THE DYNAMICAL THEORY OF THE TIDES IN A ZONAL OCEAN

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1. *Introduction.*

This paper is an extension of one already published,* and deals with the tides in an ocean bounded by two circles of latitude on a rotating globe.

The introduction of two boundaries into the problem causes very great complexity; and it has been found necessary to omit the discussion of many points of interest, in particular, the nature and magnitude of the free periods of oscillation.

I am again much indebted to Dr. T. H. Havelock for his advice and assistance.

Originally, in discussing the results, I had made numerical comparisons between the dynamical tides and the uncorrected "equilibrium" tides. But, at the suggestion of a referee, to whom the paper was submitted, I have substituted the "corrected equilibrium" values. A remarkable agreement, exhibited and explained on p. 223 results. At his suggestion I have also added a note making a comparison between the dynamical tide heights found in the previous paper for a polar basin, and the corresponding "corrected equilibrium" values.

2. *The Equations for the Tides and their Solutions.*

The equations which express the small oscillations of a liquid on a

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* *Proc. London Math. Soc.*, p. 31 of the present volume.

rotating sphere are, in the usual notation,

$$\left.\begin{aligned} u &= \frac{\sigma}{4m(f^2-\mu^2)}\left\{\sqrt{(1-\mu^2)}\,\frac{d\zeta'}{d\mu} - \frac{s}{f}\,\frac{\mu\zeta'}{\sqrt{(1-\mu^2)}}\right\} \\ v &= \frac{\sigma}{4m(f^2-\mu^2)}\left\{\frac{\mu}{f}\sqrt{(1-\mu^2)}\,\frac{d\zeta'}{d\mu} - \frac{s\zeta'}{\sqrt{(1-\mu^2)}}\right\} \\ \frac{\sigma a\zeta}{h} &= -\frac{d}{d\mu}\left\{u\sqrt{(1-\mu^2)} - \frac{sv}{\sqrt{(1-\mu^2)}}\right\} \end{aligned}\right\}. \tag{1}$$

In these we have taken out the time-and-longitude factor, and put $\mu = \cos\theta =$ sine of latitude.

The part of the tide-producing potential which depends upon μ is, for the three species :

(a) $H_0(\mu^2-\frac{1}{3})$,

(b) $H_1\mu\sqrt{(1-\mu^2)}$,

and (c) $H_2(1-\mu^2)$.

The form of these quantities suggests the substitution

$$\zeta = (1-\mu^2)^{s/2}\,z,$$

$$\bar{\zeta} = (1-\mu^2)^{s/2}\,\bar{z},$$

and

$$\zeta' = (1-\mu^2)^{s/2}\,z',$$

where $s = 0$, 1 or 2, according to the species of tide.

The first pair of equations (1) then becomes

$$\left.\begin{aligned} u\sqrt{(1-\mu^2)} &= \frac{\sigma(1-\mu^2)^{s/2}}{4m(f^2-\mu^2)}\left\{(1-\mu^2)\frac{dz'}{d\mu} - s\mu z'\left(1+\frac{1}{f}\right)\right\} \\ \frac{v}{\sqrt{(1-\mu^2)}} &= \frac{\sigma(1-\mu^2)^{s/2}}{4m(f^2-\mu^2)}\left\{\frac{\mu}{f}(1-\mu^2)\frac{dz'}{d\mu} - sz'\left(1+\frac{\mu^2}{f}\right)\right\} \end{aligned}\right\}. \tag{2}$$

On substituting these in the third equation (1), we have

$$\beta z = \frac{d}{d\mu}\left[\frac{s\mu z'\left(1+\frac{1}{f}\right) - (1-\mu^2)\frac{dz'}{d\mu}}{f^2-\mu^2}\right] + \frac{s^2z' + s\mu\left(1-\frac{1}{f}\right)\frac{dz'}{d\mu}}{f^2-\mu^2}; \tag{3}$$

where

$$\beta = \frac{4ma}{h} = \frac{4\omega^2a^2}{gh}.$$

The coefficients of equation (3) are regular in the vicinity of $\mu = 0$;

but they have singularities at the points

$$\mu = \pm f,$$

$$\mu = \pm 1,$$

and

$$\mu = \infty.$$

The points $\mu = \pm f$ may be shown to be "apparent singularities."* At the points $\mu = \pm 1$, we have indices 0 and 1; while the point $\mu = \infty$ does not appear in the problem before us.

In choosing the form of the solution, the nature of the zone must be considered. If the pole $\mu = +1$ is included, a series of the type

$$z = (\mu-1)[A_0+A_1(\mu-1)+A_2(\mu-1)^2+\ldots]$$

would be most useful. As such a series would usually be convergent up to the other singularity $\mu = -1$, this would be a comprehensive form including all polar basins, even those covering more than a hemisphere. The period equation would be given by the expression which states that the meridional velocity is zero along a certain circle of latitude. For a zone bounded by two circles of latitude, however, this series is not convenient; as, since the indices differ by an integer, the companion integral will contain a logarithm. In spite of the obvious advantages of such a series in the case of a polar basin, no use has been made of it; for the recurrence relation between the coefficients is very complex and difficult to handle: even in the simplest case (the tides of the first species) this relation contains four terms.

If series of powers of μ be used, we get two comparatively simple integrals, one an odd and the other an even function of μ. These are convergent up to the singularities $\mu = \pm 1$, and so can be applied to any zone bounded by two circles of latitude. But if we wish to include the poles, a second and rather troublesome condition must be added. In what follows, series of the latter class alone are used; and the basins are, consequently, such as do not include the poles.

By putting, in (3),

$$s\mu z'\left(1+\frac{1}{f}\right)-(1-\mu^2)\frac{dz'}{d\mu} = X(f^2-\mu^2), \tag{4}$$

* *Loc. cit.*, p. 34.

we find
$$\beta z = \frac{sX + \dfrac{s}{f^2}\dfrac{dz'}{d\mu}}{\mu\left(1+\dfrac{1}{f}\right)} + \frac{dX}{d\mu}. \tag{5}$$

Now assume
$$z' = \sum_0^\infty A_n \mu^n,$$
$$\bar{z} = \sum_0^\infty B_n \mu^n,$$
and
$$X = \sum_0^\infty C_n \mu^n.$$

We then find
$$\left\{s+n\left(1+\frac{1}{f}\right)\right\} C_n = \left(1+\frac{1}{f}\right)\beta(A_{n-1}+B_{n-1}) - \frac{s}{f^2}(n+1)A_{n+1}:$$

and, on substituting in (4) and eliminating the C's,
$$A_{n+2}\frac{(n+1)(n+2)}{s+(n+1)(1+1/f)}$$
$$+A_n\left\{-s-\frac{ns(1-1/f)+n(n-1)}{s+(n-1)(1+1/f)}+\frac{\beta f^2}{s+(n+1)(1+1/f)}\right\}$$
$$-A_{n-2}\frac{\beta}{s+(n-1)(1+1/f)}$$
$$= -B_n\frac{\beta f^2}{s+(n+1)(1+1/f)} + B_{n-2}\frac{\beta}{s+(n-1)(1+1/f)}. \tag{6}$$

If the ocean is bounded by circles of latitude, we must have, at each, $u = 0$; or
$$\sum_{n=0}^\infty\left[\frac{\beta(1+1/f)(A_{n-1}+B_{n-1}) - s/f^2\,(n+1)\,A_{n+1}}{s+n(1+1/f)}\right]\mu^n = 0. \tag{7}$$

But if the ocean extends up to and includes the pole, this condition is to be replaced by the statement that the series $\Sigma A_n\mu^n$ is convergent with its first and second derivatives for $\mu = \pm 1$.

