# TETRAHEDRA IN RELATION TO SPHERES AND QUADRICS

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## *Introduction.*

This paper arose out of the following question :—"What are the relations between the inscribed and circumscribed spheres of a tetrahedron ?" In other words, given two spheres $S$, $S'$, what are the necessary and sufficient conditions that there should be *real* tetrahedra which are inscribed in $S$ and circumscribed to $S'$ ?

It is proved (§§ 8, 9) that the sole condition is

$$(R+r)(R-3r) \geqslant d^2,$$

when $R$, $r$ are the radii of the spheres and $d$ is the distance between their centres; corresponding conditions for the escribed spheres are found, and in particular it is shewn that when two spheres are external to each other there are always real tetrahedra whose vertices lie on the one and whose faces touch the other.

The earlier sections are concerned with some general properties of quadrics which may be stated thus :—If the vertices of a tetrahedron lie on a quadric $S$ and its faces touch a quadric $S'$, then, when one face $\pi$ is given, the opposite vertex lies on a certain plane section $\pi'$ of $S$, and when $\pi$ varies the envelope of $\pi'$ is a quadric through the common curve of $S$ and $S'$.

At the end of the paper will be found some simpler arguments that corroborate the general conclusions previously obtained ; these simpler arguments are, in fact, more forcible in some ways than the earlier ones, but they are not so far-reaching.*

* I may add that the invariant of § 5 occurs in quite a different connexion in a paper by M. G. Fontené relating to certain polyhedra that have a hole through them, and are inscribed in the one quadric and circumscribed to the other. See *Nouv. Ann.*, 1904, p. 289. The quadric envelope of § 3 is alluded to by Kohn, *l.c. infra*, § 2. None of the papers quoted has reference to the reality of the tetrahedra. I cannot find that this question has been discussed before.

1. When a tetrahedron $LMNP$ has its vertices on a fixed quadric $S$, its position depends on eight arbitrary constants, so we are led to expect that the problem of finding a tetrahedron whose vertices lie on one quadric $S$ and whose faces touch a second quadric $S'$ will admit of $\infty^4$ solutions. This, as will presently appear, is usually the case; but, denoting the face $LMN$ by $\pi$, we cannot choose the vertex $P$ and the face $\pi$ arbitrarily. In fact denote by $\sigma$ and $\sigma'$ the conics in which $\pi$ cuts $S$ and the tangent cone from $P$ to $S'$: then, if there were such a tetrahedron, there would be one triangle $LMN$ whose corners lie on $\sigma$ and whose sides touch $\sigma'$; hence there would be an infinite number of such triangles, and hence an infinite number of the tetrahedra for which the vertex $P$ and the opposite face $\pi$ are the same.

2. I proceed to shew that when the face $\pi$ (it is of course a tangent plane to $S'$) is given, the locus of the vertex $P$ is in general a plane section of the quadric $S$. In addition to being on the quadric, $P$ must be such that the conics $\sigma$ and $\sigma'$ satisfy the poristic condition already alluded to. Suppose, then, that a generator of $S'$ in the plane $\pi$ cuts $S$ in $H$ and $K$. Then, since $HK$ is a tangent to the conic $\sigma'$, the other tangents from $H$ and $K$ to $\sigma'$ must meet on $\sigma$; but, if $h$ and $k$ are the other generators of $S'$ through $H$ and $K$, these two tangents are the lines in which the planes $Ph$, $Pk$ cut the plane $\pi$, and it follows that these two planes must meet on the conic $\sigma$. Hence, the two pencils $h(P)$, $k(P)$ being homographic, $P$ must lie on a quadric containing the lines $h$, $k$ and the conic $\sigma$. As this quadric has the plane section $\sigma$ in common with $S$, the remaining part of the intersection is also a plane section of $S$, and this is the required locus of the vertex $P$.

COR. I.—The quadric locus for $P$ must also contain the other generators at the points $H'$, $K'$ in which $S$ is cut by the other generator of $S'$ in the plane $\pi$.

