OBITUARY NOTICES

HIERONYMUS GEORG ZEUTHEN.

By the death of Prof. Hieronymus Georg Zeuthen, on January 6th, 1920, in the eighty-first year of his age, the Society has lost one of the oldest of its honorary members—a member of forty-five years standing—for it was in January 1875 that Drs. Klein, Kronecker, and Zeuthen were elected foreign members of the Society. The writer of this notice had not the privilege of personal acquaintance with Prof. Zeuthen, and wishes gratefully to acknowledge his obligation to the kindness of Prof. C. Juel, of Copenhagen, who has allowed him to quote from the memoir of Prof. Zeuthen which was read before the Royal Danish Academy of Science, and has also supplied a list of Prof. Zeuthen's publications.

Zeuthen was born in Jutland in 1839, and entered the University of Copenhagen as a student in 1857. His earliest productions were papers contributed to the Danish *Tidsskrift for Matematik*, which was founded in 1859; these were written during his student days or the years immediately following. His first work of importance was his dissertation for his Doctorate. He had gone, in 1863, to Paris to study under Chasles, the mathematician who undoubtedly exerted a greater influence upon him than any other. Chasles is the founder of Enumerative Geometry and of the Theory of Characteristics, and it was in these subjects that Zeuthen's powers first revealed themselves. His earliest work in this field was his Doctor's Thesis of 1865, translated and published in the *Nouvelles Annales de Mathématiques* in 1866, with the title "A New Method of Determining the Characteristics of Systems of Conics"—a work whose merit was immediately recognised. Zeuthen next studied surfaces of the second order and determined the characteristics in the elementary systems of such surfaces. It may be mentioned that, on learning that Chasles was writing on the same subject, Zeuthen withheld his results from publication, sending them in a closed envelope to the Danish Academy of Science, with the instructions that it should not be opened until after the publication of Chasles' treatise. Continuing investigations of a similar kind Zeuthen produced in 1873 his comprehensive "General Properties of Systems of Plane Curves with Application to determine Characteristics in the Elementary Systems of the Fourth Order." The subject is one which has never attracted so much attention in this country as it has abroad. As a type of the results which Zeuthen obtained we may extract an example from Chap. XV of Pascal's *Repertorio di Mathematiche Superiori*, Vol. II, 1900. Since nine conditions determine a plane cubic curve, it was to be expected that a finite number of such curves will pass through r given points and touch $9-r$ given lines. Zeuthen determined the number of such curves, corresponding to values 9, 8, 7, ..., 0 of r, viz. 1, 4, 16, 64, 256, 976, 3424, 9766, 21004, 33616.

A variety of such results will be found quoted in this chapter; reference should also be made to Zeuthen's article "Abzählende Methoden," in the *Encyklopädie der Mathematischen Wissenschaften*, Bd. III, Heft 2 (1906), pp. 257–312. For quadric surfaces, determined by nine conditions of passing through certain points, touching certain planes, and touching certain lines, thirty separate cases have to be considered and the numbers of solutions in various cases range from 1 up to 128.

Other important works by Zeuthen of this period bear testimony to the brilliance of his powers. They include several which deal with the genus (or deficiency) of algebraic curves and allied matters. There is his beautiful geometrical proof, *Comptes Rendus*, Vol. 70 (1870), p. 743, that the genera of two curves whose points are in (1, 1)-correspondence must be equal (a theorem already proved by Riemann from consideration of Riemann surfaces, and algebraically by Clebsch and Gordan). Zeuthen's proof was obtained independently of a very similar proof published a few months earlier by Bertini [*Giorn. di Mat., Battaglini*, Vol. 7 (1869), p. 105]. The method of proof is as follows. If a moving point M of a given curve C and a moving point M_1 of a second given curve C_1 are in (1, 1)-correspondence, the intersection of the lines AM and A_1M_1 joining M and M_1 to two fixed points A and A_1 traces an algebraic curve; and by considering the class of this curve as calculated from the number of tangents to it from A and A_1, respectively, the theorem that the genus of C is equal to that of C_1 follows at once. It would be difficult to devise any proof more simple and fundamental than this. Zeuthen proceeded to use his method to extend the theorem to cases where the points of two curves are in multiple correspondence, and so established what is known as "Zeuthen's extended theorem upon genus". Here Zeuthen's geometrical method led to a result which had not previously been recognised, although it was remarked later that the theorem could be obtained from the classical theory. It is therefore fitting that the theorem should bear Zeuthen's name. Continuing to work in the same field Zeuthen applied the principles of correspondence to solid geometry, and in 1871 discovered a number which is invariant in any point for point transformation of one algebraic surface into another. The value of this discovery was not, and in fact could not be, recognised at the time: but more than twenty years later, when such properties of surfaces were investigated by more modern methods by the Italian school of mathematicians, the invariant was re-discovered in 1895 by Segre, and now is known as the Zeuthen-Segre invariant of the surface. See Prof. H. F. Baker's Presidential Address to this Society (1912), *Proc. London Math. Soc.*, Vol. 12, p. 33, or *Encyklopädie*, Vol. III, Cap. 6, *b.*, p. 701.

