ON THE DERIVATES OF A FUNCTION

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1. Theorems relating to perfectly general functions are rare.* In the present connexion there is the theorem, given by myself in the *Acta*,† that *the points, if any, at which the upper derivate on one side is less than the lower derivate on the other side, form a countable set.* In this theorem no restriction whatever is laid on the primitive function, except tacitly that it is everywhere finite, and the derivates may be ordinary or taken with respect to any monotone increasing function $g(x)$ without constant stretches.‡

I propose now to enunciate and prove three fundamental theorems, already known to be true for special classes of functions, concerning the derivates of a function $f(x)$, which, if not perfectly general, is nevertheless, for mathematical purposes, practically unrestricted in character. I only assume that

(*a*) *$f(x)$ is a measurable function ;*

and (*b*) *$f(x)$ is finite everywhere.*

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* I would recall the theorem communicated by my husband to the British Association at Leicester in 1907, and, in an extended form, to the Mathematical Congress at Rome in 1908, that, except at a countable set of points, the limits of $f(x)$ on the right and on the left [that is the $\lim_{h\to 0}$ limits of $f(x+h)$ and $f(x-h)$, $(0 < h)$] are the same, and $f(x)$ lies between the greatest and least of these limits (inclusive) on each side. See W. H. Young, "On the Distinction of Right and Left at Points of Discontinuity," 1907, *Quarterly Journal*, Vol. 39, pp. 77 and 82. "Sulle due funzioni a piu valori costituite dai limite d'una funzione d'una variabile reale a destra e a sinistra di ciascun punto," 1909, *Rend. della R. Acc. dei Lincei*, Ser. 5, Vol. XVII, pp. 582-587. These surprising theorems formed the starting point of our investigations, and suggested that similar general results could be obtained for derivates; till now, however, we had not been able to justify such a supposition.

† "A Note on Derivates and Differential Coefficients," 1912, *Acta Math.* (1914), Vol. 37, p. 144.

‡ Cp. W. H. Young, "On Integrals and Derivates with respect to a Function," 1914, *Proc. London Math. Soc.*, Ser. 2, Vol. 15, p. 61, footnote. I take this occasion to call attention to a misprint: this footnote (p. 61) and the following footnote (p. 62) have been interchanged.

The enunciations of these theorems are as follows:—

THEOREM 1.—*The points at which the upper derivate on one side is* $+\infty$ *and the lower derivate on the other side is not* $-\infty$, *form a set of content zero.**

In other words, if $f^{+}(x) = +\infty$, then $f_{-}(x) = -\infty$, except at a set of content zero; and if $f^{-}(x) = +\infty$, then $f_{+}(x) = -\infty$, except at a set of content zero.

THEOREM 2.—*The points at which $f(x)$ has an infinite right-hand (left-hand) differential coefficient form a set of content zero.*†

As an immediate corollary we have the generalisation of Lusin's theorem:—

The points, if any, at which a measurable function $f(x)$ has an infinite differential coefficient, form a set of zero content.

THEOREM 3.—*The points at which one of the upper derivates and one of the lower derivates of $f(x)$ are finite and not equal form a set of content zero.*‡

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* In my essay on "Infinite Derivates," to which was awarded the Gamble Prize at Girton College, Cambridge, last year, I obtained the theorem that, *when $f(x)$ is a continuous function, the points at which $f^{+}(x) = +\infty$, and both $f_{-}(x)$ and $f_{+}(x)$ different from $-\infty$, form a set of content zero;* the reasoning was subsequently seen by me to give the theorem as it stands above, when $f(x)$ is continuous; in this form it is given by Denjoy for a continuous function in his interesting memoir, "Sur les nombres dérivés des fonctions continues," 1915, *Jour. de Math.*, Sér. 7, Vol. 1, pp. 105–240, *Application III*, p. 195. This appeared almost exactly at the time of the award of the Gamble Prize, for which, of course, the publication of my result had been delayed. After perusing M. Denjoy's memoir, I was unwilling to put the corresponding part of my prize essay into print as it stood; with some slight alterations it is appearing in the *Quarterly Journal of Mathematics*, and the present communication, in which M. Denjoy's results and my own are considerably extended, is a further result. I take this opportunity, however, to call attention to the importance of Denjoy's results; if we assume them, the theorems of the present paper can be deduced with extreme ease by the use of the C-property (see below, § 2).

† This theorem, given by Denjoy for a continuous function, *Application I*, p. 187, is an extension of the original theorem by Lusin (1911), *Recueil de la Société math. de Moscou*, Vol. XXVIII, 2, in Russian. Both Denjoy and I use a Lemma, introduced by Lusin in this connexion. For a function of bounded variation this theorem is, of course, included in Lebesgue's theorem quoted in the following footnote.

‡ The original of this theorem would seem to be Lebesgue's theorem that *a continuous function of bounded variation has summable derivates, which are accordingly finite almost everywhere, and possesses a finite differential coefficient, except at a set of content zero* (1901), *Leçons sur l'intégration*, p. 128. W. H. Young removed the restriction as to continuity (1910), "On Functions of Bounded Variation" (1910), *Quarterly Journal*, Vol. 42, § 2, p. 79. Montel, on the other hand, retaining the continuity and the finitude of the derivates almost

In other words the four sets at which

(i) $f^+(x)$ and $f_-(x)$, (ii) $f^+(x)$ and $f_+(x)$, (iii) $f^-(x)$ and $f_-(x)$,

$f^-(x)$ and $f_+(x)$,

have finite unequal values, are all sets of zero content.

The three main theorems enable us to state for any measurable function which is finite everywhere the striking properties enunciated for a continuous function by Denjoy* :—

If we neglect a set of points whose content is certainly zero, the derivates of $f(x)$ at any point x belong to one of the three following cases :—

(I) *There is an ordinary finite differential coefficient $f'(x)$.*

(II) *The upper derivates on each side are $+\infty$ and the lower derivates on each side are $-\infty$.*

(III) *The upper derivate on one side is $+\infty$, the lower derivate on the other side is $-\infty$, and the two remaining extreme derivates are finite and equal.*

Each of these cases may occur at points of a set of any content, as is shown by examples, and all three cases may present themselves simultaneously for a single function $f(x)$, or one or more of the cases may be altogether absent.

From these three main theorems we can draw a number of interesting corollaries, of which the following may be specially mentioned :—

THEOREM 4.—*There is no geometrical distinction of right and left with*

everywhere, removed the restriction as to bounded variation (1913), *Comptes Rendus*. More generally he enunciates the theorem that the content of the set of points at which all the derivates are finite is the same as that of the set of points where a finite differential coefficient exists (it being tacitly assumed that the function is continuous). Montel's proof consists in an adaptation of a method given by my husband and myself in these *Proceedings*, Ser. 2, Vol. 9, p. 331, for the proof of Lebesgue's theorem. It will be seen that it is this same form of proof, amplified but not altered, which I have used in the present paper. Montel uses, in addition, a Lemma due to Denjoy. Finally, Denjoy, in his recent memoir, gives two theorems which are together equivalent to the above theorem when $f(x)$ is continuous (*Applications II and V*, pp. 188 and 192). It is hardly necessary to say that no previous writer except my husband, from whom the idea is taken, has contemplated other than ordinary derivates.

