\mathfrak{J}_{[n+2]}+\dots$$
$$+p^{3(3s-1)+n(n+2s)}[2n+4s]\frac{[2n+2s-2]\dots[2n+2]}{\{2s\}!}\mathfrak{J}_{[n+2s]}+\dots$$
$$= \Sigma\, c_s\,\mathfrak{J}_{[n+2s]}.$$

In case $p<1$ this series is absolutely convergent.

Now suppose $\qquad f(x) = a_0+a_1x+a_2x^2+\dots$

an absolutely convergent series; then, on substituting for each power of x its expression in J^p functions ($p>1$) or $\mathfrak{J}^p$ functions ($p<1$), we obtain in the case of the J functions a series $\sum_{0}^{\infty} b_s J_{[s]}$, where

$$b_s = \{2s\}!\left[a_s+\frac{a_{s-2}}{[2s-2][2]}+\frac{a_{s-4}}{[2s-2][2s-4][2][4]}+\dots\right].$$

If the series are both absolutely convergent, we may put

$$\Sigma\, a_s x^s = \Sigma\, b_s J_{[s]}.$$

An important special case is when the series $\Sigma a_s x^s$ satisfies Cauchy's first test of convergence,

$$a_s x / a_{s-1} < k,$$

where k is a definitely assigned proper fraction.

In this case

$$L \frac{b_s J_{[s]}}{b_{s-1} J_{[s-1]}}$$

$$= L \frac{\{2s\}!\left[a_s + \frac{1}{[2s-2]} a_{s-2} + \dots\right]\left\{\frac{x^s}{\{2s\}!} - \frac{x^{s+2}}{\{2s+2\}!\,\{2\}!} + \dots\right\}}{\{2s-2\}!\left[a_{s-1} + \frac{1}{[2s-4]} a_{s-3} + \dots\right]\left\{\frac{x^{s-1}}{\{2s-2\}!} - \frac{x^{s+2}}{\{2s\}!\,\{2\}!} + \dots\right\}}$$

$$= \frac{a_s}{a_{s-1}} x,$$

and the series $\Sigma b_s J_{[s]}$ is absolutely convergent.

Similar arguments will suffice to show that we may expand $f(x)$ in a series

$$\Sigma b_s \mathfrak{J}_{[s]} \quad (p < 1).$$

In the same manner, if in series (β) and (γ) we replace each term on the right sides of these equations by a series of powers of x of the form

$$J_{[n]} \mathfrak{J}_{[m]} = \sum_{r=0}^{\infty} (-1)^r \frac{\{2m+2n+4r\}!}{\{2m+2n+2r\}!\,\{2m+2r\}!\,\{2n+2r\}!\,\{2r\}!} x^{m+n+2r},$$

the terms involving x^{2n+2r} in (β) are $r+1$ in number, and the sum reduces identically to zero by a property of Heine's series, and similar arguments to those used above would suffice to give us a generalization of Neumann's expansion

$$\Sigma a_s x^s = \Sigma (b_{2s} J_s^2 + b_{2s+1} J_s J_{s+1}).$$

I do not, however, discuss this case, my aim being to justify in one case the expansion (41) of the previous section.