Space will not permit more than a mention of Zeuthen's work upon cubic surfaces, or of his contribution to Vol. 10, Ser. 1, of the *Proceedings* of this Society in 1879. An arresting paper is that entitled "Sur les différentes formes de courbes planes du quatrième ordre" (*Math. Annalen*, Vol. 7, pp. 410–432), in which Zeuthen first examines the distinction (pointed out by v. Staudt) between the odd and even branches of a curve, and proves in a very simple manner the theorems concerning the intersections of two branches. He then shows that of the twenty-eight double tangents of a curve without nodes there are always four which are real, and either touch the *same* branch twice or are isolated, *i.e.* are real lines having two imaginary contacts with the curve; the eight points of contact of these four double tangents lie on a conic. All other real double tangents touch two different branches, each two branches external to one another necessarily giving rise to four double tangents. It is hardly to be doubted that this paper largely inspired the striking discoveries of Klein and Harnack, published in two famous papers in Vol. 10 of the *Math. Annalen*; and Klein's results as to the form of cubic surfaces are closely connected with it.

From about 1880 onwards Zeuthen's interests turned more and more towards the history of mathematics, chiefly, but by no means wholly, in classical times. He had published a short paper in 1876 on "Brahmegupta's trapeziums", but from 1880 he found a richer field for study in tracing the development of Greek mathematics. Thus it is probable that the name of Zeuthen is better known at the present day as a historian of mathematics than as an original discoverer in the subject. We will not here attempt to give a detailed account of his many writings upon Archimedes, Euclid, Apollonius, Diophantus, &c., or of those dealing with the later times of Descartes, Cardan, Fermat, Newton, Barrow. His most important historical work, *Die Lehre von den Kegelschnitten in Altertum*, was published in 1886.

To the end of his life Zeuthen continued to publish papers on mathematical, and for the most part geometrical or historical, subjects. In the year 1919 (a year before his death) he published two papers—one on the origin of Algebra, and the other on the explanation of a paradox in Enumerative Geometry.

Almost the whole of Zeuthen's life was passed in Copenhagen, where he was for many years Professor at the University. The number of Zeuthen's publications amounts to nearly two hundred, and include besides the numerous articles in various periodicals, elementary textbooks, textbooks for students at the University or Polytechnic, papers read at various International Congresses in mathematics or philosophy, and (in addition to the history of Conic Sections already referred to) a *History of Mathematics* (1883), and a *History of Mathematics in the Sixteenth and Seventeenth Centuries* (1903). Until the last year of his life he was Secretary to the Danish Academy of Science.

H. W. R.

(6) If $v = \frac{x}{1+}\,\frac{x^3+x^6}{1\;+}\,\frac{x^6+x^{12}}{1\;+}\,\frac{x^9+x^{18}}{1\;+}\,\&c.$

then

$$\text{i.}\quad x\left(1+\frac{1}{v}\right) = \frac{1+x+x^3+x^6+x^{10}+\&c}{1+x^9+x^{27}+x^{54}+x^{90}+\&c}$$

$$\text{ii.}\quad x^3\left(1+\frac{1}{v^3}\right) = \left(\frac{1+x+x^3+x^6+x^{10}+\&c}{1+x^3+x^9+x^{18}+x^{30}+\&c}\right)^4$$

(7) If n is any odd integer,

$$\frac{1}{\cosh\frac{\pi}{2n}+\cos\frac{\pi}{2n}} - \frac{1}{3\left(\cosh\frac{3\pi}{2n}+\cos\frac{3\pi}{2n}\right)} + \frac{1}{5\left(\cosh\frac{5\pi}{2n}+\cos\frac{5\pi}{2n}\right)} \dots \&c = \frac{\pi}{8}.$$

. . .