* Denjoy, *loc. cit.*, p. 105.

regard to the four extreme derivates $f^+(x)$, $f_+(x)$, $f^-(x)$, $f_-(x)$, *except at a set of content zero.**

Also the two following generalisations of Montel's theorems, which are immediate corollaries from Theorem 3 :—

If the derivates of $f(x)$ *are all finite, except possibly at a set of content zero, then* $f(x)$ *has a finite differential coefficient almost everywhere.*

The set of points at which the derivates of $f(x)$ *are finite is the same as the set of points at which the differential coefficient exists and is finite, exception being possibly made of a set of content zero.*

The proofs here given of the main theorems are closely modelled on that formerly given in these *Proceedings* by my husband and myself for the case of a monotone function $f(x)$; they do not involve integration, nor transfinite processes, nor Cantor's numbers.

In conclusion I relax the condition (*b*) laid on our measurable function, and only assume that

(*c*) $f(x)$ *is finite at a set of points of positive content.*

The three main theorems then remain unaltered, provided we always abstract from consideration the points at which the primitive function is infinite.

If we have the condition that

(*c bis*) $f(x)$ *is finite at a set of points of positive content dense everywhere*

(so that the derivates are everywhere determinate), the Theorems 1 and 3, as well as Lusin's theorem, remain true as they stand, and the second theorem takes the following form :—

THEOREM 2 *bis.—The points at which* $f(x)$ *has an infinite right-hand (left-hand) differential coefficient consist of the infinities of* $f(x)$, *and possibly also a set of content zero.*

To obtain these results I require to elaborate slightly the work of Egoroff and Lusin, used in the earlier part of this paper.

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* In 1908 my husband published the theorem that *there is no distinction of right and left with respect to the derivates of a continuous function except at a set of the first category.* We suspected at that time that this set was of content zero, but failed to obtain a proof. Two discussions were given; both require the continuity of the primitive function. This set contains the above set of content zero, which is accordingly of the first category. See W. H. Young, "Oscillating Successions of Continuous Functions," 1908, *Proc. London Math. Soc.*, Ser. 2, Vol. 6, pp. 305–7.

2. The generality of the present results is due to the application of Egoroff's theorem.* The enunciation of this important theorem is as follows:—

If a succession of measurable functions converges at all the points of an interval (a, b), *except possibly at a set of content zero, it is always possible to remove from the fundamental interval* (a, b) *a set of content as small as we please, and such that with respect to the complementary set, the succession converges uniformly.*

Hence it immediately follows that it is a necessary and sufficient condition that the limiting function $f(x)$ of a sequence of measurable functions should be finite at every point of the fundamental interval, except possibly at a set of content zero, that we should always be able to find a perfect set, whose content differs from that of the fundamental segment by less than e, with respect to which the sequence converges uniformly, e being any chosen positive quantity.

In consequence of Egoroff's theorem, as Lusin pointed out,† every measurable function which is finite, except possibly at a set of content zero, has what he calls the C-property; that is to say, we can remove a suitable set of points, of content as small as we please, from the fundamental segment, leaving over a complementary set, with respect to which the function is continuous. This property is indeed almost evident for measurable functions assuming only a finite number of values, whence it easily follows for all bounded measurable functions, and again for all measurable functions finite everywhere or almost everywhere, using the theorem, easily proved, that the limit of a sequence of functions possessing the C-property has itself the C-property.

If instead of a fundamental segment we are working only in a set of positive content, say greater than q, the C-property still holds for a measurable function, finite almost everywhere. For we only have to remove a set of content less than q from the continuum, leaving over a complementary set with respect to which our function is continuous, and this complementary set will be bound to have in common with the fundamental set a sub-set of positive content, with respect to which the function

* D. Egoroff, "Sur les suites de fonctions mesurables," 1911, *Comptes Rendus*, Vol. 153. The argument is reproduced in W. H. Young's paper, "On Successions whose Oscillation is usually Finite," 1912, *Quarterly Journal of Math.*, Vol. 44, pp. 131–133. The reader will observe an obvious clerical error of $+\infty$ for $-\infty$ in the enunciation and proof on p. 131, and in the enunciations of the three following theorems, see below, § 13.

† N. Lusin, *loc. cit.*, Summary in "Sur les propriétés des fonctions mesurables," 1912, t. 154, see below, § 17. Conversely a function which has the C-property is easily seen to be a measurable function finite everywhere or almost everywhere.

is continuous. It is by means of this property that I am able to apply to our very general function the reasoning which Lusin, Denjoy, and myself had used in the case of a continuous function.

3. An essential part of our mathematical machinery is a Lemma in the Theory of Sets of Points, introduced by Lusin.*

LUSIN'S LEMMA.—*If S is a set of positive content, it contains a perfect sub-set, dense nowhere,*† *and throughout of positive content, that is such that the part of it in every interval containing one of its points is of positive content.*

If the given set is not dense nowhere, we can change it into such a set by excluding from it all the points which are internal to a set of intervals, dense everywhere and of sufficiently small content. We shall therefore assume that the set S is nowhere dense, so that this condition as to the required sub-set is certainly fulfilled. If the given set is not closed, we may replace it by its nucleus. Thus we may assume that S is closed and dense nowhere.

Let the intervals whose end-points are the rational points be arranged in countable order $d_1, d_2, \ldots$, and consider all those of these intervals which contain points of S forming a sub-set of zero content; such intervals we shall call *zero intervals*. The content of the sub-set of S inside any finite or infinite number of the zero intervals is then zero, since the content is not greater than the sum of the contents in the separate zero intervals. Hence the sub-set S' of S, consisting of the points of S not internal to the zero intervals, is a sub-set of the same content as S. It is moreover clearly a closed sub-set, since none of its limiting points can be internal to the zero intervals, and they all belong to S, since S is closed.

Now the black intervals of S' are tiled over by the zero intervals, and therefore each contains only a sub-set of S of content zero. Moreover the end-points of these black intervals are not internal to zero intervals. Hence no two of these black intervals can abut, for, if so, any one of the intervals $d_1, d_2, \ldots$, one of whose end-points lay in one of the two abutting black intervals, and the other in the other, would be a zero interval, and

* Enunciated without proof in the paper quoted above on p. 361. He gave me a proof in MS. based on Lebesgue integration. D. Mirimanoff gave me another, on the lines of his elegant proof of the theorem of Cantor-Bendixson, "Sur quelques points de la théorie des ensembles," 1914, *Enseignement Math.*, Vol. 16, pp. 29, 30. Denjoy sketches a proof on p. 131 of his recent memoir. All the proofs are perfectly straightforward. The one here given is taken from my Gamble Prize Essay.