COR. II.—It might happen that the quadric locus for $P$ coincided with $S$. In this case $S$ and $S'$ must have four common generators, and the quadric that contains them and the conic $\sigma$ will coincide with $S$ for all positions of $\pi$.

$P$ and $\pi$ can now be chosen arbitrarily, and there are $\infty^5$ tetrahedra inscribed in $S$ and circumscribed to $S'$.* In other cases there are $\infty^4$,

* This theorem is due to M. G. Fontené, *Nouvelles Annales*, 1900, p. 67. It has also been discussed by Kohn, *Monatshefte für Math. und Phys.*, 1900, p. 102, and by Humbert, *Bull. Soc. Math. Fran.*, 1904, p. 135. The same writers naturally notice the fact that the locus of $P$, when $\pi$ is given, is usually a plane section of $S$.

since when $\pi$ is given there are $\infty^1$ positions for $P$, and corresponding to each such position there are $\infty^1$ tetrahedra.

3. I call the plane that contains $P$ when $\pi$ is given $\pi'$, and proceed to prove that when $\pi$ varies the envelope of $\pi'$ is a quadric of the pencil

$$S+\lambda S' = 0.$$

For this purpose I use elliptic parameters on the curve common to $S$ and $S'$. They can be so chosen that if $a$, $b$, $c$, $d$ are four coplanar points

$$a+b+c+d \equiv 0,$$

while if $ab$ is a generator of a fixed quadric of the pencil

$$a+b = k,$$

a constant, and if $cd$ is a generator of the opposite system

$$c+d = -k.$$

Now, referring back, let the generators $h$, $k$ cut $S$ again in $\alpha$, $\beta$, and let the corresponding generators $h'$, $k'$, derived from $H'K'$ the other generator of $S'$ in the plane $\pi$, cut $S$ again in $\alpha'$, $\beta'$. Then we have

$$H+K = k, \quad H+\alpha = -k, \quad K+\beta = -k,$$

$$H'+K' = -k, \quad H'+\alpha' = k, \quad K'+\beta' = k,$$

and hence

$$\alpha+\beta = -3k,$$

$$\alpha'+\beta' = 3k.$$

It follows that the lines $\alpha\beta$, $\alpha'\beta'$ are generators of a fixed quadric of the pencil, and so the plane $\alpha\beta\alpha'\beta'$, which is manifestly the plane $\pi'$, always touches this fixed quadric.

4. Some further remarks will be useful.

(i) It is clear that, when $\pi$ is given, the plane $\pi'$ and its envelope depend only on the quadric $S'$ and its curve of intersection with $S$—not on the quadric $S$ in particular.

(ii) The plane $\pi$ is touched by two quadrics of the pencil besides $S'$, and the plane $\pi'$ by two besides the envelope. These remaining tangent quadrics are the same for $\pi$ and $\pi'$. In fact, if we write

$$H+H' = \lambda,$$

we have $\alpha+\alpha' = -\lambda, \quad \beta+\beta' = +\lambda, \quad K+K' = -\lambda,$

shewing that the lines $HH'$, $KK'$, $\alpha\alpha'$, $\beta\beta'$ are generators of the same quadric, and hence that $\pi$ and $\pi'$ touch this quadric. Similarly for the four lines

$$HK', \quad H'K, \quad \alpha\beta', \quad \alpha'\beta.$$

It follows, for example, that if $\pi'$ touches $S$, and $\pi$ does not touch $S$, then $S$ is itself the envelope.

(iii) If the plane $\pi$ is itself tangent to the envelope of $\pi'$, the point of contact of $\pi'$ is on the common curve of the pencil.

(iv) If the point of contact of $\pi$ with $S'$ is on the envelope of $\pi'$, then $\pi'$ is also tangent to $S'$.