(10) If $F(\alpha, \beta, \gamma, \delta, \epsilon) = 1 + \frac{\alpha}{\lfloor 1} \cdot \frac{\beta}{\delta} \cdot \frac{\gamma}{\epsilon} + \frac{\alpha(\alpha+1)}{\lfloor 2} \cdot \frac{\beta(\beta+1)}{\delta(\delta+1)} \times \frac{\gamma(\gamma+1)}{\epsilon(\epsilon+1)} + \&c.$

then

$$F(\alpha, \beta, \gamma, \delta, \epsilon) = \frac{\Gamma(\delta)\,\Gamma(\delta-\alpha-\beta)}{\Gamma(\delta-\alpha)\,\Gamma(\delta-\beta)} \cdot F(\alpha, \beta, \epsilon-\gamma, \alpha+\beta-\delta+1, \epsilon)$$
$$+ \frac{\Gamma(\delta)\,\Gamma(\epsilon)\,\Gamma(\alpha+\beta-\delta)\,\Gamma(\delta+\epsilon-\alpha-\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\epsilon-\gamma)\,\Gamma(\delta+\epsilon-\alpha-\beta)}$$
$$\times F(\delta-\alpha, \delta-\beta, \delta+\epsilon-\alpha-\beta-\gamma, \delta-\alpha-\beta+1, \delta+\epsilon-\alpha-\beta).$$

. . .

(13) $\frac{a}{1+n+}\,\frac{a^2}{3+n+}\,\frac{(2a)^2}{5+n+}\,\frac{(3a)^2}{7+n+}\dots$

$$= 2a\int_0^1 z^{\frac{n}{\sqrt{(1+a^2)}}}\,\frac{dz}{\{\sqrt{(1+a^2)}+1\}+z^2\{\sqrt{(1+a^2)}-1\}}.$$

(14) If $F(\alpha, \beta) = \alpha + \frac{(1+\beta)^2+k}{2\alpha\;+}\,\frac{(3+\beta)^2+k}{2\alpha\;+}\,\frac{(5+\beta)^2+k}{2\alpha\;+}\dots,$

then

$$F(\alpha, \beta) = F(\beta, \alpha).$$

(15) If $F(\alpha, \beta) = \frac{\alpha}{n+}\,\frac{\beta^2}{n+}\,\frac{(2\alpha)^2}{n+}\,\frac{(3\beta)^2}{n+}\dots$

then

$$F(\alpha, \beta) + F(\beta, \alpha) = 2F\{\tfrac{1}{2}(\alpha+\beta), \sqrt{(\alpha\beta)}\}$$

. . .

(17) If $F(k) = 1 + \left(\frac{1}{2}\right)^2 k + \left(\frac{1.3}{2.4}\right)^2 k^2 + \dots$ and $F(1-k) = \sqrt{(210)}\,F(k)$,

then $k = (\sqrt{2}-1)^4(2-\sqrt{3})^2(\sqrt{7}-\sqrt{6})^4(8-3\sqrt{7})^2(\sqrt{10}-3)^4(4-\sqrt{15})^4$
$\times(\sqrt{15}-\sqrt{14})^2(6-\sqrt{35})^2.$

. . . .

(20) If $F(\alpha) = \int_0^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt{\{1-(1-\alpha)\sin^2\phi\}}} \Big/ \int_0^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt{\{1-\alpha\sin^2\phi\}}}$

and $F(\alpha) = 3F(\beta) = 5F(\gamma) = 15F(\delta),$

then i. $[(\alpha\delta)^{\frac{1}{2}}+\{(1-\alpha)(1-\delta)\}^{\frac{1}{2}}][(\beta\gamma)^{\frac{1}{2}}+\{(1-\beta)(1-\gamma)\}^{\frac{1}{2}}] = 1$

. . .

v. $(\alpha\beta\gamma\delta)^{\frac{1}{2}}+\{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac{1}{2}}$
$+\{16\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac{1}{6}} = 1$

. . .

(21) If $F(\alpha) = 3F(\beta) = 13F(\gamma) = 39F(\delta)$

or $F(\alpha) = 5F(\beta) = 11F(\gamma) = 55F(\delta)$

or $F(\alpha) = 7F(\beta) = 9F(\gamma) = 63F(\delta)$

then $$\frac{\{(1-\alpha)(1-\delta)\}^{\frac{1}{4}}-(\alpha\delta)^{\frac{1}{4}}}{\{(1-\beta)(1-\gamma)\}^{\frac{1}{4}}-(\beta\gamma)^{\frac{1}{4}}} = \frac{1+\{(1-\alpha)(1-\delta)\}^{\frac{1}{4}}+(\alpha\delta)^{\frac{1}{4}}}{1+\{(1-\beta)(1-\gamma)\}^{\frac{1}{4}}+(\beta\gamma)^{\frac{1}{4}}}$$

. . .