† In the present paper the property of being dense nowhere is not required.

would contain the common end-point, which, as we saw, is impossible. Thus the black intervals are non-abutting, and therefore S' is a perfect set.

Now, if P is any point of S', and d any interval containing P, there will be one of the intervals d_1, d_2, ..., containing P inside d, and this will not be a zero interval, and will therefore contain a sub-set of S of positive content. Thus also the part of S' in d has positive content. This proves that S' is throughout of positive content.

We have thus proved the Lemma.

4. The adaptation to our present purposes of the proof, to which reference has already been made, requires something to take the place of Dini's theorem that *the upper and lower bounds of any derivate are the same as those of the incrementary ratio, provided the function itself is continuous.* Generalisations of this theorem have lately been given in a communication to this Society,* but they are not sufficiently general for our present purpose. The Lemmas I use are fourfold. It will, however, only be necessary to enunciate one of them; the others are got by interchanging right and left.

LEMMAS (*vice* Dini's Theorem).—

I. *If S is a perfect set, with respect to which a function*† $f(x)$ *is continuous, and at each point x of which*

$$f^+ < A,$$

then we can find an interval (a, b), containing a part of the set S, such that

$$[f(x+h)-f(x)]/h = R(x, x+h) \leqslant A \quad (h > 0),$$

provided x and x+h both lie in (a, b), and x is a point of the set S.

II. *If S is a perfect set, with respect to which a function $f(x)$ is continuous, and at each point x of which*

$$B < f_+(x),$$

* W. H. Young, "On Integrals and Derivates with respect to a Function," 1914, *Proc. London Math. Soc.*, Ser. 2, Vol. 15, p. 43.

† Unrestricted in character. If we are forming the derivates of $f(x)$ with respect to a monotone function $g(x)$ without constant stretches, we must assume here that both $f(x)$ and $g(x)$ are continuous with respect to the set S. This means, at most, the avoidance of a countable set of points in assigning S, and does not therefore affect the efficiency of the Lemmas in their application.

then we can find an interval (a, b), containing a part of the set S, such that

$$[f(x)-f(x-h)]/h = R(x, x-h) \geqslant B \quad (h > 0),$$

provided x and $x-h$ both lie in (a, b), and x is a point of the set S.

It will be sufficient to prove the former of these lemmas.

Suppose it is not true. Then taking any interval (a_1, b_1) of length $< e$, containing part of the set S, there must be inside it a pair of points x_1 and x_1+h_1, of which x_1 is a point of S, such that

$$R(x_1, x_1+h_1) > A, \quad (h_1 > 0). \tag{2}$$

Since $f(x)$ is continuous at the point x_1 with respect to the perfect set S, there is an interval d_1, with x_1 as one of its end-points (left or right, according as x is or is not a right-hand end-point of a black interval of the set S), containing part of the set S, and such that for each point x inside d_1,

$$R(x, x_1+h_1) > A, \tag{3}$$

d_1 being of length less than (x_1, x_1+h_1), and lying in (a_1, b_1), so that

$$0 < x_1 - x < 2e. \tag{4}$$

Now in d_1 take an interval (a_2, b_2) of length less than $\frac{1}{2}e$, containing part of S, and repeat the argument, and so on. We get a sequence of intervals (a_1, b_1), (a_2, b_2), (a_3, b_3), ..., each contained in the preceding, and with lengths which diminish constantly down towards zero. There is therefore a single point X internal to all of them.

X must be a point of the perfect set S, since in the successive intervals (a_r, b_r) we always have a part of S. Also for each index r, we have a point x_r+h_r such that, by the relations corresponding to (3) and (4),

$$R(X, x_r+h_r) > A, \tag{5}$$

and

$$0 < x_r+h_r-X < 2^{-r+1}\,e. \tag{6}$$

That is we have a sequence of points on the right of X with X as limiting point, for which the incrementary ratio to the point X is greater than A. Hence, by the definition of the right-hand upper derivate,

$$f^{+}(x) \geqslant A.$$

But this is contrary to (1), therefore our supposition is untenable, and the statement first made is true.

5. We shall require one more lemma, but it is well known, and need not be proved here. We only recall the enunciation :—

YOUNG'S FIRST LEMMA.*—*If with each point of an interval as left-hand end-point we are given an interval, or several intervals forming a finite or infinite set, we can find a finite number of these, nowhere overlapping, and such that the sum of the complementary intervals is less than any pre-assigned positive quantity e.*

6. We now go on to the enunciation and proof of the first of our main theorems :—

THEOREM 1.—*If $f(x)$ is any finite measurable function, the points at which the upper right-hand derivate $f^+(x)$ has the value $+\infty$ are identical with the points at which the lower left-hand derivate has the value $-\infty$, exception being made, at most, of a set† of content zero.*

The points in question are none other than the points at which

$$f^+(x) = +\infty, \quad f_-(x) > k_i, \tag{1}$$

for all integers i, where $\qquad k_1 > k_2 > k_3 > \ldots$

is a monotone decreasing sequence of constants with $-\infty$ for limit.

Denoting the set of points at which (1) holds by G_i, we see that the set G_i contains G_{i-1}, for each integer i, and the set G, which we are going to consider, is the outer limiting set of the sets G_i. Therefore, it will be sufficient to prove that G_i always has the content zero, since if this is true for all the sets G_i, it is also true for the outer limiting set.

Now if we write $\qquad g(x) = f(x) - k_i x,$

we have $\qquad g^+(x) = f^+(x) - k_i, \quad g_-(x) = f_-(x) - k_i.$

Therefore the set G_i is also the set of points at which

$$g^+(x) = +\infty, \quad g_-(x) > 0.$$

Also the function $g(x)$ is, like $f(x)$, measurable and finite. Thus our problem is reduced to proving that, when k_i is zero, the content of G_i is zero. We proceed therefore to prove that *the points at which* $f^+(x) = +\infty$, $f_-(x) > 0$ *form a set of content zero.*

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* "On the Existence of a Differential Coefficient," *loc. cit.*, p. 327. Another proof is given in "The Reduction of Sets of Intervals," 1914, *Proc. London Math. Soc.*, Ser. 2, Vol. 14, p. 121.

† This set is of course measurable.

Assume, if possible, that these points form a set S_1 of positive content. Then since $f(x)$ has the C-property (§ 2), we can remove a sub-set of sufficiently small content, leaving over a complementary sub-set S_2, with respect to which $f(x)$ is continuous.

By Lusin's Lemma (§ 3), we can now find a perfect sub-set S_3 of S_2, which is throughout of positive content. At every point of S_2, we have

$$f_-(x) > 0,$$

by (2); hence, by one of the lemmas, *vice* Dini's theorem (§ 4), we can find an interval (a, b), containing a part E of S_2 (which is accordingly of positive content E), such that the incrementary ratio of $f(x)$ over any interval in (a, b) whose right-hand end-point belongs to the set E is $\geqslant 0$.