The values of B entering into the tide-raising potential are confined to B_0, B_1, and B_2; hence, in discussing the convergence of the series

$\Sigma A_n \mu^n$, we may omit the right-hand side of (6). We then find

$$A_{n+2}/A_n = \left\{ s + \frac{ns(1-1/f)+n(n-1)}{s+(n-1)(1+1/f)} - \frac{\beta f^2}{s+(n+1)(1+1/f)} - \frac{\beta}{s+(n-1)(1+1/f)} \frac{1}{A_n/A_{n-2}} \right\} \frac{s+(n+1)(1+1/f)}{(n+1)(n+2)}.$$

Whatever finite value f may have, it is clear that A_{n+2}/A_n tends to the limit 0 or 1. In the first event, the series $\Sigma A_n \mu^n$ is valid up to and including the poles. In the second we have, more precisely,

$$\operatorname*{Limit}_{m\to\infty} A_{2m+2}/A_{2m} = 1 - \frac{1}{m} + \frac{\omega_m}{m^2},$$

where ω_m is less than a fixed finite quantity for all values of m greater than a given value. Hence, in this case, the series is only valid so long as $|\mu| < 1$; that is, up to, but not at, the pole. It is evident then, that if we wish to determine a series that will express the height of the tide at the poles we must determine the arbitrary quantities so that the condition

$$\operatorname*{Limit}_{n\to\infty} A_{n+2}/A_n = 0, \tag{8}$$

is fulfilled.

It is further clear from (6) that two independent series are represented, viz.,

(i) $A_0 + A_2\mu^2 + A_4\mu^4 + \ldots + A_{2r}\mu^{2r} + \ldots,$

and (ii) $A_1\mu + A_3\mu^3 + A_5\mu^5 + \ldots + A_{2r+1}\mu^{2r+1} + \ldots.$

The zonal oceans that can be discussed by means of the preceding analysis fall into two sets which are best examined separately as follow:

(i) *A Zonal Ocean bounded by Two Parallels of Latitude Symmetrically Placed with regard to the Equator.*

For convenience, write

$$L_n = \left\{ -s - \frac{ns(1-1/f)+n(n-1)}{s+(n-1)(1+1/f)} + \frac{\beta f^2}{s+(n+1)(1+1/f)} \right\} \frac{s+(n+1)(1+1/f)}{(n+1)(n+2)},$$

$$M_n = \frac{\beta}{s+(n-1)(1+1/f)} \frac{s+(n+3)(1+1/f)}{(n+3)(n+4)},$$

and

$$\alpha'_n = \frac{-s/f^2 \,.\, n\mu_1^{n-1}}{s+(n-1)(1+1/f)} + \frac{\beta(1+1/f)\,\mu_1^{n+1}}{s+(n+1)(1+1/f)}. \tag{9}$$

Also let the boundaries be the circles of latitude $\text{arc}\sin\mu_1$ and $\text{arc}\sin(-\mu_1)$.

In seeking for comprehensive expressions that will represent the facts of all three tide-species, the differences of the three forms of tide-producing potential must be noted. For the first species B_0 and B_2 both appear, for the second B_1, and for the third B_0, only. The introduction of the two coefficients for the long period tides complicates the general process, so that a slight variation of the method, given later, has been found advisable. But for the second and third species, where only one coefficient appears, we shall give but one expression, and we shall suppose that only B_0 is present. To cover species 2, this will simply necessitate placing the additional quantities in the right-hand side of the equations for the odd terms instead of the right-hand side of the equations for the even terms. With this understanding the following analysis may be regarded as comprehending the two cases $s = 1$ and $s = 2$.

Let $$B_0 = H_s = \kappa A_0.$$

Then from (6) and (7), the equations to be satisfied are

$$\left.\begin{array}{r}\left(\frac{\kappa\beta f^2}{2}+L_0\right)A_0 \qquad\qquad + A_2 = 0\\ \left\{\frac{-\beta(s+3+3/f)\kappa}{(s+1+1/f)\,3.4}-M_0\right\}A_0+L_2A_2+A_4 = 0\\ -M_2A_2+L_4A_4+A_6 = 0\\ \dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\end{array}\right\}, \qquad (10a)$$

$$\left.\begin{array}{r}L_1A_1+A_3 = 0\\ -M_1A_1+L_3A_3+A_5 = 0\\ -M_3A_3+L_5A_5+A_7 = 0\\ \dots\quad\dots\quad\dots\quad\dots\quad\dots\end{array}\right\}, \qquad (10b)$$

$$\left.\begin{array}{r}\left\{\frac{(1+1/f)\beta\kappa\mu_1}{s+1+1/f}+\alpha_0'\right\}A_0+\alpha_1'A_1+\alpha_2'A_2+\dots = 0\\ \left\{-\frac{(1+1/f)\beta\kappa\mu_1}{s+1+1/f}-\alpha_0'\right\}A_0+\alpha_1'A_1-\alpha_2'A_2+\dots = 0\end{array}\right\}. \qquad (10c)$$

This group of equations represents both the free and the forced oscillations. For, in the case of the former, $\kappa = 0$ and f can then be determined; in the case of the latter, f is known and κ (or A_0) can be determined.

It is easily seen that, as in the case of an ocean wholly covering the globe, there are two possible systems of free oscillations: the first is characterized by symmetry with regard to the equator and has a non-zero height there; the second is asymmetric with regard to the equator and has a zero height there.

By a process of step-by-step elimination of the quantities A from (10a) and (10c), we have, in the limit, the following infinite determinantal equation

$$\begin{vmatrix} \dfrac{(1+1/f)\beta\kappa\mu_1}{s+1+1/f}+a_0' & a_2' & a_4' & a_6' & \dots \\ \dfrac{\kappa\beta f^2}{2}+L_0 & 1 & 0 & 0 & \dots \\ -\dfrac{(s+3+3/f)\beta\kappa}{(s+1+1/f)3.4}-M_0 & L_2 & 1 & 0 & \dots \\ 0 & -M_2 & L_4 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = 0. \qquad (11)$$

Taking advantage of the notation

$$\Delta_n \equiv \begin{vmatrix} a_n' & a_{n+2}' & a_{n+4}' & \dots \\ L_n & 1 & 0 & \dots \\ -M_n & L_{n+2} & 1 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix},$$

to which may be added the relation between successive Δ's,

$$\Delta_n' = a_n' - L_n\Delta_{n+2}' + M_n\Delta_{n+4}';$$

we find the following reduced form of equation (11)

$$\Delta_0' = -\frac{\beta\kappa\mu_1(1+1/f)}{s+1+1/f} + \frac{\beta f^2\kappa}{2}\Delta_2' - \frac{(s+3+3/f)\beta\kappa}{(s+1+1/f)\,3.4}\Delta_4',$$

or $$A_0\Delta_0' = \left[-\frac{\beta\mu_1(1+1/f)}{s+1+1/f} + \frac{\beta f^2}{2}\Delta_2' - \frac{(s+3+3/f)\beta}{(s+1+1/f)\,3.4}\Delta_4'\right] H_s. \qquad (12)$$

Equation (12) shows that the free periods are given by the transcendental equation

$$\Delta_0' = 0.$$

A corresponding equation can be written down from (10b) and (10c) for the other system of free oscillations.