(23) $(1+e^{-\pi\sqrt{1353}})(1+e^{-3\pi\sqrt{1353}})(1+e^{-5\pi\sqrt{1353}})\dots$

$$= \sqrt[4]{2}\, e^{-\frac{1}{24}\pi\sqrt{1353}} \times \sqrt{\left\{\sqrt{\left(\frac{569+99\sqrt{33}}{8}\right)}+\sqrt{\left(\frac{561+99\sqrt{33}}{8}\right)}\right\}}$$
$$\times\sqrt{\left\{\sqrt{\left(\frac{25+3\sqrt{33}}{8}\right)}+\sqrt{\left(\frac{17+3\sqrt{33}}{8}\right)}\right\}} \times \sqrt[4]{\left(\frac{\sqrt{128}+11}{\sqrt{2}}\right)}$$
$$\times \sqrt[8]{(10+3\sqrt{11})} \times \sqrt[8]{(26+15\sqrt{3})} \times \sqrt[12]{\left(\frac{6817+321\sqrt{451}}{\sqrt{2}}\right)}$$

17 April 1913

". . . I am a little pained to see what you have written. . . .* I am not in the least apprehensive of my method being utilized by others. On the contrary my method has been in my possession for the last eight years and I have not found anyone to appreciate the method. As I wrote in my last letter I have found a sympathetic friend in you and I am willing to place unreservedly in your hands what little I have. It was on

* Ramanujan might very reasonably have been reluctant to give away his secrets to an English mathematician, and I had tried to reassure him on this point as well as I could.

account of the novelty of the method I have used that I am a little diffident even now to communicate my own way of arriving at the expressions I have already given. . . .

. . . I am glad to inform you that the local University has been pleased to grant me a scholarship of £60 per annum for two years and this was at the instance of Dr. Walker, F.R.S., Head of the Meteorological Department in India, to whom my thanks are due. . . . I request you to convey my thanks also to Mr. Littlewood, Dr Barnes, Mr. Berry and others who take an interest in me. . . ."

III.

It is unnecessary to repeat the story of how Ramanujan was brought to England. There were serious difficulties; and the credit for overcoming them is due primarily to Prof. E. H. Neville, in whose company Ramanujan arrived in April 1914. He had a scholarship from Madras of £250, of which £50 was allotted to the support of his family in India, and an exhibition of £60 from Trinity. For a man of his almost ludicrously simple tastes, this was an ample income: and he was able to save a good deal of money which was badly wanted later. He had no duties and could do as he pleased; he wished indeed to qualify for a Cambridge degree as a research student, but this was a formality. He was now, for the first time in his life, in a really comfortable position, and could devote himself to his researches without anxiety.

There was one great puzzle. What was to be done in the way of teaching him modern mathematics? The limitations of his knowledge were as startling as its profundity. Here was a man who could work out modular equations, and theorems of complex multiplication, to orders unheard of, whose mastery of continued fractions was, on the formal side at any rate, beyond that of any mathematician in the world, who had found for himself the functional equation of the Zeta-function, and the dominant terms of many of the most famous problems in the analytic theory of numbers; and he had never heard of a doubly periodic function or of Cauchy's theorem, and had indeed but the vaguest idea of what a function of a complex variable was. His ideas as to what constituted a mathematical proof were of the most shadowy description. All his results, new or old, right or wrong, had been arrived at by a process of mingled argument, intuition, and induction, of which he was entirely unable to give any coherent account.

It was impossible to ask such a man to submit to systematic instruction, to try to learn mathematics from the beginning once more. I was afraid too that, if I insisted unduly on matters which Ramanujan found irksome, I might destroy his confidence or break the spell of his inspiration. On the other hand there were things of which it was impossible that he should remain in ignorance. Some of his results were wrong, and in particular those which concerned the distribution of primes, to which he attached the greatest importance. It was impossible to allow him to go through life supposing that all the zeros of the Zeta-function were real. So I had to try to teach him, and in a measure I succeeded, though obviously I learnt from him much more than he learnt from me. In a few years' time he had a very tolerable knowledge of the theory of functions and the analytic theory of numbers. He was never a mathematician of the modern school, and it was hardly desirable that he should become one; but he knew when he had proved a theorem and when he had not. And his flow of original ideas showed no symptom of abatement.