We may clearly choose a and b to be points of E, then $f(b)$ and $f(a)$ are finite. Now adjoin to each point x which is a left-hand end-point, or an internal point, of a black interval of the set E, the part r_x of that black interval on the right of the point x. For each such interval r_x then the incrementary ratio of $f(x)$ is $\geqslant 0$, and therefore also the increment of $f(x)$ is $\geqslant 0$.

To each of the remaining points of (a, b) other than b, we adjoin all the intervals $(x, x+h)$ to the right of the point x and in (a, b), such that

$$R(x, x+h) > Q, \tag{1}$$

where Q is any chosen positive quantity, as great as we please, say

$$Q > 2[f(b)-f(a)]/E. \tag{2}$$

Now we apply Young's First Lemma. We get a finite number of the intervals $(x, x+h)$ and r_x, nowhere overlapping, and such that the sum of the complementary intervals, say t_x, is less than e, where e is any chosen positive quantity, say $< \frac{1}{2}E$.

Since a and b, as well as the left-hand end-points of each of the chosen intervals, belong to the set E, it follows that the right-hand end-point of each t_x belongs to E; hence the increment of $f(x)$ over each t_x is > 0.

But the intervals t_x with the chosen intervals r_x and $(x, x+h)$ form a finite number of abutting intervals, reaching from a to b, therefore the sum of the increments of $f(x)$ over them is $f(b)-f(a)$. Omitting therefore the increments over the intervals t_x and r_x, as being $\geqslant 0$, we have, by (1),

$$\begin{aligned} f(b)-f(a) &\geqslant \Sigma\{f(x+h)-f(x)\} \\ &\geqslant \Sigma hR(x, x+h) > Q\Sigma h > Q(E-e) > \tfrac{1}{2}QE, \end{aligned} \tag{3}$$

since the sum of the intervals t_x is less than e, and the intervals r_x contain no point of E inside them, so that the intervals $(x, x+h)$ contain a sub-set of E of content greater than $E-e$, that is greater than $\frac{1}{2}E$.

But the last inequality (3) is in contradiction to (2), and is therefore impossible. Thus our assumption is untenable, which proves the theorem.

7. The second main theorem is an immediate result of the preceding in conjunction with my theorem from the *Acta.*

THEOREM 2.—*If $f(x)$ is a measurable function which is finite, except possibly at a set of content zero, the points, if any, at which it has an infinite right (left) hand differential coefficient (with determinate sign), form a set of content zero.*

It is evidently sufficient to prove at length that the points at which

$$f^+(x) = f_+(x) = f'_+(x) = +\infty \tag{1}$$

form a set G of content zero. The other parts of the theorem result at once by the exchange of right and left, upper and lower, $+\infty$ and $-\infty$.

Now, by my theorem in the *Acta*, except at a countable set S_2 of points of G,

$$f^-(x) = +\infty, \tag{2}$$

since

$$f_+(x) = +\infty.$$

Hence, by (2) and the preceding theorem, except at a set S_1 of content zero,

$$f_+(x) = -\infty;$$

but this latter relation is not true at *any* point of G, therefore G itself is a set of content zero, which proves the theorem.

8. We next proceed to prove the third theorem :—

THEOREM 3.—*If $f(x)$ is a finite measurable function, the points, if any, at which one of the upper derivates and one of the lower derivates are finite and different from one another, form a set of content zero.*

In virtue of my theorem from the *Acta* (§ 1), the proof is the same in all the different cases which may occur. In fact, if the upper derivate in question is not on the same side as the lower derivate, *e.g.*, if we are concerned with $f_-(x)$ and $f^+(x)$, we only have to remove a countable set of points, and at the remaining points we shall have

$$f_-(x) \leqslant f^+(x),$$

precisely as if the two derivates referred to the same side.

In the course of the proof we use the Lemmas *vice* Dini's Theorem, and it will, of course, be an appropriate pair of these Lemmas which have to be called into play.

We shall accordingly only give the proof in the case when the two derivates are both on the right.

Assume, if possible, that the set S of points at which $f^+(x)$ and $f_+(x)$ are finite and distinct has positive content.

Let S_r denote the sub-set of S at whose points $f^+(x)$ and $f_+(x)$ lie between $-r$ and r, both bounds excluded, r being any positive integer. Then each set S_r contains its predecessor S_{r-1}, and the outer limiting set is S itself. Therefore for some value of r the content of S_r must be positive, in order that the content of S, which is $\underset{r\to\infty}{\mathrm{Lt}}\ S_r$, should be positive. Let r denote the least integer for which this is the case.

Now write $$f(x) = g(x)+rx,$$

then the derivates of $g(x)$ are got from the corresponding derivates of $f(x)$ by adding r; also $g(x)$ is a function of the same character as $f(x)$. Therefore, replacing $f(x)$ by $g(x)$, and then altering the nomenclature and denoting $g(x)$ by $f(x)$, we see that S_r is the set of points at which

$$0 < f_+(x) < f^+(x) < A, \tag{α}$$

where $A = 2r$.

Since $f(x)$ possesses the C-property (§ 2), and S_r is of positive content, we can remove a sub-set of sufficiently small content, leaving over a complementary sub-set S'_r, of positive content, with respect to which $f(x)$ is continuous.

By Lusin's Lemma (§ 3), we can then find a perfect sub-set S''_r of S'_r, which is throughout of positive content.

Finally, since at each point of S''_r we have

$$f^+(x) < A, \quad \text{and} \quad f_+(x) > 0,$$

we can, by the Lemmas *vice* Dini's Theorem (§ 4), choose a fundamental interval (a, b), so as to contain a part G of S''_r, and such that for each pair of points x and $x+h$ in it, we have

$$\left.\begin{aligned} R(x, x+h) \leqslant A \quad (h > 0) \\ 0 \leqslant R(x, x+h) \quad (h > 0) \end{aligned}\right\}, \tag{β}$$

provided x belongs to the set G.

We may clearly assume a and b to be points of G; thus $f(b)$ and $f(a)$ are both finite.*

* By these preliminary reductions we have reduced the problem to precisely the same as in the case discussed in "The Existence of a Differential Coefficient," *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 329, 330, when $f(x)$ is a monotone increasing function. The remainder of the proof may be taken, word for word, from this earlier paper, the tacit use of Dini's theorem on p. 330, line 14, being replaced by the lemmas of § 4 of the present memoir.

Let G_k denote the sub-set of G at which

$$f^+(x) - f_+(x) > k. \tag{1}$$

Then, if k assumes in succession the values A, $\frac{1}{2}A$, $\frac{1}{4}A$, ..., each set G_k is contained in the following, and the outer limiting set is G. Hence again, in order that our assumption should be tenable G_k must, for one of these values of k, have positive content. Let k then have the greatest such value.

Now the set G_k is itself the sum of the finite number of sets $H_{k,y}$, at which, besides (1), we have

$$\tfrac{1}{2}(y-1)\,k \leqslant f_+(x) < \tfrac{1}{2}yk, \tag{2}$$

y denoting any positive integer up to that for which $\frac{1}{2}(y-1)k = A$. Hence at least one of these sets must have positive content; let y have the least of the values for which this is true. Then there is a perfect sub-set of $H_{k,y}$ of positive content; let us call this E.