(ii) *A Zonal Sea bounded by Two Circles of Latitude not Symmetrically Placed with regard to the Equator.*

Let the circles of latitude be arc sin (μ_1) and arc sin (μ_2). The equations to be satisfied are then (10a), (10b), and

$$\left.\begin{aligned}\left\{\frac{(1+1/f)\beta\kappa\mu_1}{s+1+1/f}+\alpha_0'\right\}A_0+\alpha_1'A_1+\alpha_2'A_2+\ldots=0\\ \left\{\frac{(1+1/f)\beta\kappa\mu_2}{s+1+1/f}+\alpha_0''\right\}A_0+\alpha_1''A_1+\alpha_2''A_2+\ldots=0\end{aligned}\right\}. \qquad (13)$$

By the same process as before, we find that, in order that (10a), (10b), and (13) may be consistent, we must have

$$A_0\begin{vmatrix}\dfrac{(1+1/f)\beta\kappa\mu_1}{s+1+1/f}+\alpha_0' & \alpha_2' & \alpha_4' & \alpha_6' & \ldots\\ \dfrac{\beta\kappa f^2}{2}+L_0 & 1 & 0 & 0 & \ldots\\ -\dfrac{(s+3+3/f)\beta\kappa}{(s+1+1/f)\,3.4}-M_0 & L_2 & 1 & 0 & \ldots\\ 0 & -M_2 & L_4 & 1 & \ldots\\ \ldots & \ldots & \ldots & \ldots & \ldots\end{vmatrix}$$

$$+A_1\begin{vmatrix}\alpha_1' & \alpha_3' & \alpha_5' & \alpha_7' & \ldots\\ L_1 & 1 & 0 & 0 & \ldots\\ -M_1 & L_3 & 1 & 0 & \ldots\\ 0 & -M_3 & L_5 & 1 & \ldots\\ \ldots & \ldots & \ldots & & \ldots\end{vmatrix}=0; \qquad (14)$$

together with a similar equation, in which μ_1 is replaced by μ_2.

In the same manner as before these reduce to

$$\left.\begin{aligned}A_0\Delta_0'+A_1\Delta_1'&=\left[-\frac{(1+1/f)\beta\mu_1}{s+1+1/f}+\frac{\beta f^2}{2}\Delta_2'-\frac{(s+3+3/f)\beta.\Delta_4'}{(s+1+1/f)\,3.4}\right]H_s\\ A_0\Delta_0''+A_1\Delta_1''&=\left[-\frac{(1+1/f)\beta\mu_2}{s+1+1/f}+\frac{\beta f^2}{2}\Delta_2''-\frac{(s+3+3/f)\beta.\Delta_4''}{(s+1+1/f)\,3.4}\right]H_s\end{aligned}\right\}. \qquad (15)$$

For the free oscillations, the right-hand members disappear, and there remains a pair of simultaneous transcendental equations which determine f and the ratio A_1/A_0. There will be a double infinity of free periods. Equations (10a) and (10b) complete the solution by determining all the coefficients A in terms of A_0, which is left arbitrary. In the case of the forced oscillations, (15) determine A_0 and A_1 uniquely, and the solution is completed again in the same manner.

If one boundary be the equator itself, say $\mu_2 = 0$, we see from (9) that all the quantities α_n'', excepting α_1'', are zero; and

$$\alpha_1'' = -\frac{s/f^2}{s} = -\frac{1}{f^2}.$$

For such a basin, the period equations will be (14), and

$$A_0 \begin{vmatrix} 0 & 0 & 0 & \dots \\ L_0 & 1 & 0 & \dots \\ -M_0 & L_2 & 1 & \dots \\ \dots & \dots & & \dots \end{vmatrix} + A_1 \begin{vmatrix} -1/f^2 & 0 & 0 & \dots \\ L_1 & 1 & 0 & \dots \\ -M_1 & L_3 & 1 & \dots \\ \dots & \dots & & \dots \end{vmatrix} = 0. \qquad (16)$$

From (14) and (16) it is at once clear that $A_1 = 0$, and

$$\Delta_0' = 0. \qquad (17)$$

Hence there is only a *single* system of free periods, and the corresponding oscillations are symmetrical. Equation (17) is precisely that already given for the free periods in Case (i) above. The effect of the boundary at the equator is, then, to cut out the asymmetrical system of oscillations.

Numerical examples have been worked out for two cases: a narrow basin bounded by circles of latitude 30° and 14° 30′, wholly on one side of the equator; and one bounded by circles of latitude 30° and −14° 30′, including the equator. From these a fair idea may be obtained of the effect of such boundaries in modifying the tides; and a comparison with the equilibrium and complete ocean theories can be made without involving hopelessly complex arithmetical work.

3. *Tides of the First Species.*

In the general solution worked out in § 2, it was assumed that the tide-producing potential contained only one term, and we took it to be B_0. In the case of the tides of the first species, however, the potential is $H_0(\mu^2 - \frac{1}{3})$, and hence two terms, B_0 and B_2, appear. The general method, with the two quantities B, becomes very complicated; but a slight modification, applicable to the first species only, will enable us to simplify it.

In equations (2) and (3), put $s = 0$. We then find

$$\frac{d}{d\mu}\left[\frac{(1-\mu^2)\frac{d\zeta'}{d\mu}}{f^2-\mu^2}\right]+\beta(\zeta'+\bar{\zeta}) = 0, \tag{18}$$

and

$$u = \frac{\sigma\sqrt{(1-\mu^2)}}{4m(f^2-\mu^2)}\,\frac{d\zeta'}{d\mu}. \tag{19}$$

If we now assume, in the vicinity of $\mu = 0$,

$$\frac{1}{f^2-\mu^2}\,\frac{d\zeta'}{d\mu} = \sum_{n=0}^{\infty} A_n\mu^n, \tag{20}$$

and substitute in (18), we find

$$\zeta' = D+f^2A_0\mu+f^2A_1\frac{\mu^2}{2}+\frac{f^2A_2-A_0}{3}\mu^3+\ldots+(f^2A_{n-1}-A_{n-3})\frac{\mu^n}{n}+\ldots; \tag{21}$$

and the general relation between the coefficients is

$$A_{n+1}+A_{n-1}\left(\frac{\beta f^2}{n(n+1)}-1\right)-\frac{\beta}{n(n+1)}A_{n-3} = -\frac{\beta}{n+1}B_n. \tag{22}$$

This holds for all values of n except $n = 0$, when the relation is

$$A_1+\beta D = -\beta B_0. \tag{23}$$

Since we are excluding the possibilities $\mu = \pm 1$, the boundary conditions are, from (19),

$$\Sigma A_n\mu_1^n = 0 \quad\text{and}\quad \Sigma A_n\mu_2^n = 0.$$

If we use the notation of § 2, but note that now

$$L_n = \frac{\beta f^2}{(n+1)(n+2)}-1 \quad\text{and}\quad M_n = \frac{\beta}{(n+3)(n+4)},$$

and form the determinants as before, we find

$$\left.\begin{aligned} A_1\Delta_1'+A_0\Delta_0' &= \frac{\beta}{3}H_0\Delta_3' \\ \text{and}\qquad A_1\Delta_1''+A_0\Delta_0'' &= \frac{\beta}{3}H_0\Delta_3'' \end{aligned}\right\}. \tag{24}$$

For the free oscillations we reject the right-hand members of (24), and solve the pair of simultaneous transcendental equations for f and A_1/A_0. It hardly seems possible to do this with any satisfaction. When we deal with the narrow basin ($\mu_1 = \frac{1}{2}$, $\mu_2 = \frac{1}{4}$), the values of f are very large, and even a rough approximation is difficult to obtain. In the other example chosen, however, ($\mu_1 = \frac{1}{2}$, $\mu_2 = -\frac{1}{4}$), the values of f are smaller and some idea of their order of value may be obtained by rejecting the fourth and higher powers of μ_1, and the third and higher powers of μ_2.