I should add a word here about Ramanujan's interests outside mathematics. Like his mathematics, they showed the strangest contrasts. He had very little interest, I should say, in literature as such, or in art, though he could tell good literature from bad. On the other hand, he was a keen philosopher, of what appeared, to followers of the modern Cambridge school, a rather nebulous kind, and an ardent politician, of a pacifist and ultra-radical type. He adhered, with a severity most unusual in Indians resident in England, to the religious observances of his caste; but his religion was a matter of observance and not of intellectual conviction, and I remember well his telling me (much to my surprise) that all religions seemed to him more or less equally true. Alike in literature, philosophy, and mathematics, he had a passion for what was unexpected, strange, and odd; he had quite a small library of books by circle-squarers and other cranks.

It was in the spring of 1917 that Ramanujan first appeared to be unwell. He went into the Nursing Home at Cambridge in the early summer, and was never out of bed for any length of time again. He was in sanatoria at Wells, at Matlock, and in London, and it was not until the autumn of 1918 that he showed any decided symptom of improvement. He had then resumed active work, stimulated perhaps by his election to the Royal Society, and some of his most beautiful theorems were discovered about this time. His election to a Trinity Fellowship was a further encouragement; and each of those famous societies may well congratulate themselves that they recognised his claims before it was too late. Early in 1919 he had recovered, it seemed, sufficiently for the voyage home to India, and the best medical opinion held out hopes of a permanent restoration. I was rather alarmed by not hearing from him for a considerable time; but a letter reached me in February 1920, from which it appeared that he was still active in research.

University of Madras
12th January 1920

"I am extremely sorry for not writing you a single letter up to now. . . . I discovered very interesting functions recently which I call 'Mock' ϑ-functions. Unlike the 'False' ϑ-functions (studied partially by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as the ordinary ϑ-functions. I am sending you with this letter some examples. . . .

Mock ϑ-functions

$$\phi(q) = 1 + \frac{q}{1+q^2} + \frac{q^4}{(1+q^2)(1+q^4)} + \dots$$

$$\psi(q) = \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^3)} + \frac{q^9}{(1-q)(1-q^3)(1-q^5)} + \dots$$

. . .

Mock ϑ-functions (of 5th order)

$$f(q) = 1 + \frac{q}{1+q} + \frac{q^4}{(1+q)(1+q^2)} + \frac{q^9}{(1+q)(1+q)(1+q^3)} + \dots$$

. . .

Mock ϑ-functions (of 7th order)

$$\text{(i)} \quad 1 + \frac{q}{1-q^2} + \frac{q^4}{(1-q^3)(1-q^4)} + \frac{q^9}{(1-q^4)(1-q^5)(1-q^6)} + \dots$$

. . ."

He said little about his health, and what he said was not particularly discouraging; and I was quite unprepared for the news of his death.

IV.

Ramanujan published the following papers in Europe:—

(1) "Some definite integrals", *Messenger of Mathematics*, Vol. 44 (1914), pp. 10-18.
(2) "Some definite integrals connected with Gauss's sums", *ibid.*, pp. 75-85.
(3) "Modular equations and approximations to π", *Quarterly Journal of Mathematics*, Vol. 45 (1914), pp. 350-372.
(4) "New expressions for Riemann's functions ζ(s) and Ξ(t)", *ibid.*, Vol. 46 (1915) pp. 253-261.
(5) "On certain infinite series". *Messenger of Mathematics*, Vol. 45 (1915), pp. 11-15.
(6) "Summation of a certain series", *ibid.*, pp. 157-160.
(7) "Highly composite numbers", *Proc. London Math. Soc.*, Ser. 2, Vol. 14 (1915) pp. 347-409.

(8) "Some formulæ in the analytic theory of numbers", *Messenger of Mathematics*, Vol. 45 (1916), pp. 81–84.
(9) "On certain arithmetical functions", *Trans. Cambridge Phil. Soc.*, Vol. 22 (1916), No. 9, pp. 159–184.
(10) "Some series for Euler's constant", *Messenger of Mathematics*, Vol. 46 (1916), pp. 73–80.
(11) "On the expression of numbers in the form $ax^2+by^2+cz^2+dt^2$", *Proc. Cambridge Phil. Soc.*, Vol. 19 (1917), pp. 11–21.
*(12) "Une formule asymptotique pour le nombre des partitions de n", *Comptes Rendus*, 2 Jan. 1917.
*(13) "Asymptotic formulæ concerning the distribution of integers of various types", *Proc. London Math. Soc.*, Ser. 2, Vol. 16 (1917), pp. 112–132.
*(14) "The normal number of prime factors of a number n", *Quarterly Journal of Mathematics*, Vol. 48 (1917), pp. 76–92.
*(15) "Asymptotic formulæ in Combinatory Analysis", *Proc. London Math. Soc.*, Ser. 2, Vol. 17 (1918), pp. 75–115.
*(16) "On the coefficients in the expansions of certain modular functions", *Proc. Roy. Soc.*, (A), Vol. 95 (1918), pp. 144–155.
(17) "On certain trigonometrical sums and their applications in the theory of numbers", *Trans. Camb. Phil. Soc.*, Vol. 22 (1918), pp. 259–276.
(18) "Some properties of $p(n)$, the number of partitions of n", *Proc. Camb. Phil. Soc.*, Vol. 19 (1919), pp. 207–210.
(19) "Proof of certain identities in Combinatory Analysis", *ibid.*, pp. 214–216.
(20) "A class of definite integrals", *Quarterly Journal of Mathematics*, Vol. 48 (1920), pp. 294–309.
(21) "Congruence properties of partitions", *Math. Zeitschrift*, Vol. 9 (1921), pp. 147–153.