Let e be any chosen small positive quantity satisfying the inequality

$$\tfrac{1}{2}kE > e[(y+1)\,k + 2A]. \tag{3}$$

Now divide the fundamental interval (a, b) into a finite number of compartments, namely,

(i) *Black intervals of E, so chosen that the sum of the remaining black intervals is less than e; and*

(ii) *Complementary compartments*, whose sum is accordingly $\geqslant E$ and $\leqslant E+e$, since they contain the whole set E as well as these remaining black intervals. Let there be n of these compartments. We shall consider each of these separately.

To each point x which is an end-point of a black interval of E in the compartment considered, we adjoin that black interval as interval r_x. To each point x of the compartment considered, which is a left-hand end-point or internal point of a black interval of the set E, we adjoin the part r_x of that black interval on the right of the point x. For each such interval r_x then the incrementary ratio of $f(x)$ is $\geqslant 0$ and $\leqslant A$, since the right-hand end-point of r_x belongs to the set E.

At each of the remaining points of the compartment, since it is a point of E, (1) and (2) hold, therefore

$$f_+(x) < \tfrac{1}{2}yk, \tag{4}$$

and

$$\tfrac{1}{2}(y+1)\,k < f^+(x). \tag{5}$$

Therefore we can find intervals $(x, x+h_1)$ and $(x, x+h_2)$, on the right of x, and in our compartment, such that

$$f(x+h_1)-f(x) < \tfrac{1}{2}ykh_1, \tag{6}$$

and

$$\tfrac{1}{2}(y+1)kh_2 < f(x+h_2)-f(x). \tag{7}$$

By Young's First Lemma we can choose out a finite number of the intervals $(x, x+h_1)$ and r_x, nowhere overlapping, and such that the sum of the complementary intervals, say $t_{x,1}$, filling up the compartment, is less than e/n. Then the incrementary ratio of $f(x)$ over each r_x or $t_{x,1}$ is $\leqslant A$, since the right-hand end-point of each such interval belongs to the set E.

Let us do this in each compartment, then the sum of the chosen intervals r_x is, by our choice of the compartments, less than e, and so is the sum of the intervals $t_{x,1}$, since there are n compartments in which they lie, and in each the sum of these intervals is less than e/n.

Now let P_1, p_1, and P denote respectively the sum of the increments of $f(x)$ over the chosen intervals $(x, x+h_1)$, over all the chosen intervals r_x and the intervals $t_{x,1}$, and over the compartments (i). Then, since all these intervals together form a finite number of abutting intervals reaching from a to b,

$$f(b)-f(a) = P_1+p_1+P.$$

But, by (6),

$$P_1 < \tfrac{1}{2}yk(E+e),$$

since the content of the chosen intervals $(x, x+h_1)$ is less than that of the compartment (ii) in which they lie, that is less than $E+e$. Also

$$p_1 < 2Ae,$$

since, as we saw the incrementary ratio over each interval is $\leqslant A$, and the sum of the intervals is less than $2e$.

Hence

$$f(b)-f(a) < \tfrac{1}{2}yk(E+e)+2Ae+P. \tag{8}$$

Similarly, working with the intervals $(x, x+h_2)$, instead of $(x, x+h_1)$, and denoting the sum of the increments of $f(x)$ over the chosen intervals $(x, x+h_2)$, and over the chosen intervals r_x and the intervals $t_{x,2}$, respectively, by P_2 and p_2, we have

$$f(b)-f(a) = P_2+p_2+P.$$

But, by (7),

$$\tfrac{1}{2}(y+1)k(E-e) < P_2,$$

since these intervals contain all the points of E, except a sub-set of content $< e$.

Also $$0 \leqslant p_2,$$

since, the right-hand end-point of each of the intervals r_x and $t_{x,2}$ belonging to the set E, the incrementary ratio over each of these intervals is $\leqslant 0$. Hence

$$\tfrac{1}{2}(y+1)\,k(E-e)+P<f(b)-f(a). \tag{9}$$

Combining (8) and (9),

$$\tfrac{1}{2}(y+1)\,k(E-e)<\tfrac{1}{2}yk(E+e)+2Ae,$$

whence, *a fortiori*, $$\tfrac{1}{2}kE<e[(y+1)k+2A],$$

which is in contradiction to (3).

Thus our assumption is untenable.*

This proves the theorem.

9. If we now examine the cases which may occur at any point x with regard to the extreme derivates $f^+(x)$, $f_+(x)$, $f^-(x)$ and $f_-(x)$ of a measurable function, which is finite almost everywhere, we may distinguish five cases:—

(1) *All the four derivates are finite:* in this case either there is an ordinary finite differential coefficient, or else, by Theorem 3, the point x belongs to a certain set of content zero S_3; in the former case there is no distinction of right and left, since all four derivates are equal and finite.

(2) *One, and only one, of the derivates is infinite:* in this case, by Theorem 1, the point x belongs to a certain set S_1 of content zero.

(3) *Two, and only two, of the derivates are infinite:* in this case, unless the point x belongs to S_1, it must be the upper derivate on one side which is $+\infty$, and the lower derivate on the other side which is $-\infty$; the remaining two derivates being both finite, must be equal, unless the point x belongs to S_3; we notice that, if x does not belong to S_1 or S_3,

* As an example of the simple verbal changes required in the other cases of the enunciation, suppose it is $f_-(x)$ and $f^+(x)$ that are to be considered. The relations (β) then become

$$R(x, x-h) \leqslant A, \quad R(x, x+h) > 0 \quad (h>0).$$

In the construction of the set of intervals (6), however, we must change right into left, and the relation (6) becomes

$$f(x)-f(x-h) < \tfrac{1}{2}kh_1 \quad (h>0).$$

Thus the intervals $t_{x,1}$, have their *left*-hand end-points belonging to the set S, and therefore, as before, it is the former of the two inequalities (β) which applies. We get the same inequality (8) as before, and the rest of the argument is unaltered.

there is again no distinction of right and left; in fact, the extreme derivates on the right are represented geometrically by the vertical through the point x, and a certain other direction, and those on the left by the vertical and the same other direction; on both sides the vertical is directed upwards or downwards, as the case may be, but this gives no geometrical distinction of right and left.

(4) *Three, and only three, of the derivates are infinite:* in this case the point x belongs to S_1 or a countable set S_2, by Theorems 1 and 2.

(5) *All four derivates are infinite:* in this case, if x does not belong to S_1 or S_2, the two upper derivates are $+\infty$, and the two lower derivates are $-\infty$; this latter case again presents no geometrical distinction of right and left.