For the forced oscillations, A_1 and A_0 are determined from (24). It is to be noted that the determinants required are readily found from the recurrence relation

$$\Delta'_n = \mu_1^n - L_n \Delta'_{n+2} + M_n \Delta'_{n+4}.$$

Afterwards (22) and (23) enable us to find D and the remaining coefficients A; and, finally, the value of $\zeta, = \zeta' + \bar{\zeta}$, is determined from (21).

(i) *Zonal Ocean bounded by Circles of Latitude* 14°30′ *and* 30°.

The following values of the first (and lowest) frequency were obtained by the method just indicated. They must be regarded as very rough approximations indeed.

It is to be noted that $\beta f^2 \left(= \frac{4\omega^2 a^2}{gh} \frac{\sigma^2}{4\omega^2}\right)$ is independent of ω, and as M is not involved in the approximation we are using, these values of the frequencies are obtained without reference to the rotation. This in itself, however, will not greatly vitiate the results, as the effect of the rotation is known to be small in low latitudes, reducing to zero at the equator.

The value we obtain is $\beta f^2 = 54$. Hence

$$\beta = \quad 5, \quad 10, \quad 20, \quad 40;$$

$$\text{Depth} = 58{,}080, \; 29{,}040, \; 14{,}520, \; 7{,}260 \text{ feet};$$

$$f\left(= \frac{\sigma}{2\omega}\right) = \quad 3{\cdot}3, \quad 2{\cdot}3, \quad 1{\cdot}6, \quad 1{\cdot}2.$$

As an example of a forced tide we take the case of the lunar fortnightly, for which $f^2 = {\cdot}00133$.

The expressions for the lunar fortnightly tides in the same ocean are the following; each expression is to be multiplied by the factor $\cos(\sigma t + \epsilon)$.

$\beta = 5$; *depth* = 58,080 *feet* :

$$\zeta/H_0 = -{\cdot}146 - {\cdot}0002\mu + 1{\cdot}000\mu^2 + {\cdot}0518\mu^3 - {\cdot}182\mu^4 + {\cdot}0310\mu^5 + {\cdot}157\mu^6 + {\cdot}0317\mu^7 + {\cdot}0946\mu^8 + {\cdot}0273\mu^9 + {\cdot}0869\mu^{10} \ldots .$$

$\beta = 10$; *depth* = 29,040 *feet* :

$$\zeta/H_0 = -\cdot146 - \cdot0005\mu + 1\cdot001\mu^2 + \cdot104\mu^3 - \cdot365\mu^4 + \cdot0621\mu^5 + \cdot313\mu^6 + \cdot0799\mu^7 + \cdot1437\mu^8 + \cdot0737\mu^9 + \cdot1598\mu^{10} \ldots .$$

$\beta = 20$; *depth* = 14,520 *feet* :

$$\zeta/H_0 = -\cdot146 - \cdot0009\mu + 1\cdot002\mu^2 + \cdot207\mu^3 - \cdot725\mu^4 + \cdot1211\mu^5 + \cdot629\mu^6 + \cdot236\mu^7 + \cdot1045\mu^8 + \cdot229\mu^9 + \cdot263\mu^{10} \ldots .$$

$\beta = 40$; *depth* = 7,260 *feet* :

$$\zeta/H_0 = -\cdot147 - \cdot0017\mu + 1\cdot004\mu^2 + \cdot417\mu^3 - 1\cdot456\mu^4 + \cdot246\mu^5 + 1\cdot385\mu^6 + \cdot7702\mu^7 - \cdot418\mu^8 + \cdot782\mu^9 + \cdot458\mu^{10} \ldots .$$

(ii) *Zonal Ocean bounded by Circles of Latitude* 30° *and* −14° 30′.

The approximate frequency equation is

$$\beta f^2 = 14\cdot3.$$

Hence we have the following rough values of the lowest frequencies for the different depths :

$\beta =$	5,	10,	20,	40;
Depth =	58,080,	29,040,	14,520,	7,260 feet;
$f\left(=\frac{\sigma}{2\omega}\right) =$	1·7,	1·2,	·85,	·60.

It is at once obvious that there is no possibility of synchronism of the lunar fortnightly tide with any of the free tides.

Expressions for the lunar fortnightly tides in the same zonal ocean the series must each be multiplied by the factor $\cos(\sigma t + \epsilon)$.

$\beta = 5$; *depth* = 58,080 *feet* :

$$\zeta/H_0 = -\cdot0619 + \cdot0001\mu + 1\cdot0002\mu^2 - \cdot0171\mu^3 - \cdot0779\mu^4 - \cdot0102\mu^5 + \cdot226\mu^6 - \cdot0103\mu^7 + \cdot160\mu^8 - \cdot009\mu^9 + \cdot143\mu^{10} \ldots .$$

$\beta = 10$; *depth* = 29,040 *feet* :

$$\zeta/H_0 = -\cdot0613 + \cdot0001\mu + 1\cdot0004\mu^2 - \cdot0336\mu^3 - \cdot155\mu^4 - \cdot0202\mu^5 + \cdot449\mu^6 - \cdot0279\mu^7 + \cdot302\mu^8 - \cdot0257\mu^9 + \cdot306\mu^{10} \ldots .$$

$\beta = 20$; *depth* = 14,520 *feet* :

$$\zeta/H_0 = -\cdot 0601 + \cdot 0002\mu + 1\cdot 0007\mu^2 - \cdot 0655\mu^3 - \cdot 302\mu^4 - \cdot 0389\mu^5 + \cdot 911\mu^6 - \cdot 0747\mu^7 + \cdot 546\mu^8 - \cdot 0724\mu^9 + \cdot 688\mu^{10} \dots$$

$\beta = 40$; *depth* = 7,260 *feet* :

$$\zeta/H_0 = -\cdot 0578 + \cdot 0005\mu + 1\cdot 0014\mu^2 - \cdot 123\mu^3 - \cdot 581\mu^4 - \cdot 0718\mu^5 + 1\cdot 84\mu^6 - \cdot 228\mu^7 + \cdot 796\mu^8 - \cdot 231\mu^9 + 1\cdot 72\mu^{10} \dots$$

4. *Tides of Second Species.*

In preparing the formulæ for these, we must bear in mind that the general formulæ previously given must be modified by the transference of the tide-producing terms into the odd, instead of the even coefficients.