Of these those marked with an asterisk were written in collaboration with me, and (21) is a posthumous extract from a much larger unpublished manuscript in my possession.† He also published a number of short notes in the *Records of Proceedings* at our meetings, and in the *Journal of the Indian Mathematical Society*. The complete list of these is as follows:

Records of Proceedings at Meetings.

*(22) "Proof that almost all numbers n are composed of about $\log\log n$ prime factors", 14 Dec. 1916.
*(23) "Asymptotic formulæ in Combinatory Analysis", 1 March, 1917.
(24) "Some definite integrals", 17 Jan., 1918.
(25) "Congruence properties of partitions", 13 March, 1919.
(26) "Algebraic relations between certain infinite products", 13 March, 1919.

Journal of the Indian Mathematical Society.

(A) Articles and Notes.

(27) "Some properties of Bernoulli's numbers", Vol. 3 (1911), pp. 219–235.
(28) "On Q. 330 of Prof. Sanjana", Vol. 4 (1912), pp. 59–61.
(29) "A set of equations" Vol. 4 (1912), pp. 94–96.

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† All of Ramanujan's manuscripts passed through my hands, and I edited them very carefully for publication. The earlier ones I rewrote completely. I had no share of any kind in the results, except of course when I was actually a collaborator, or when explicit acknowledgment is made. Ramanujan was almost absurdly scrupulous in his desire to acknowledge the slightest help.

(30) "Irregular numbers", Vol. 5 (1913), pp. 105–107.
(31) "Squaring the circle", Vol. 5 (1913), pp. 132–133.
(32) "On the integral $\int_0^x \arctan t . \frac{dt}{t}$", Vol. 7 (1915), pp. 93–96.
(33) "On the divisors of a number", Vol. 7 (1915), pp. 131–134.
(34) "The sum of the square roots of the first n natural numbers", Vol. 7 (1915), pp. 173–175.
(35) "On the product $\pi\left[1+\frac{x^2}{(a+nd)^2}\right]$", Vol. 7 (1915), pp. 209–212.
(36) "Some definite integrals", Vol. 11 (1919), pp. 81–88.
(37) "A proof of Bertrand's postulate", Vol. 11 (1919), pp. 181–183.
(38) (Communicated by S. Narayana Aiyar), Vol. 3 (1911), p. 60.

(B) Questions proposed and solved.

Nos. 260, 261, 283, 289, 294, 295, 298, 308, 353, 358, 386, 427, 441, 464, 489, 507, 541, 546, 571, 605, 606, 629, 642, 666, 682, 700, 723, 724, 739, 740, 753, 768, 769, 783, 785.

(C) Questions proposed but not solved as yet.

Nos. 284, 327, 359, 387, 441, 463, 469, 524, 525, 526, 584, 661, 662, 681, 699, 722, 738, 754, 770, 784, 1049, 1070, and 1076.

Finally, I may mention the following writings by other authors, concerned with Ramanujan's work.

"Proof of a formula of Mr. Ramanujan", by G. H. Hardy (*Messenger of Mathematics*, Vol. 44, 1915, pp. 18–21).
"Mr. S. Ramanujan's mathematical work in England", by G. H. Hardy (Report to the University of Madras, 1916, privately printed).
"On Mr. Ramanujan's empirical expansions of modular functions", by L. J. Mordell (*Proc. Camb. Phil. Soc.*, Vol. 19, 1917, pp. 117–124).
"Life sketch of Ramanujan" (editorial in the *Journal of the Indian Math. Soc.*, Vol. 11, 1919, p. 122).
"Note on the parity of the number which enumerates the partitions of a number", by P. A. MacMahon (*Proc. Camb. Phil. Soc.*, Vol. 20, 1921, pp. 281–283).
"Proof of certain identities and congruences enunciated by S. Ramanujan", by H. B. C. Darling (*Proc. London Math. Soc.*, Ser. 2, Vol. 19, 1921, pp. 350–372).
"On a type of modular relation", by L. J. Rogers (*ibid.*, pp. 387–397).