10. Disregarding the set of content zero constituted by S_1, S_2, and S_3, we have, by the preceding article, only three possibilities at a point x:—

(1) *The upper derivates on each side are $+\infty$, and the lower derivates $-\infty$, that is,*

$$f^+(x) = f^-(x) = +\infty, \quad f_+(x) = f_-(x) = -\infty.$$

If $f(x)$ is a continuous function, every possible direction going out from the point x on the right or on the left, including the upward and downward verticals, will be the limiting directions of chords in the vicinity of the point x; such a point I call *an inextricable knot or tangle* on the locus $y = f(x)$.

(2) *The upper derivate on one side is $+\infty$, the lower derivate on the other side is $-\infty$, and the two remaining extreme derivates are finite and equal;* that is either

$$f^+(x) = +\infty, \quad f_-(x) = -\infty, \quad f_+(x) = f^-(x) = c,$$

or

$$f^-(x) = +\infty, \quad f_+(x) = -\infty, \quad f_-(x) = f^+(x) = c.$$

If $f(x)$ is a continuous function, the limiting directions will then on each side fill up the angle between the upward (downward) vertical and a certain other line; such a point I call *a partial knot or tangle* on the locus.

(3) *There is an ordinary finite differential coefficient:* such a point is *an ordinary point* on the locus.

11. We see then also that we have the following theorem as an imme-

diate corollary to our main theorems :—

THEOREM 4.—*There is no geometrical distinction of right and left with regard to the four extreme derivates $f^+(x)$, $f_+(x)$, $f^-(x)$, $f_-(x)$, of a measurable function $f(x)$, finite almost everywhere, except possibly at a set of content zero.*

Indeed not only is there no distinction of right and left, as pointed out in § 8, except at the points of the sets S_1, S_2, and S_3, but at certain points of S_2 also there is no distinction of right and left; these are the points at which both a right-hand and a left-hand differential coefficient exist, both infinite and opposite in sign. If $f(x)$ is continuous, such a point is *a cusp* on the locus $y = f(x)$, with vertical tangent pointing upwards or downwards; this tangent is the unique limiting direction of chords from the point x to neighbouring points on the locus.

We notice that, when $f(x)$ is continuous, there is no distinction of right and left not only with respect to the extreme derivates, but also with regard to the intermediate derivates at an ordinary point, a cusp, or an inextricable knot, but that this is not the case at a partial knot.

12. In connexion with these general theorems I examined* anew the well-known continuous and non-differentiable function of Weierstrass

$$y = f(x) = \sum_{n=0}^{\infty} b^n \cos a^n x\pi,$$

where a is an odd integer, and b a positive quantity less than unity, these two parameters being connected by the inequality

$$ab > 1 + 3\pi/2.$$

Weierstrass had indeed shown that at *every* point $f(x)$ has one of its upper derivates $+\infty$ and the lower derivate on the other side $-\infty$. Thus a proper differential coefficient, whether finite, or infinite with determinate sign, cannot exist at any point.

By our general theory, however, it is possible that an infinite right-hand or left-hand differential coefficient might exist. Such a point might be a cusp, with vertical tangent, upward or downward, but, by my theorem in the *Acta*, such cusps can, at most, be countably infinite. The remaining points, by Theorem 2, can only form a set of content zero.

I proved that in point of fact such cusps do exist, and form a countably

* These results were communicated to the Société Helvétique on the occasion of its Centenary at Geneva last August. (See *L'Enseignement Math.*, Vol. XVII, p. 348.) The details of the work were incorporated in my Gamble Prize Essay.

infinite set dense everywhere; they are given by

$$x = Ma^{-m},$$

where M and m are any positive integers.

I showed, moreover, that at *every* point except at the cusps, *both* the upper derivates are $+\infty$, or *both* the lower derivates are $-\infty$. The set S_3 is therefore the null set.

Now in the case when $f(x)$ is a continuous function, W. H. Young* has proved that the points at which the two upper derivates are not equal to one another, and the two lower derivates not equal to one another, form a set of the first category. This set, being in Weierstrass's case, the set S_1+S_2, and in the general case of a continuous $f(x)$ the sets $S_1+S_2+S_3$, is therefore a set of the first category and content zero.

I showed that, in Weierstrass's case, this exceptional set is either identical with or contained in a certain set, which is easily proved to be of the first category and content zero. The points of this set are characterised by

$$x = j_0 \cdot j_1 \ldots j_{m-1} j_m j_{m+1} \ldots \textit{ad inf.}$$

when x is expressed in the scale whose base is the parameter a, when for some integer m, depending on x, all the digits $j_{m+1}, j_{m+2}, \ldots$ are even.

It appears from this investigation that, regarded as a curve $y = f(x)$, Weierstrass's curve is not really a curve without tangents, but has, in every interval an infinity of cusps with vertical tangents.

Cellérier, on the other hand, who found independently of Weierstrass, and possibly even before him,† a non-differentiable function, has defined for us *a curve entirely without tangents.* The equation to this curve is

$$y = f(x) = \sum_{n=1}^{\infty} a^{-n} \sin a^n x,$$

where a is a large positive even integer.‡

12. Now let us examine what modifications are necessary in our theorems, if we remove from $f(x)$ the conditions (a) and (b). We must

* See the paper quoted above in § 1, and also "On the Derivates of Non-Differentiable Functions," 1908, *Messenger of Mathematics*, Vol. 38, pp. 65–69.

† C. Cellérier, "Note sur les principes fondamentales d'analyse" (date uncertain, Prof. Raoul Pictet of Geneva says, according to his memory of a conversation with Cellérier, it must have been as early as 1860), published after the author's death, *Bull. des Sc. Math.* (2), Tome 14 (1890).

‡ It will be seen that I had thus already solved the problem left to the reader by Denjoy on p. 210 of his memoir, and cleared up the point he leaves in doubt as to the possession of unilateral differential coefficients in the case of Weierstrass's function.

then, of course, revise the definitions of the derivates, since the expression for the incrementary ratio has no meaning when $f(x)$ and $f(x+h)$ are both infinite with the same sign. I shall therefore at once put aside all functions which are infinite at every point of any interval. Thus *the set of points at which $f(x)$ is finite is supposed to be dense everywhere.*

Considering then only such pairs of points x and $x+h$ for which the incrementary ratio $R(x, x+h)$ has a definite value, the derivates may be defined as before, and we shall still have everywhere

$$f_+(x) \leqslant f^+(x), \quad f_-(x) \leqslant f^-(x). \tag{1}$$

The discussion given by me in the *Acta* then only requires this modification that the points at which $f(x)$ is infinite form an exception. At such a point we have evidently

$$f(x) = +\infty, \quad f_+(x) = f^+(x) = -\infty, \quad f_-(x) = f^-(x) = +\infty, \tag{2}$$

or $$f(x) = -\infty, \quad f_+(x) = f^+(x) = +\infty, \quad f_-(x) = f^-(x) = -\infty.$$

We have accordingly the following extended form of my theorem in the *Acta*:—

THEOREM.—*The set of points at which the upper derivate on one side is less than the lower derivate on the other side contains the set of points at which the primitive function $f(x)$ is infinite; the remaining points, if any, form a countable set.*