If in the quantities of (9) we put $s = 1$, and take the tide-producing potential as containing one term only $B_1, = H_1$, we find, in place of (10*a*) and (10*b*), after putting $H_1 = \kappa A_1$,

$$\left.\begin{array}{l} A_0 L_0 + A_2 = 0 \\ -A_0 M_0 + A_2 L_2 + A_4 = 0 \\ \dots \quad \dots \quad \dots \quad \dots \end{array}\right\}, \qquad (25a)$$

$$\left.\begin{array}{l} A_1\left(L_1 + \frac{\beta\kappa}{24}\right) + A_3 = 0 \\ A_1\left(-M_1 - \frac{\beta\kappa . 13}{140}\right) + L_3 A_3 + A_5 = 0 \\ -A_3 M_3 + L_5 A_5 + A_7 = 0 \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \end{array}\right\}. \qquad (25b)$$

The boundary condition (7) also becomes

$$\Sigma a_n' A_n = -\tfrac{3}{7}\beta H_1 \mu_1^2; \qquad (26)$$

where $$a_n' = \frac{-1/f^2\, n}{1+(n-1)(1+1/f)}\mu_1^{n-1} + \frac{\beta(1+1/f)}{1+(n+1)(1+1/f)}\mu_1^{n+1}.$$

In place of equations (15), we find

$$\left.\begin{array}{l} A_0\Delta_0' + A_1\Delta_1' = H_1\left(-\frac{3\beta}{7}\mu_1^2 + \frac{\beta}{24}\Delta_3' - \frac{13\beta}{140}\Delta_5'\right) \\ A_0\Delta_0'' + A_1\Delta_1'' = H_1\left(-\frac{3\beta}{7}\mu_2^2 + \frac{\beta}{24}\Delta_3'' - \frac{13\beta}{140}\Delta_5''\right) \end{array}\right\}. \qquad (27)$$

These equations are treated in precisely the same manner as those of the first species.

For the forced tides, we have taken as typical the luni-solar diurnal for which f is rigorously $\frac{1}{2}$. The numerical work is also simplified thereby. It is to be noticed that equations (25a) and (25b) are then satisfied by putting

$$A_0 = A_2 = A_4 = \ldots = A_3 = A_5 = \ldots = 0, \quad \text{and} \quad \kappa = -1;$$

for then both $L_1 + \frac{\beta\kappa}{24}$ and $M_1 + \frac{\beta\kappa . 13}{140}$ equal zero. Hence there is a solution

$$z' = -A_1\mu = -H_1\mu,$$

or

$$\zeta' = z'\sqrt{(1-\mu^2)} = -H_1\mu\sqrt{(1-\mu^2)};$$

and

$$\zeta = \zeta' + \bar{\zeta} = 0.$$

This is Laplace's celebrated theorem that the luni-solar diurnal tide is evanescent when the ocean is of uniform depth and covers the whole globe. It is to be noted, however, that the condition (26) is not satisfied by this solution.

For the free periods we reject the right-hand side of (27) and eliminate A_0 and A_1. The result is

$$\Delta_0'\Delta_1'' - \Delta_1'\Delta_0'' = 0.$$

The complicated way in which f enters into each of these determinants makes it hopeless to attempt to find even rough values of the free periods. As shown later on, all that can be done is to ascertain approximately those critical depths at which a free mode coincides with a selected forced mode.

The forced tides are, however, not so difficult to obtain.

(i) *Zonal Ocean bounded by Circles of Latitude* 30° *and* 14°20′.

Luni-solar diurnal tides; each expression must be multiplied by the factor $\cos(\sigma t + \phi + \epsilon)$.

$\beta = 5$; *depth* = 58,080 *feet*:

$$\zeta/H_1 = [-7{\cdot}26 - 9{\cdot}99\mu^2 - 14{\cdot}8\mu^4 - 16{\cdot}3\mu^6 - 17{\cdot}5\mu^8 - 18{\cdot}2\mu^{10} + \ldots$$
$$+ {\cdot}756\mu - {\cdot}1575\mu^3 + {\cdot}202\mu^5 + {\cdot}170\mu^7 + {\cdot}185\mu^9 \ldots]\sqrt{(1-\mu^2)}.$$

$\beta = 10$; *depth* = 29,040 *feet*:

$$\zeta/H_1 = [-{\cdot}5845 - {\cdot}438\mu^2 - 1{\cdot}49\mu^4 - 1{\cdot}54\mu^6 - 1{\cdot}82\mu^8 - 1{\cdot}97\mu^{10} + \ldots$$
$$+ {\cdot}858\mu - {\cdot}358\mu^3 + {\cdot}510\mu^5 + {\cdot}339\mu^7 + {\cdot}417\mu^9 \ldots]\sqrt{(1-\mu^2)}.$$

$\beta = 20$; *depth* = 14,520 *feet*:

$$\zeta/H_1 = [-\cdot411+\cdot206\mu^2-1\cdot63\mu^4-1\cdot04\mu^6-1\cdot75\mu^8-1\cdot93\mu^{10}+\ldots$$
$$+\cdot661\mu-\cdot553\mu^3+\cdot848\mu^5+\cdot322\mu^7+\cdot608\mu^9\ldots]\sqrt{(1-\mu^2)}.$$

$\beta = 40$; *depth* = 7,260 *feet*:

$$\zeta/H_1 = [-\cdot264+\cdot793\mu^2-2\cdot20\mu^4+\cdot228\mu^6-1\cdot98\mu^8-1\cdot61\mu^{10}+\ldots$$
$$+\cdot395\mu-\cdot639\mu^3+1\cdot19\mu^5-\cdot0415\mu^7+\cdot833\mu^9\ldots]\sqrt{(1-\mu^2)}.$$

(ii) *Zonal Ocean bounded by Circles of Latitude* 30° *and* −14° 30′.

Luni-solar diurnal tides: each expression must be multiplied by the factor $\cos(\sigma t+\phi+\epsilon)$.

$\beta = 5$; *depth* 58,080 *feet*:

$$\zeta/H_1 = [-1\cdot33-1\cdot83\mu^2-2\cdot72\mu^4-3\cdot24\mu^6-3\cdot45\mu^8-3\cdot59\mu^{10}+\ldots$$
$$+1\cdot094\mu-\cdot229\mu^3+\cdot310\mu^5+\cdot131\mu^7+\cdot155\mu^9\ldots]\sqrt{(1-\mu^2)}.$$

$\beta = 10$; *depth* 29,040 *feet*:

$$\zeta/H_1 = [-\cdot241-\cdot181\mu^2-\cdot569\mu^4-\cdot595\mu^6-\cdot699\mu^8-\cdot759\mu^{10}+\ldots$$
$$+1\cdot184\mu-\cdot493\mu^3+\cdot703\mu^5+\cdot470\mu^7+\cdot542\mu^9\ldots]\sqrt{(1-\mu^2)}.$$

$\beta = 20$; *depth* 14,520 *feet*:

$$\zeta/H_1 = [-\cdot196+\cdot0980\mu^2-\cdot344\mu^4-\cdot169\mu^6-\cdot323\mu^8-\cdot350\mu^{10}+\ldots$$
$$+1\cdot472\mu-1\cdot23\mu^3+1\cdot76\mu^5+\cdot628\mu^7+1\cdot204\mu^9\ldots]\sqrt{(1-\mu^2)}.$$

$\beta = 40$; *depth* 7,260 *feet*:

$$\zeta/H_1 = [-\cdot258+\cdot774\mu^2-2\cdot15\mu^4+\cdot305\mu^6-1\cdot904\mu^8-1\cdot50\mu^{10}\ldots$$
$$+2\cdot914\mu-4\cdot86\mu^3+8\cdot74\mu^5-\cdot502\mu^7+5\cdot96\mu^9\ldots]\sqrt{(1-\mu^2)}.$$

5. *Tides of Third Species.*

The formulæ already found, viz., (10*a*), (10*b*), (13) and (15), are ready for use in the case of the third species, if we put $s = 2$. Only one term appears in the tide-producing potential, viz., $B_0 = H_2$. The procedure is in every case precisely the same as before.