5. I shall next find the equation of the envelope of $\pi'$ when the equations of $S$ and $S'$ are

$$S \equiv ax^2+by^2+cz^2+dw^2 = 0,$$
$$S' \equiv x^2+y^2+z^2+w^2 = 0.$$

For this purpose take $\pi$ to be

$$lx+my+nz+rw = 0.$$

The sections by $\pi$ of $S$ and the tangent cone from $P$ to $S'$ being $\sigma$ and $\sigma'$, we must have

$$\Theta'^2 = 4\Delta'\Theta,$$

the usual poristic condition. Eliminating $w$, a straightforward calculation shews that $P$ must lie on the quadric whose equation is

$$\{\Sigma lx(b+c+d-a)\}^2$$
$$= 4\Sigma x^2\{l^2(cd+ab+bc)+(ac+ad)m^2+(ad+al)n^2+(ab+ac)r^2\}$$
$$+8\{admnyz+\ldots+bclrxw+\ldots\};$$

and thence that the equation of $\pi'$ is

$$\alpha lx+\beta my+\gamma nz+\delta rw = 0,$$

wherein $\alpha = 8a^2-4a(a+b+c+d)+4(bc+ca+ab+ad+bd+cd)$
$$-(a+b+c+d)^2,$$

etc. Since $\quad l^2+m^2+n^2+r^2 = 0,$

the envelope of $\pi'$ is

$$\alpha^2x^2+\beta^2y^2+\gamma^2z^2+\delta^2w^2 = 0.$$

Of course we must have $a^2 = Pa + Q$,

where $P$, $Q$ are symmetrical in $a$, $b$, $c$, $d$.

If we denote the discriminant of $S+\lambda S'$ by

$$\Delta+\lambda\Theta+\lambda^2\Phi+\lambda^3\Theta'+\lambda^4\Delta',$$

the zeros of this expression in $\lambda$ are $-a$, $-b$, $-c$, $-d$; and $\Delta$, $\Theta$, &c., are now the fundamental invariants of the two quadrics.

An easy calculation now shews that the envelope is

$$PS+QS' = 0,$$

where $$P = 8\Delta'(\Theta'^3-4\Phi\Theta'\Delta'+8\Theta\Delta'^2),$$

and $$Q = (4\Phi\Delta'-\Theta'^2)^2-64\Delta\Delta'^3.$$

If we take $S'$ to be a proper quadric, the condition that $S$ itself should be the envelope of $\pi'$ is $$(4\Phi\Delta'-\Theta'^2)^2 = 64\Delta\Delta'^3;$$

further the point of contact of $\pi'$ with its envelope is always the pole of $\pi$ for the quadric $$\alpha x^2+\beta y^2+\gamma z^2+\delta w^2 = 0.$$

Hence, when $S$ is the envelope of the planes $\pi'$, we can choose each vertex to be the point of contact of the position of $\pi'$ that corresponds to the opposite face, and all such tetrahedra are now self-conjugate for the quadric last mentioned, say $F$.

In verification we may note that

$$\alpha+\beta+\gamma+\delta = 0,$$

so that there are $\infty^3$ tetrahedra inscribed in $F$ that are self-conjugate for $S'$, and, further, $S$ and $S'$ are reciprocals with respect to $F$. These results are analogous to known ones relating to triangles inscribed in one conic and circumscribed to another.

6. I now proceed to discuss the reality of the tetrahedra when the two quadrics are spheres, and begin by outlining the method.

(I) Suppose one of the faces $\pi$ is fixed: then the opposite vertex will lie on a plane $\pi'$. It is necessary that $\pi'$ should cut $S$ in a real circle in some positions anyhow. Now the condition for a real section will be found by expressing the condition that a certain expression, which when equated to zero is the condition that $\pi'$ should touch $S$, is positive. We know (§ 4) that $\pi'$ only touches $S$ either when $S$ is the envelope or when $\pi$ touches $S$, so that the result will be largely independent of the position of $\pi$. In fact the expression arising will be the product of two factors, one of which will re-express the fact that $\pi$ should meet $S$ in a real curve, and the other the condition (§ 5) that $S$ should be the envelope.

(II) It is not enough that the vertex $P$ should be real: to give real tetrahedra it must be external to $S'$—it will be seen that a real section by $\pi'$ is always external to $S'$.

(III) Even when the plane $\pi$ is given and there are real vertices corresponding outside $S'$, does it follow that some of the triangles inscribed in the circle $\sigma$ and circumscribed to the conic $\sigma'$ (both of which are real) are always real? The answer is "yes", though the argument is a little tedious.