It is plainly impossible for me, within the limits of a notice such as this, to attempt a reasoned estimate of Ramanujan's work. Some of it is very intimately connected with my own, and my verdict could not be impartial; there is much too that I am hardly competent to judge; and there is a mass of unpublished material, in part new and in part anticipated, in part proved and in part only conjectured, that still awaits analysis. But it may be useful if I state, shortly and dogmatically, what seems to me Ramanujan's finest, most independent, and most characteristic work.

His most remarkable papers appear to me to be (3), (7), (9), (17), (18), (19), and (21). The first of these is mainly Indian work, done before he came to England; and much of it had been anticipated. But there is much that is new, and in particular a very remarkable series of algebraic approximations to π. I may mention only the formulæ

$$\pi = \frac{63}{25}\,\frac{17+15\sqrt{5}}{7+15\sqrt{5}}, \qquad \frac{1}{2\pi\sqrt{2}} = \frac{1103}{99^2},$$

correct to 9 and 8 places of decimals respectively.

The long memoir (7) represents work, perhaps, in a backwater of mathematics, and is somewhat overloaded with detail; but the elementary analysis of "highly composite" numbers—numbers which have more divisors than any preceding number—is exceedingly remarkable, and shows very clearly Ramanujan's extraordinary mastery over the algebra of inequalities. Papers (9) and (17) should be read together, and in connection with Mr. Mordell's paper mentioned above; for Mr. Mordell afterwards proved a great deal that Ramanujan conjectured. They contain, in particular, exceedingly remarkable contributions to the theory of the representation of numbers by sums of squares. But I am inclined to think that it was in the theory of partitions, and the allied parts of the theories of elliptic functions and continued fractions, that Ramanujan shows at his very best. It is in papers (18), (19), and (21), and in the papers of Prof. Rogers and Mr. Darling that I have quoted, that this side of his work (so far as it has been published) is to be found. It would be difficult to find more beautiful formulæ than the "Rogers-Ramanujan" identities, proved in (19); but here Ramanujan must take second place to Prof. Rogers; and, if I had to select one formula from all Ramanujan's work, I would agree with Major MacMahon in selecting a formula from (18), viz.

$$p(4)+p(9)\,x+p(14)\,x^2+\ldots = 5\,\frac{\{(1-x^5)(1-x^{10})(1-x^{15})\ldots\}^5}{\{(1-x)(1-x^2)(1-x^3)\ldots\}^6},$$

where $p(n)$ is the number of partitions of n.

I have often been asked whether Ramanujan had any special secret; whether his methods differed in kind from those of other mathematicians; whether there was anything really abnormal in his mode of thought. I cannot answer these questions with any confidence or conviction; but I do not believe it. My belief is that all mathematicians think, at bottom, in the same kind of way, and that Ramanujan was no exception. He had, of course, an extraordinary memory. He could remember the idiosyncrasies of numbers in an almost uncanny way. It was Mr. Littlewood (I believe) who remarked that "every positive integer was one of his personal friends." I remember once going to see him when he was lying ill at Putney. I had ridden in taxi-cab No. 1729, and remarked that the number (7.13.19) seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as a sum of two cubes in two different ways." I asked him, naturally, whether he knew the answer to the corresponding problem for fourth powers; and he replied, after a moment's thought, that he could see no obvious example, and thought that the first such number must be very large.* His memory, and his powers of calculation, were very unusual, but they could not reasonably be called "abnormal". If he had to multiply two large numbers, he multiplied them in the ordinary way; he would do it with unusual rapidity and accuracy, but not more rapidly or more accurately than any mathematician who is naturally quick and has the habit of computation. There is a table of partitions at the end of our paper (15). This was, for the most part, calculated independently by Ramanujan and Major MacMahon; and Major MacMahon was, in general, slightly the quicker and more accurate of the two.

It was his insight into algebraical formulæ, transformations of infinite series, and so forth, that was most amazing. On this side most certainly I have never met his equal, and I can compare him only with Euler or Jacobi. He worked, far more than the majority of modern mathematicians, by induction from numerical examples; all of his congruence properties of partitions, for example, were discovered in this way. But with his memory, his patience, and his power of calculation, he combined a power of generalisation, a feeling for form, and a capacity for rapid modification of his hypotheses, that was often really startling, and made him, in his own peculiar field, without a rival in his day.