Thus excepting the points considered in (2), and possibly also a countable set of points at which $f(x)$ is finite, we have

$$f_+(x) \leqslant f^-(x), \quad f_-(x) \leqslant f^+(x). \tag{3}$$

13. If we now examine the arguments of the present paper, it appears that, having so defined the derivates, our three main theorems remain unaltered, provided $f(x)$ is only infinite at a set of content zero. Thus, without any further ado, *we can change the condition (b) into the wider condition that $f(x)$ should be finite almost everywhere.*

14. Now let us examine what modifications are necessary in our three main theorems, if we further relax the condition (b) laid on our measurable function $f(x)$, and only demand that

(c) *$f(x)$ should be finite at a set of positive content.*

It appears on examination that we have nowhere in our reasoning used the fact that $f(x)$ is, except possibly at a set of content *precisely zero,* finite, except in applying the C-property. If instead of the fundamental interval, we work with the fundamental set obtained by abstracting from the fundamental interval the points at which $f(x)$ is infinite, the former reasoning would still apply, provided the C-property still held in this fundamental set, which, by our condition (c), is a set of positive content.

It will be seen that this reasoning is perfectly valid, as our function does really possess the C-property in this fundamental set.

Thus it appears that our three main theorems remain true in this fundamental set.

15. To obtain these results it is necessary to adapt the work of Egoroff and Lusin referred to in § 2. We have first of all the following auxiliary theorem :—

THEOREM.—*If $f_1(x)$, $f_2(x)$, ... is a succession of finite functions,* which diverges properly to $-\infty$ at a set of points S (which may be the null-set), and $G_{n,e}$ denotes the set of points x at which the upper bound $\bar{R}_n(x)$ of*

$$f_m(x)-u(x) \quad (m \geqslant n),$$

is greater than e, then the set $G_{n,e}$ contains the set $G_{n+1,e}$, for all values of n, and the inner limiting set of the sets $G_{n,e}$ is the set S.

The set $G_{n,e}$ is, in other words, the set such that, corresponding to each of its points x, there is a value of p, such that

$$f_{n+p}(x)-u(x) > e. \tag{1}$$

Hence, if x is a point of $G_{n+1,e}$, there is a value of p, such that

$$f_{n+1+p}(x)-u(x) > e,$$

which shows that x is a point of $G_{n,e}$, and that the integer $p+1$ serves our purpose to prove this.

Now at a point of the set S, the left-hand side of (1) is always $+\infty$, so that the inequality is satisfied for all integers p. Thus the set S is certainly contained in all the sets $G_{n,e}$. It remains therefore only to show that every point common to all the sets $G_{n,e}$ is a point of S.

Let then k be any point of all the sets $G_{n,e}$. Then for a certain increasing succession of integers m, we must have

$$f_m(x)-u(x) > e. \tag{2}$$

Now let us proceed to the limit with m. By the definition of $u(x)$, the

* Not necessarily measurable.

upper limit of $f_m(x)$ cannot be greater than $u(x)$. Also, by (2), it cannot be less than $u(x)$. Finally, by (2), it cannot be equal to $u(x)$ and finite. There only remains the possibility that it is equal to $u(x)$ and infinite. But in this latter case the infinite value cannot be positive, or the left-hand side of (2) would always be negative, which is a contradiction. Hence the infinite value must be negative, which proves that x is a point of the set S.

This proves the theorem.

Similarly we have the alternative theorem :—

THEOREM.—*If $l(x)$ is the lower function of $f_1(x), f_2(x), \ldots$, a succession of finite functions, which diverges properly to $+\infty$ at a set of points Z (which may be the null-set), and $H_{n,e}$ denotes the set of points x at which the upper bound of*

$$l(x)-f_m(x) \quad (m \geqslant n),$$

is greater than e, then the set $H_{n,e}$ contains the set $H_{n+1,e}$, for all values of n, and the inner limiting set of the sets $H_{n,e}$ is the set Z.

16. We can now prove the extension of Egoroff's theorem :—

THEOREM.—*If a succession of finite measurable functions converges at a set T of positive content, we can find a sub-set of T of content as small as we please, such that, with respect to the complementary sub-set of T, the succession converges uniformly, this complementary sub-set being moreover perfect.*

Let $u(x)$ and $l(x)$ be the upper and lower functions of the succession, which are known to be equal and finite at the points of T, and let their common value at each such point be denoted by $f(x)$. Let S and Z denote, as before, the sets at which $u(x) = -\infty$ and $l(x) = +\infty$ respectively. Then S and Z have no common points, and neither of them has points common with T.

Let $G_{n,e}$ denote, as before, the set of points at which the upper bound $\bar{R}_n(x)$ of

$$f_m(x)-u(x) \quad (m \geqslant n),$$

is greater than e. Then, by the first theorem of the preceding article, S is a sub-set of $G_{n,e}$, and, if the remaining sub-set be denoted by $G'_{n,e}$, each set $G'_{n,e}$ contains the next $G'_{n+1,e}$, and there is no point common to all the sets $G'_{n,e}$.

Now since the functions $f_m(x)$ are measurable, so is $u(x)$, and therefore so is $\bar{R}_n(x)$; therefore the sets $G_{n,e}$ are measurable sets. Hence the content* $G'_{n,e}$ has as limit, when n increases indefinitely, the content of the inner limiting set of the sets $G'_{n,e}$, that is zero.

Therefore if e assume in succession the values $e_1, e_2, \ldots$, forming a monotone descending sequence with zero as limit, we can, corresponding to each index i, determine an integer n_i, so that the content

$$G'_{n_i, e_i} < 2^{-k-i-1},$$

where k is as large a positive integer as we please.

Adding together all these sets G'_{n_i, e_i}, we get a set G, whose content† is not greater than the sum of the content of the composing sets G'_{n_i, e_i}, that is, less than 2^{-k}, which is as small as we please.

If x is any point not belonging to G' nor to S, it is not a point of G'_{n_i, e_i} nor of S, and is therefore not a point of G_{n_i, e_i}: that is

$$f_m(x) - u(x) \leqslant e_i \quad (m \geqslant n_i), \tag{i}$$

and this is true for all indices i.

Similarly, using the second theorem of the preceding article, we obtain a set H' of content as small as we please, such that, if x is any point not belonging to H' nor to Z,

$$l(x) - f_m(x) \leqslant e_i \quad (m \geqslant N_i), \tag{ii}$$

and this is true for all indices i.

Now if x is a point not belonging to any of the sets S, Z, G', and H', both the inequalities (i) and (ii) hold for all values of $m \geqslant m_i$, where m_i is the greater of n_i and N_i. Also if x is a point of the set T, the succession converges to $f(x)$. We have therefore

$$-e_i \leqslant f(x) - f_m(x) \leqslant e_i \quad (m \geqslant m_i), \tag{iii}$$

and this is true for all indices i. Thus the double inequality (iii) holds at all points of T which do not belong to the sets G' and H', of content as small as we please. Since T is by hypothesis of positive content, the double inequality (iii) therefore certainly holds at a sub-set of T of positive content, as near as we please to that of T.