7. It will be well to begin by calculating the invariant

$$(4\Phi\Delta'-\Theta'^2)^2-64\Delta\Delta'^3,$$

for two spheres

$$S \equiv (x-d)^2+y^2+z^2-R^2 = 0,$$

$$S' \equiv x^2+y^2+z^2-r^2 = 0,$$

whose radii are $R$, $r$, and whose centres are a distance $d$ apart.

The simple invariants being

$$\Delta' = r^2, \quad \Theta' = R^2+3r^2-d^2, \quad \Phi = 3(R^2+r^2)-2d^2,$$

$$\Theta = 3R^2+r^2-d^2, \quad \Delta = R^2,$$

the above invariant is quickly seen to be

$$(R^2+2Rr-3r^2-d^2)(R^2-2Rr+r^2-d^2)$$

$$\times (R^2-2Rr-3r^2-d^2)(R^2+2Rr+r^2-d^2). \qquad (7.1)$$

8. Suppose now that the plane $\pi$ is $z=0$, and that the two spheres are

$$S \equiv x^2+y^2+z^2+2ax+2\beta y+2\gamma z+\delta = 0,$$

$$S' \equiv x^2+y^2+z^2-2rz = 0,$$

so that

$$d^2 = (\gamma+r)^2+a^2+\beta^2,$$

$$R^2 = a^2+\beta^2+\gamma^2-\delta.$$

Proceeding directly, we find for the equation of $\pi'$

$$z\{\delta^2-4r^2(\alpha^2+\beta^2)\}-4r\alpha\delta x-4r\beta\delta y-4r\{\delta^2+2r^2(\delta-\alpha^2-\beta^2)+2r\gamma\delta\}=0.$$

We have now to set down the condition that this should cut $S$ in a real circle, *i.e.* be within a distance $R$ of the centre of this sphere. A calculation shews that the condition is

$$(R^2-\gamma^2)(R^2-2Rr-3r^2-d^2)(R^2+2Rr-3r^2-d^2)$$
$$(R^2-2Rr+r^2-d^2)(R^2+2Rr+r^2-d^2)\geqslant 0.$$

The first factor equated to zero is the condition that $\pi$ should touch $S$, and the remainder is merely the invariant (7 . 1), so that all is as could have been predicted by § 6. If any of the other factors vanish, the sphere $S$ is itself the envelope of the planes $\pi'$.

9. There will not be real tetrahedra unless

$$(R^2-2Rr-3r^2-d^2)(R^2-2Rr+r^2-d^2)(R^2+2Rr-3r^2-d^2)$$
$$(R^2+2Rr+r^2-d^2)\geqslant 0,$$

and of these factors the first is the least and the last the greatest.

Take, for clearness, the inequality first.

There are three cases—

($\alpha$) $$R^2-2Rr-3r^2-d^2>0,$$

or $$(R+r)(R-3r)-d^2>0,$$

or $$(R-r)^2>4r^2+d^2,$$

and all the other factors of course positive. Here the sphere $S$ completely encloses the sphere $S'$.

($\beta$) $$R^2+2Rr+r^2-d^2<0,$$

or $$R+r<d,$$

and all the other factors negative. Here the spheres are completely external to each other.

($\gamma$) $$(R-r)^2<d^2,$$
$$(R+r)^2>4r^2+d^2.$$

In this case the spheres intersect in a real circle.

The case $(R-r)^2>d^2$, $(R+r)^2<4r^2+d^2$ is impossible if $R>r$.

and if $R < r$ it shews that $S'$ completely encloses $S$. This is out of the question.

There are next the four cases in which the equality obtains in § 9, viz.

$$(\alpha') \quad (R-r)^2 = 4r^2+d^2,$$
$$(\beta') \quad (R+r)^2 = 4r^2+d^2,$$
$$(\gamma') \quad (R-r)^2 = d^2,$$
$$(\delta') \quad (R+r)^2 = d^2.$$

In each case $\pi'$ touches $S$. In $(\gamma')$ and $(\delta')$ $S$ and $S'$ touch, and it can be seen that $\pi'$ touches them at their point of contact, so that these cases may be ignored.