As a typical example of a forced tide of this species I have worked out the luni-solar semi-diurnal tide for the two oceans selected. For this tide

f is rigorously unity. The other semi-diurnal tides have values of f not differing much from this.

As in the case of the tides of the second species, it was found impossible to obtain even a rough estimate of the free periods.

(i) *Zonal Ocean bounded by Circles of Latitude* 30° *and* 14° 30′.

Luni-solar semi-diurnal tide; each expression must be multiplied by the factor $\cos(\sigma t + 2\phi + \epsilon)$.

$\beta = 5$; *depth* = 58,080 *feet*:

$$\zeta/H_2 = [\cdot 849 - 2\cdot 58\mu^2 - 2\cdot 53\mu^4 - 3\cdot 76\mu^6 - 4\cdot 80\mu^8 - 5\cdot 86\mu^{10} + \ldots$$
$$+ 2\cdot 53\mu + 2\cdot 95\mu^3 + 4\cdot 74\mu^5 + 6\cdot 10\mu^7 + 7\cdot 46\mu^9 \ldots](1-\mu^2).$$

$\beta = 10$; *depth* = 29,040 *feet*:

$$\zeta/H_2 = [-\cdot 554 - 1\cdot 89\mu^2 - 2\cdot 50\mu^4 - 3\cdot 61\mu^6 - 4\cdot 59\mu^8 - 5\cdot 61\mu^{10} + \ldots$$
$$- \cdot 311\mu - \cdot 104\mu^3 - \cdot 363\mu^5 - \cdot 433\mu^7 - \cdot 545\mu^9 \ldots](1-\mu^2).$$

$\beta = 20$; *depth* = 14,520 *feet*:

$$\zeta/H_2 = [-0\cdot 042 - 2\cdot 71\mu^2 - 1\cdot 40\mu^4 - 2\cdot 81\mu^6 - 2\cdot 67\mu^8 - 3\cdot 45\mu^{10} + \ldots$$
$$+ \cdot 668\mu - \cdot 890\mu^3 + 1\cdot 56\mu^5 + \cdot 769\mu^7 + 1\cdot 30\mu^9 \ldots](1-\mu^2).$$

$\beta = 40$; *depth* = 7,260 *feet*:

$$\zeta/H_2 = [-\cdot 030 + 2\cdot 49\mu^2 - 4\cdot 35\mu^4 + 2\cdot 20\mu^6 - 2\cdot 89\mu^8 - 1\cdot 02\mu^{10} + \ldots$$
$$+ 1\cdot 47\mu - 6\cdot 87\mu^3 + 8\cdot 35\mu^5 - 5\cdot 98\mu^7 + 1\cdot 81\mu^9 \ldots](1-\mu^2).$$

(ii) *Zonal Ocean bounded by Circles of Latitude* 30° *and* −14° 30′.

Luni-solar semi-diurnal tides; each expression must be multiplied by the factor $\cos(\sigma t + 2\phi + \epsilon)$.

$\beta = 5$; *depth* = 58,080 *feet*:

$$\zeta/H_2 = [-3\cdot 52 - 4\cdot 76\mu^2 - 8\cdot 89\mu^4 - 12\cdot 15\mu^6 - 15\cdot 59\mu^8 - 19\cdot 03\mu^{10} - \ldots$$
$$- 1\cdot 51\mu - 1\cdot 76\mu^3 - 2\cdot 83\mu^5 - 3\cdot 74\mu^7 - 4\cdot 67\mu^9 \ldots](1-\mu^2).$$

$\beta = 10$; *depth* = 29,040 *feet*:

$$\zeta/H_2 = [-\cdot 509 - 1\cdot 98\mu^2 - 2\cdot 50\mu^4 - 3\cdot 66\mu^6 - 4\cdot 64\mu^8 - 5\cdot 67\mu^{10} + \ldots$$
$$+ \cdot 215\mu + \cdot 0718\mu^3 + \cdot 251\mu^5 + \cdot 299\mu^7 + \cdot 377\mu^9 \ldots](1-\mu^2).$$

$\beta = 20$; *depth* $= 14{,}520$ *feet* :

$$\zeta/H_2 = [-1{\cdot}818+9{\cdot}73\mu^2-6{\cdot}06\mu^4+5{\cdot}28\mu^6-2{\cdot}019\mu^8+3{\cdot}49\mu^{10}+\ldots$$
$$+11{\cdot}4\mu-15{\cdot}2\mu^3+11{\cdot}4\mu^5+{\cdot}500\mu^7+4{\cdot}56\mu^9\ \ldots](1-\mu^2).$$

$\beta = 40$; *depth* $= 7{,}260$ *feet* :

$$\zeta/H_2 = [-{\cdot}048-2{\cdot}23\mu^2+{\cdot}083\mu^4-4{\cdot}45\mu^6-2{\cdot}47\mu^8-4{\cdot}39\mu^{10}+\ldots$$
$$+1{\cdot}48\mu-6{\cdot}91\mu^3+8{\cdot}39\mu^5-6{\cdot}02\mu^7+1{\cdot}82\mu^9\ \ldots](1-\mu^2).$$

6. *Discussion of Results.*

(i) The fortnightly tide.

We shall compare the numerical values of the tide-heights found in the preceding sections with the theory of an ocean wholly covering the globe,* and with the "corrected equilibrium" theory. For the latter we proceed to find the correction to be applied for such a zonal ocean as we are discussing, of which the boundaries are μ_1 and μ_2. The complete expression for the "equilibrium" tide is

$$\bar{\zeta} = H_0(\mu^2-\tfrac{1}{3})\cos(\sigma_0 t+\epsilon_0)+C.$$

But over the whole zone we must have

$$\iint \bar{\zeta}\, dS = 0,$$

where dS is an element of surface.

On performing the integration we find

$$C = -\frac{H_0}{3}(\mu_2^2+\mu_2\mu_1+\mu_1^2-1)\cos(\sigma_0 t+\epsilon_0).$$

The entries in the table are calculated from these expressions.

TABLE I. *Ocean bounded by Circles of Latitude* 30° *and* 14° 30′.

	$\beta = 5$		$\beta = 10$		$\beta = 20$		$\beta = 40$	
	Latitudes.							
	30°	14° 13′	30°	14° 30′	30°	14 30′	30°	14° 30′
Present Theory	·103	–·084	·102	–·083	·100	–·083	·092	–·083
Complete Ocean Theory	—	—	–·020	–·176	—	—	·021	–·095
Corrected Equilibrium Theory ..	·104	–·083	·104	–·083	·104	–·083	·104	–·083

* Darwin, *Proc. Roy. Soc.*, Vol. XLI (1886), p. 337. (Calculated for $\beta = 10$ and 40 only.)

TABLE II. *Ocean bounded by Circles of Latitude* 30° *and* −14° 30′.

	β = 5		β = 10		β = 20		β = 40	
	Latitudes.							
	30°	14° 30′	30°	14° 30′	30°	14° 30′	30°	14° 30′
Present Theory	·184	·0006	·183	·0012	·178	·0026	·170	·0048
Complete Ocean Theory	—	—	−·020	−·176	—	—	·021	−·095
Corrected Equilibrium Theory ...	·1875	zero	·1875	zero	·1875	zero	·1875	zero

(Each figure in the entries to be multiplied by H_0.)

The important fact that stands out from these tables is the close approximation of the "corrected equilibrium" tide to the true dynamical tide. This may also be exhibited analytically as follows.