It is often said that it is much more difficult now for a mathematician to be original than it was in the great days when the foundations of modern analysis were laid; and no doubt in a measure it is true. Opinions may differ as to the importance of Ramanujan's work, the kind of standard by which it should be judged, and the influence which it is likely to have on the mathematics of the future. It has not the simplicity and the inevitableness of the very greatest work; it would be greater if it were less strange. One gift it has which no one can deny, profound and invincible originality. He would probably have been a greater mathematician if he had been caught and tamed a little in his youth; he would have discovered more that was new, and that, no doubt, of greater importance. On the other hand he would have been less of a Ramanujan, and more of a European professor, and the loss might have been greater than the gain.

G. H. H.

* Euler gave $542^4 + 103^4 = 359^4 + 514^4$ as an example. See Sir T. L. Heath's *Diophantus of Alexandria*, p. 380.

PHILIP EDWARD BERTRAND JOURDAIN.

(Born October 16th, 1879; Died October 1st, 1919.)

THE death in 1919 of Philip Edward Bertrand Jourdain is a loss that will be widely felt by those who knew his work, and a cause of sincere grief to his many friends. Jourdain, in spite of severe disabilities, accomplished many things in his short life. At a very early age he showed mechanical and mathematical ability; and he went up to Trinity College, Cambridge in 1898, although he was already a cripple. His academic career shows (as is not unnatural) the strangest contrasts. He was ploughed in the Mathematical Tripos, and compelled to take a Pass Degree. He was honourably mentioned in the ensuing Smith's Prize competition, and in 1904 he was awarded the Allen studentship for research.

Apart from his own personal contributions to mathematics, Jourdain was an important figure in mathematical circles. His disinterested and efficient work in abstracting mathematical papers for the *Revue Semestrielle*, and in writing the "Recent Advances" in *Science Progress*, are examples of his labours for the advancement of mathematics. His extensive correspondence on mathematical subjects with eminent mathematicians of all nationalities shows that he was in touch with mathematical thought all over the world. The plans which he was lately elaborating for the advancement of science were to ensure the translation of all scientific papers and articles into English and French. As another example of his activities, we may refer to his attempts to republish the works of Newton. Jourdain was recognised as the leading authority on Newton, and had done a large amount of research, with a view to the publication of a new edition. This is hardly the place to describe in detail his other activities, such as his editorship of *The Monist* and the *International Journal of Ethics*, his many researches into the history of science, and his important work on Induction and Probability, which was in course of publication in *Mind*. We must, however, mention that Jourdain had in preparation a large work on *The History of Mathematical Thought*. It is quite evident that, with his intimate knowledge of the lives of the older mathematicians, his wide knowledge of foreign languages, and his keen interest in the evolution of abstract ideas, he was the ideal author of such a book.

Among the papers produced by Jourdain in his short career one of the most important is his article "On the General Theory of Functions" (*Journal für Mathematik*, Bd. 128, Heft 3). Related to this are his paper "On the Question of the Existence of Transfinite Numbers," published in the *Proceedings* of this Society (1907), and a series of articles (1909–1913) in the *Archiv der Mathematik und Physik* on "The Development of the Theory of Transfinite Numbers." These and a number of papers in the *Mathematische Annalen, Messenger of Mathematics, Quarterly Journal,* and various other periodicals, dealt with the general theory of aggregates and relations. Jourdain was also one of the large number of people who have attempted to prove "the axiom of Zermelo" or multiplicative axiom, so notorious in mathematical logic and the general theory of aggregates.

An example of the other side of his work is to be found in an early article in *The Monist,* entitled "On some Points in the Foundations of Mathematical Physics" (1908). Jourdain used his results stated in his article "On the General Theory of Functions" to attack the problems of causality in physics. This was the first of a series of papers in which Jourdain applied the conceptions of modern logic to mathematical physics. Other papers by Jourdain include a paper "On those Principles of Mechanics which depend upon Processes of Variation" (*Math. Ann.*, 1908), and two articles on "The Influence of Fourier's Theory on the Conduction of Heat on the Development of Pure Mathematics" (*Scientia,* 1917). He was also the author of two separate publications, "The Nature of Mathematics" (1912) and "The Principle of Least Action" (1913).

Jourdain's work lay in regions still unfamiliar to many mathematicians, and still distracted by controversy, and opinions will differ as to the permanent value of his accomplishment. There can be no difference of opinion as to the value of a life lived with such invincible courage and inspired by so disinterested a devotion to mathematical science.

D. M. W.