$(\alpha')$ is merely the limiting case of $(\alpha)$ and $(\beta')$ of $(\gamma)$.

It may be noted that $(\beta')$ implies $R < r+d$, so that the spheres cut in a real circle.

Only in cases $(\alpha)$, $(\beta)$, $(\gamma)$, $(\alpha')$, $(\beta')$ can there be real tetrahedra.

10. We have next to see if the vertices so found are external to $S'$.

In $(\alpha)$, $(\beta)$, $(\alpha')$ the sphere $S$ is completely external to $S'$, so that any real vertex so found must be external to $S'$.

In the other two cases the spheres meet in real points, and the same result follows from a different argument.

The condition is $\qquad (R+r)^2 \geqslant 4r^2+d^2.$

Suppose $S'$ is given, and $Q$ is a given point common to $S$ and $S'$. It is easy to see that the centre $O$ of $S$ must not be inside the paraboloid of revolution whose focus is $Q$, and whose vertex is $O'$, the centre of $S'$; for, if $OO'$ subtends an angle $\theta$ at $Q$, we have

$$R \cos^2 \tfrac{1}{2}\theta \geqslant r.$$

If now $O'$ and the line of centres are given as well as $Q$, all the condition implies is that there is a point $\Omega$ on the line such that $O$ may be anywhere not internal to the segment $\Omega O'$. Further $O'$ and $\Omega$ are on opposite sides of the radical plane, and $\Omega$ is the centre of the envelope sphere. It is now easy to deduce that no tangent plane $\pi'$ to the envelope sphere can meet $S$ in points that are inside $S'$.

To sum up: in case any one of the following sets of conditions is satisfied, viz.,

$$(\alpha) \quad (R-r)^2 \geqslant 4r^2+d^2, \qquad (\beta) \quad R+r < d,$$
$$(\gamma) \quad (R+r)^2 \geqslant 4r^2+d^2, \quad R-r < d$$

and the plane $\pi$ cuts $S$ in real points, there are real positions of $P$ on $S$ that are external to $S'$, such that the two conics $\sigma$ and $\sigma'$ are real and satisfy the poristic condition

$$\Theta'^2 = 4\Delta'\Theta \quad (\S\ 5).$$

11. I shall now shew that in such cases it is practically certain that there are real triangles whose sides touch $\sigma'$, and whose vertices lie on $\sigma$. If $P$ is any point on $\sigma$, the triangle with a vertex at $P$ will be real if the tangents from $P$ to $\sigma'$ are real. Now, if the conics cut (but do not touch) in a real point $A$, there are points $P$ on $\sigma$ near $A$ which are external to $\sigma'$; when the conics have two conjugate imaginary points of intersection a real projection will bring them into circles, and then the poristic condition shews either that $\sigma$ encloses $\sigma'$ or that $\sigma$ and $\sigma'$ meet in two real points. It follows that some of the triangles will be real except possibly when $\sigma$ and $\sigma'$ touch in two real points. This is an actual exception because the poristic condition then shews that at the common points the curvature of $\sigma$ is four times that of $\sigma'$, so that $\sigma'$ encloses $\sigma$ completely, and there are no real triangles. This sole exception is obviously a case of such rare occurrence that it might be ignored in a discussion like that of the tetrahedra, but I add one or two remarks.

From the reality of the conics $\sigma$ and $\sigma'$ follows the reality of the triangles (or some of them) except when $\sigma$ and $\sigma'$ have double contact. The condition of double contact is a double one of equality, and hence in general none of the positions of $P$ found will satisfy it. A more minute argument confirms the conclusion, but, as the possible exception is a trivial one and the argument is only destined to prove a negative, I shall suppress it.

12. We have thus proved that the necessary and sufficient conditions that there should be real tetrahedra whose vertices lie on a sphere $S$ and whose faces touch a sphere $S'$, are

($\alpha$) $S$ encloses $S'$, and $(R-r)^2 \geqslant 4r^2+d^2$,

or ($\beta$) $S$ and $S'$ are external to each other,

or ($\gamma$) $S$ and $S'$ meet in a real circle, and $(R+r)^2 \geqslant 4r^2+d^2$.