From equations (1) we find for the tide of first species

$$\left.\begin{aligned} u &= \frac{\sigma}{4m(f^2-\mu^2)}\sqrt{(1-\mu^2)}\left\{\frac{d\zeta}{d\mu}-\frac{d\bar{\zeta}}{d\mu}\right\} \\ \frac{\sigma a\zeta}{h} &= -\frac{d}{d\mu}\{u\sqrt{(1-\mu^2)}\} \end{aligned}\right\}. \tag{28}$$

From these we find

$$\frac{h}{4ma}\frac{d^2}{d\mu^2}u\sqrt{(1-\mu^2)} = \frac{-u(f^2-\mu^2)}{\sqrt{(1-\mu^2)}}-\frac{\sigma}{4m}\frac{d\bar{\zeta}}{d\mu}.$$

On integrating,

$$\frac{h}{4ma}u\sqrt{(1-\mu^2)}+\int d\mu\int d\mu\left\{\frac{u(f^2-\mu^2)}{\sqrt{(1-\mu^2)}}\right\} = -\frac{\sigma}{4m}\int\bar{\zeta}\,d\mu+E_1\mu+E_2.$$

If U be the maximum value of $|u|$ in the zone under consideration, then

$$\int d\mu\int d\mu\left\{\frac{u(f^2-\mu^2)}{\sqrt{(1-\mu^2)}}\right\} \leqslant U\int d\mu\int d\mu\left\{\frac{(f^2-\mu^2)}{\sqrt{(1-\mu^2)}}\right\}$$

$$\leqslant U\left\{\frac{f^2\mu^2}{2}+\left(\frac{f^2}{2}-1\right)\frac{\mu^4}{12}+\dots\right\}.$$

Now by the original supposition of infinitely small motions upon which equations (1) were based, U is a quantity whose square may be neglected.

If then f^4 and μ^4 be at least of the same order as U, the term we are discussing may be neglected in comparison with that preceding it. Hence

$$\frac{h}{4ma}u\sqrt{(1-\mu^2)} = -\frac{\sigma}{4m}\int\bar{\zeta}d\mu+E_1\mu+E_2.$$

On applying the conditions $u=0$ at $\mu=\mu_1$ and $\mu=\mu_2$, we have

$$E_2 = 0,$$

and
$$E_1 = \frac{\sigma}{4m(\mu_2-\mu_1)}\int_{\mu_1}^{\mu_2}\bar{\zeta}\,d\mu.$$

Finally, by the second of equations (28),

$$\zeta = \bar{\zeta}-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\bar{\zeta}\,d\mu.$$

This is rigorously the expression for the "corrected equilibrium" tide height.

The term we have neglected was compared with the term

$$\frac{h}{4ma}u\sqrt{(1-\mu^2)},$$

and we tacitly assumed that the fraction $\frac{h}{4ma}$ was not small enough to reduce the order of the term. If, however, h be reduced greatly in value, the approximation we have found will be less satisfactory. This is the explanation of the fact that in the table the agreement of the theories is less close as β increases, that is, as h decreases.

(ii) Diurnal tide.

The tables give the tide heights at the corresponding latitudes.

TABLE III. *Ocean bounded by Circles of Latitude* 30° *and* 14° 30′.

	$\beta=5$		$\beta=10$		$\beta=20$		$\beta=40$	
	Latitudes.							
	30°	14° 30′	30°	14° 30′	30°	14° 30′	30°	14° 30′
Present Theory...	$-9{\cdot}23H_1$	$-7{\cdot}51H_1$	$-{\cdot}364H_1$	$-{\cdot}396H_1$	$-{\cdot}168H_1$	$-{\cdot}240H_1$	$-{\cdot}045H_1$	$-{\cdot}129H_1$
Equilibrium Theory* ...	$+{\cdot}433H_1$	$+{\cdot}242H_1$	$+{\cdot}433H_1$	$+{\cdot}242H_1$	$+{\cdot}433H_1$	$+{\cdot}242H_1$	$+{\cdot}433H_1$	$+{\cdot}242H_1$

* The correction to the "equilibrium" theory in this case is zero.

TABLE IV. *Ocean bounded by Circles of Latitude* 30° *and* −14° 30′.

	$\beta = 5$		$\beta = 10$		$\beta = 20$		$\beta = 40$	
	Latitudes.							
	30°	−14° 30′	30°	−14° 30′	30°	−14° 30′	30°	−14° 30′
Present Theory...	$-1{\cdot}30H_1$	$-1{\cdot}67H_1$	$+{\cdot}193H_1$	$-{\cdot}532H_1$	$+{\cdot}475H_1$	$-{\cdot}524H_1$	$+{\cdot}843H_1$	$-{\cdot}843H_1$
Equilibrium Theory* ...	$+{\cdot}433H_1$	$-{\cdot}242H_1$	$+{\cdot}433H_1$	$-{\cdot}242H_1$	$+{\cdot}433H_1$	$-{\cdot}242H_1$	$+{\cdot}433H_1$	$-{\cdot}242H_1$

The luni-solar diurnal tide is, as already mentioned, evanescent in the complete ocean theory.

In Table III, the high magnitudes of the tide heights when $\beta = 5$ are due to approximate synchronism. While the complexity of the period equation makes it almost impossible to get even an approximate idea of the values of the free periods, yet with a little trouble we may find rough values of β (or h) which would cause synchronism between the free oscillations and the forced oscillations of a given character. So that, if in the equations [derived from (27)]

$$\left.\begin{aligned} A_0\Delta_0'+A_1\Delta_1' &= 0 \\ A_0\Delta_0''+A_1\Delta_0'' &= 0 \end{aligned}\right\}, \tag{29}$$

we substitute the suitable values for μ_1 and μ_2, put $f = \frac{1}{2}$, and take only the first terms in each determinant, we get an algebraic equation for β. By this method, great exactness is not attainable, but a sufficient indication of the position of the critical depths can be found.

Corresponding to Table III, we find $\beta = 3{\cdot}1$, 338.

Corresponding to Table IV, we find $\beta = 3{\cdot}8$, 46.

As will be shown presently, the first root is usually too small; the second is probably considerably wrong, and is only useful for showing that the higher roots indicate depths quite out of the range here considered. To give greater confidence to the results, I have worked out the values of $\Delta_0'\Delta_1''-\Delta_1'\Delta_0''$ for $\beta = 4$ and $\beta = 5$. The former gives $+{\cdot}4515$, and the latter $-{\cdot}2242$; showing that the critical depth lies between those given by $\beta = 4$ and 5. For the larger basin, the values of the same expression for $\beta = 4$ and $\beta = 5$ are, respectively, $+{\cdot}9169$ and $-1{\cdot}2239$. The critical depth is again between those given by $\beta = 4$ and $\beta = 5$. But as the variation of the function between values given by $\beta = 4$ and 5

* See note on previous page.

is much more rapid in the latter than in the former case, the effect of a small departure from the critical depth in the latter case is to reduce much more rapidly the height of the tide; hence in Table IV, for $\beta = 5$, the heights are not far from normal.

Neither the theory of the ocean completely covering the globe nor the equilibrium theory offers any useful indication of the correct tide heights.

(iii) Semi-diurnal tide.

TABLE IV. *Ocean bounded by Circles of Latitude* 30° *and* 14° 30′.