Since this sub-set is of positive content, it contains a perfect sub-set (its nucleus) of the same content. Therefore the inequality (iii) holds at every point of this perfect sub-set.

But the double inequality (iii) expresses exactly the uniform conver-

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* *Quâ* inner content.

† *Quâ* outer content. This and the preceding footnote show why the attempt to prove this theorem for non-measurable functions fails.

gence of the succession to $f(x)$ with respect to this perfect sub-set. This therefore proves the theorem.

17. Hence we may state the extended form of the necessary and sufficient condition of Weyl-Egoroff that a succession should converge almost everywhere.

CONDITION.—*A necessary and sufficient condition that a succession of finite measurable functions should converge except possibly at a set of content k, less than that of the fundamental segment, is that it should be possible to find a set of content less than $k+e$, with respect to whose complementary set the succession converges uniformly, whatever sufficiently small positive quantity e may be.*

The necessity of the condition is indeed proved in the preceding theorem. To prove the sufficiency, suppose the condition fulfilled, and let $e_1 > e_2 > \ldots$ be any monotone descending sequence with zero as limit. Then we can find sets G_1, G_2, ..., respectively of content less than $k+e_1$, $k+e_2$, ..., with respect to whose complementary set the succession converges uniformly. If E denote the set of all the points not belonging to any of these sets, each point of E will belong to the complementary set of at least one of them, so that the succession will converge there. On the other hand, the content of the set complementary to E, which is a set contained in all the sets G_1, G_2, ..., is not greater than the content of each of these sets, and is therefore $< k+e_i$, for all indices i, that is, it is $\leqslant k$. This proves the sufficiency of the condition.

18. I denote by the (C, k)-property of a function $f(x)$ that we can remove from the fundamental segment a suitable set of content less than $k+e$, where e is any sufficiently small positive quantity, and with respect to the remaining set the function $f(x)$ is continuous.

If $f(x)$ *is a measurable function, and infinite at a set of content* k, less than the content of the fundamental segment, the set of its infinities must of course be included in the set removed, if $f(x)$ possesses the (C, k)-property. Thus for such a function the (C, k)-property is the same as the C-property in the fundamental set obtained by removing from the continuum the infinities of $f(x)$.

What we require, therefore, is simply to prove that *such a function $f(x)$ possesses the (C, k)-property.*

Let then $f(x)$ denote a measurable function, finite except possibly at a set S of content k, less than the length of the fundamental segment (a, b).

Let $f_i(x)$ denote the function which is equal to $f(x)$, wherever

$|f(x)| \leqslant i$, and has the value $+i$ or $-i$ elsewhere, the sign being the same as that of $f(x)$. Then the function $f_i(x)$ is certainly measurable, for with the exception of the values $+i$ and $-i$, the sets of points at which it has any value are the same as for $f(x)$, and the sets at which it has the value $+i$ or $-i$ are measurable, since they are the sets at which $f(x)$ is $\geqslant +i$ and $\leqslant -i$ respectively.

Also giving i all positive integral values, we get a sequence of bounded measurable functions converging or diverging to $f(x)$ everywhere. Hence we may apply the extension of Egoroff's theorem given above (§ 16), T being now the set of points where $f(x)$ is finite. We can, therefore, find a sub-set T_e of T of content less than $\frac{1}{2}e$, where e is as small as we please, such that with respect to the remaining sub-set of T the sequence converges uniformly, this remaining set, say U, being moreover perfect.

If therefore we show (as Lusin has already done), that a bounded measurable function has the C-property, that is the $(C, 0)$-property, it follows at once that our function $f(x)$ has the (C, k)-property, for it will be continuous with respect to a sub-set U' of U, of content as near as we please to that of U, with respect to which each of the functions $f_i(x)$ is continuous. Indeed we only have to find a set S_i of content less than $2^{-i-1}e$ corresponding to $f_i(x)$, in accordance with the C-property, and form the set of all the sets S_i, for all indices i, and denote the sub-set of U containing no point of this latter set by U'.

19. It remains therefore only to give Lusin's proof of the fact that any bounded measurable function possesses the C-property.

Now, as is well known, if $f(x)$ is a bounded measurable function, it is the limit of a monotone ascending sequence of measurable functions, each having only a finite number of values. For instance, if K be the lower and K' the upper bound of $f(x)$, and we write

$$e_n = (K-K')/n,$$

and
$$k_r = K + re_n \quad [r = 0, 1, \ldots, (n-1)],$$

we may define the constituent $f_n(x)$ of the required sequence as follows:—

$$\text{Wherever} \quad k_{n-1} \leqslant f < K', \quad \text{we have} \quad f_n = k_{n-1},$$

$$k_{n-2} \leqslant f < k_{n-1}, \quad \text{,,} \quad f_n = k_{n-2},$$

and so on, finally, where

$$K \leqslant f < k_1, \quad \text{,,} \quad f_n = K.$$

It is unnecessary to discuss this further, as it is familiar.

Hence, again, as before, it is only necessary to prove that such constituent functions possess the C-property.

But a measurable function which has only a finite number of values is evidently the sum of a finite number of functions each having only the values 0 and 1, each multiplied by a suitable constant. Hence the property only requires to be proved for such a simple function.

Now such a function is 1 at a measurable set S_1, and 0 at the complementary measurable set S_0. In S_1 take a closed set S_1' of content as near as we please to that of S_1, and in S_0 a closed set S_0', of content as near as we please to that of S_0. Then with respect to the set $S_1'+S_0'$ our function is continuous, which proves that it has the C-property.

This completes the proof of the assertion that a measurable function $f(x)$, which is infinite at a set of content k, less than that of the fundamental segment, possesses the (C, k)-property, and therefore possesses the C-property in the set got by removing from the fundamental segment all the infinities of $f(x)$.

20. Using the (C, k)-property, therefore, in place of the C-property, we get, without change in the reasoning, extensions of our three main theorems, in which the points referred to are not points at which $f(x)$ is infinite.

As, however, at a point at which $f(x) = +\infty$ $(h > 0)$,

$$[f(x+h)-f(x)]/h = -\infty,$$

and

$$[f(x)-f(x-h)]/h = +\infty$$

$[f(x+h)$ being different from $+\infty]$, it follows that, supposing $f(x)$ not to be infinite throughout any interval, at a point where $f(x) = +\infty$, we have the right-hand differential coefficient $-\infty$, and the left-hand differential coefficient $+\infty$. At a point where $f(x) = -\infty$, the signs of these infinities are, of course, reversed. Thus such points do not affect the results in Theorem 1, Theorem 3, and Lusin's Theorem, which remain true for our more general function. Theorem 2, however, takes the following form.

THEOREM 2, *bis*.—*The points at which $f(x)$ has an infinite right-hand (left-hand) differential coefficient consist of the infinities of $f(x)$ and possibly of an additional set of content zero.*