In general there are $\infty^4$ of the tetrahedra, but in case of the sign of equality there are $\infty^3$, and then (§ 5) the tetrahedra are self-conjugate with respect to a fixed quadric.

13. There are three types of spheres that touch the faces of a tetrahedron, viz. :—

($\alpha$) the inscribed sphere;

($\beta$) the doubly escribed spheres, which touch two of the faces on the opposite side to the inscribed sphere, and the other two on the same side;

($\gamma$) the ordinary escribed spheres, which touch one face on the opposite side to the inscribed sphere, and the other three faces on the same side.

It seems natural to suppose that this classification coincides with that in § 12. To establish this we have only to prove that an escribed sphere ($\gamma$) always cuts the circum-sphere in a real point, while an escribed sphere ($\beta$) is always external to the circum-sphere—it is obvious that the inscribed sphere is enclosed by the circum-sphere. The proofs required are simple enough if we assume a known theorem,* to the effect that the spheres touching the faces of a tetrahedron $ABCD$ can be divided into four pairs, such that the centres of a pair are collinear with $D$, and that a pair are touched by the same sphere through the circle $ABC$. There will be no need to give the argument in detail—it is simple when the theorem alluded to is assumed and that theorem is not quite simple.

14. It may be of some interest to give an argument relating to the inscribed sphere, which does not depend on the, sometimes elaborate, apparatus used before.

We have two spheres $S$ and $S'$, of which the first encloses the second, and we want the necessary and sufficient condition that there should be real tetrahedra inscribed in $S$ and circumscribed to $S'$.

The case in which the spheres are concentric is particularly simple. Any real tetrahedron satisfying the conditions must now have its opposite edges equal in pairs. For if it be $ABCD$, the circumcentre of the face $ABC$, being the point of contact of the inscribed sphere with that face, must be internal to the face, and the triangle $ABC$ must thus be acute-angled.

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* *Camb. Phil. Soc. Trans.*, Vol. 16, p. 166.

Again, the circles $ABC$, $ABD$ being equal, the angles $ACB$, $ADB$ are equal or supplementary; as both are acute they must be equal.

It follows that the angles at any corner, say $D$, are equal to the angles in the opposite face $ABC$, and thence we deduce that the tetrahedron must be isosceles, as stated.

15. Conversely, if a tetrahedron is isosceles, the faces are acute-angled and the circum-centre and in-centre both coincide with the centroid. Also the radius of the inscribed sphere is one-fourth of each perpendicular of the tetrahedron, and hence at most one-third of the radius of the circumscribing sphere.

16. Take now two concentric spheres $S$ and $S'$ of radii $R$ and $r$ ($R > r$). A plane $\pi$ touching $S'$ at $H$ will cut $S$ in a circle $\sigma$ whose centre is $H$, and the tangent cone from any point $R$ on $S$ to $S'$ in a conic $\sigma'$, of which $H$ is a focus. If $z$ is the distance of $R$ from $\pi$ and $t$ the tangent from $R$ to $S'$, the major axis $2a$ of $\sigma'$ is given by

$$2a = 2t \frac{r}{z-2r}.$$

When this major axis is equal to the radius of the circle $\sigma$, which is $t$, there will be triangles inscribed in $\sigma$ whose sides touch $\sigma'$ (§ 2). Thus there will be tetrahedra with one vertex at $R$, and the other three in the plane $\pi$, which are inscribed in $S$ and circumscribed to $S'$. The condition for this is

$$z = 4r;$$

so $R$ must lie on a plane parallel to $\pi$ and at a distance $4r$ from it. This plane cuts $S$ in a real circle only if $3r < R$: it touches $S$ if $3r = R$.

All this corroborates §§ 8, 9, and shews that, if $3r \leqslant R$, there are real tetrahedra having $S$ for circumscribing sphere and $S'$ for inscribed sphere. There are $\infty^4$ if $3r < R$ and $\infty^3$ if $3r = R$. The tetrahedra are all isosceles (of course regular if $3r = R$) and their altitudes are $4r$.