	$\beta = 5$		$\beta = 10$		$\beta = 20$		$\beta = 40$	
	Latitudes.							
	30°	14° 30′	30°	14° 30′	30°	14° 30′	30°	14° 30′
Present Theory...	$1{\cdot}36H_2$	$\cdot 348H_2$	$-1{\cdot}09H_2$	$-\cdot 734H_2$	$-\cdot 436H_2$	$-\cdot 063H_2$	$+\cdot 363H_2$	$+\cdot 361H_2$
Complete Ocean Theory	$1{\cdot}25H_2$	$1{\cdot}75H_2$	$5{\cdot}86H_2$	$8{\cdot}56H_2$	$-\cdot 445H_2$	$-1{\cdot}56H_2$	$+6{\cdot}90H_2$	$-1{\cdot}02H_2$
Equilibrium Theory* ...	$+\cdot 75H_2$	$+\cdot 94H_2$	$+\cdot 75H_2$	$+\cdot 94H_2$	$+\cdot 75H_2$	$+\cdot 94H_2$	$+\cdot 75H_2$	$+\cdot 94H_2$

TABLE V. *Ocean bounded by Circles of Latitude* 30° *and* −14° 30′.

	$\beta = 5$		$\beta = 10$		$\beta = 20$		$\beta = 40$	
	Latitudes.							
	30°	−14° 30′	30°	−14° 30′	30°	−14° 30′	30°	−14° 30′
Present Theory...	$-4{\cdot}98H_2$	$-3{\cdot}23H_2$	$-\cdot 837H_2$	$-\cdot 655H_2$	$+3{\cdot}36H_2$	$-3{\cdot}61H_2$	$-\cdot 443H_2$	$-\cdot 429H_2$
Complete Ocean Theory	$1{\cdot}25H_2$	$1{\cdot}75H_2$	$5{\cdot}86H_2$	$8{\cdot}56H_2$	$-\cdot 445H_2$	$-1{\cdot}56H_2$	$6{\cdot}90H_2$	$-1{\cdot}02H_2$
Equilibrium Theory* ...	$\cdot 75H_2$	$\cdot 94H_2$	$\cdot 75H_2$	$\cdot 94H_2$	$\cdot 75H_2$	$\cdot 94H_2$	$\cdot 75H_2$	$\cdot 94H_2$

The values of β roughly indicating critical depths for these tides are, for the first ocean, 2·2 and 84; for the second, 2·8 and 18. These numbers account for the higher magnitude of the tides, in Table V, where $\beta = 5$ and $\beta = 20$, than in the other cases.

Again, there seems no relationship between the figures given by the three theories.

* The correction to the ordinary "equilibrium" theory is zero in this case.

It is important to notice that the effect of the boundaries on the diurnal tide (which is zero in the complete ocean theory) is to make it to correspond in magnitude with the semi-diurnal and in many cases to predominate. This is fully exhibited in the following table, where we have taken the ratio $H_1/H_2 = 4\cdot5$.* The cases of exaggerated tides due to approximate synchronism are omitted.

		Lat. 30°.		Lat. 14° 30′.	
First example.		Diurnal.	Semi-diurnal.	Diurnal.	Semi-diurnal.
	$\beta = 10$	$1\cdot6H_2$	$1\cdot09H_2$	$1\cdot7H_2$	$\cdot73H_2$
	$\beta = 20$	$\cdot76H_2$	$\cdot44H_2$	$1\cdot02H_2$	$\cdot063H_2$
	$\beta = 40$	$\cdot20H_2$	$\cdot36H_2$	$\cdot58H_2$	$\cdot36H_2$
		Lat. 30°.		Lat. −14° 30′.	
Second example.		Diurnal.	Semi-diurnal.	Diurnal.	Semi-diurnal.
	$\beta = 10$	$\cdot87H_2$	$\cdot84H_2$	$2\cdot4H_2$	$\cdot66H_2$
	$\beta = 40$	$3\cdot8H_2$	$\cdot44H_2$	$3\cdot8H_2$	$\cdot43H_2$

Note on the "Corrected Equilibrium" Theory of the Tides in a Polar Basin.

In a previous paper (*Proceedings*, p. 31 of the present volume) I have worked out the dynamical theory of the tides in a polar basin. In comparing the results obtained with the "equilibrium" results, the uncorrected forms of the latter were taken. For the tides of the second and third species the correction is zero, but in the case of those of the first species the correction makes an important difference.

From p. 223 of the present paper, the corrected form of the long period "equilibrium" tide is

$$\bar{\zeta} = H_0 \left\{ (\mu^2 - \tfrac{1}{3}) - \tfrac{1}{3}(\mu_1^2 + \mu_1\mu_2 + \mu_2^2 - 1) \right\},$$

where μ_1 and μ_2 are the boundaries of the zone. If this be changed into a form suitable for use in a polar basin by putting $\nu^2 = 1 - \mu^2$, and by taking $\mu_2 = 1$, $\mu_1 = \sqrt{(1-\nu_1^2)}$, the result is

$$\bar{\zeta} = H_0 \left\{ (\tfrac{2}{3} - \nu^2) - \tfrac{1}{3}(1 - \nu_1^2) - \tfrac{1}{3}\sqrt{(1-\nu_1^2)} \right\}.$$

---

* Darwin, *Scientific Papers*, Vol. I, pp. 20 and 21 (tides K_1 and K_2).

We now repeat the tables on p. 62 of the former paper, substituting the corrected form of the equilibrium tide.

Boundary $\nu_1 = \frac{1}{4}$, *Lat.* 75° 30′.

	Tide Height at Pole.		Tide Height at Boundary.	
	Dynamical Theory.	Corrected Equilibrium Theory.	Dynamical Theory.	Corrected Equilibrium Theory.
$\beta = 5$	$\cdot 0306 H_0$	$\cdot 0315 H_0$	$-\cdot 0307 H_0$	$-\cdot 0310 H_0$
$\beta = 10$	$\cdot 0298 H_0$	$\cdot 0315 H_0$	$-\cdot 0305 H_0$	$-\cdot 0310 H_0$
$\beta = 20$	$\cdot 0284 H_0$	$\cdot 0315 H_0$	$-\cdot 0293 H_0$	$-\cdot 0310 H_0$
$\beta = 40$	$\cdot 0260 H_0$	$\cdot 0315 H_0$	$-\cdot 0270 H_0$	$-\cdot 0310 H_0$

Boundary $\nu_1 = \frac{1}{2}$, *Lat.* 60°.

	Tide Height at Pole.		Tide Height at Boundary.	
	Dynamical Theory.	Corrected Equilibrium Theory.	Dynamical Theory.	Corrected Equilibrium Theory.
$\beta = 5$	$\cdot 117 H_0$	$\cdot 128 H_0$	$-\cdot 115 H_0$	$-\cdot 122 H_0$
$\beta = 10$	$\cdot 105 H_0$	$\cdot 128 H_0$	$-\cdot 112 H_0$	$-\cdot 122 H_0$
$\beta = 20$	$\cdot 0887 H_0$	$\cdot 128 H_0$	$-\cdot 103 H_0$	$-\cdot 122 H_0$
$\beta = 40$	$\cdot 0668 H_0$	$\cdot 128 H_0$	$-\cdot 0899 H_0$	$-\cdot 122 H_0$

It is noticeable that the "corrected equilibrium" theory gives a fair approximation to the dynamical theory when the basin is small and the depth great. The approximation is not so good (for values of $\nu = \mu$) as in the case of a zonal ocean near the equator. If the matter be examined analytically in the same manner as is done on p. 224 of this paper, the reasons for the less satisfactory approximation become obvious.