17. Passing to the general case in which $S$ and $S'$ are not concentric, I first shew that if there is one real tetrahedron inscribed in $S$ and circumscribed to $S'$, then there exists such a tetrahedron, one of whose faces is an arbitrary tangent plane $\pi$ of $S'$.

Suppose $LMNP$ is the given tetrahedron: the plane $LMN$ will cut $S$ and the tangent cone from $P$ to $S'$ in two conics $\sigma$ and $\sigma'$, of which $\sigma$ encloses $\sigma'$ completely. Hence there will be a tetrahedron with one corner at $P$ and another anywhere on the circle $\sigma$. The vertices $L$, $M$,

$N$, $P$ cannot all be on the same side of the plane $\pi$; if $L$ and $M$, say, are on opposite sides, it is clear that the sections of $S$ by the planes $LMN$ and $\pi$ have two real points in common. It follows that we can bring one vertex, say $L$, of the tetrahedron into the plane $\pi$ without moving $P$.

Repeating the process and keeping $L$ fixed, we can bring a second vertex, say $M$, into the plane $\pi$. The two remaining vertices are one on each of the tangent planes through the line $LM$ to $S'$; thus one of them is on $\pi$, and $\pi$ is now a face.

The argument would apply equally well to two ellipsoids: it depends merely on Poncelet's Porism, and the fact that one surface is entirely inside the given tetrahedron, while the other is all but completely external to it.

18. We are thus justified in finding the necessary and sufficient condition for the existence of real tetrahedra by taking one face to be any tangent plane to the inscribed sphere $S'$. Naturally one of those whose point of contact is on the line joining the centres is chosen.

Call this plane $\varpi$ and the opposite vertex $P$: the radii of the spheres are $R$, $r$, and the distance between the centres is $d$: $z$ is the distance of $P$ from $\varpi$, and $t$ the tangent from $P$ to $S'$. Then the major axis $2a$ of the conic $\sigma'$, in which $\varpi$ cuts the tangent cone from $P$ to $S'$, is given by

$$a = t\,\frac{r}{z-2r}. \tag{II}$$

The section of $S$ by $\varpi$ is a circle $\sigma$ of radius $\rho$, say, whose centre is the point $O$ when $\varpi$ touches $S'$, and the poristic relation between $\sigma$ and $\sigma'$ is

$$\rho = 2a.$$

Now, using $O$ for origin, the equations of the spheres will be

$$S \equiv x^2+y^2+z^2-2cz-\rho^2 = 0,$$

$$S' \equiv x^2+y^2+z^2-2rz \qquad = 0.$$

The point $P$ must satisfy the conditions

$$\rho^2(z-2r)^2 = 4r^2(x^2+y^2+z^2-2rz),$$

from (II), and
$$x^2+y^2+z^2-2cz-\rho^2 = 0.$$

We thus get an equation for $z$ which reduces to

$$\rho^2 z = 4\rho^2 r+8r^2c-8r^3, \tag{III}$$

and the distance of $P$ from $z = 0$ must not be greater than $c+R$. Thus we have

$$\rho^2(R+c) \geqslant 4\rho^2 r + 8r^2c - 8r^3,$$

and, of course,

$$R^2 = \rho^2 + c^2.$$

We deduce

$$(R^2-c^2)(R+c) \geqslant 4(R^2-c^2)\,r + 8r^2c - 8r^3;$$

and, since

$$d + r = c,$$

this gives

$$\{R^2-(r+d)^2\}\{R+d-3r\} \geqslant 8r^2d,$$

or

$$R^3 - R^2(3r-d) - R(r+d)^2 + (r-d)(3r^2+d^2) \geqslant 0,$$

or

$$(R-r+d)(R^2-2Rr-3r^2-d^2) \geqslant 0.$$

Clearly $S'$ must be inside $S$, so that the first factor is positive, and we are left with

$$(R+r)(R-3r) \geqslant d^2,$$

or

$$(R-r)^2 \geqslant 4r^2+d^2 \quad \text{(cf. §§ 8, 9)}.$$