ON THE MODIFICATION OF A TRAIN OF WAVES AS IT ADVANCES INTO SHALLOW WATER

By W. BURNSIDE.

[Received June 23rd, 1914.—Read November 12th, 1914.]

1. In his memoir "On the Motion of Waves in a Variable Canal of small Breadth and Depth" (*Camb. Phil. Trans.*, 1838), Green shewed that if h is the depth of the canal, the height of a wave travelling along it varies as $h^{-\frac{1}{4}}$ and the length as $h^{\frac{1}{2}}$. Incidentally Green's analysis shews that if the depth varies slowly enough, there is practically no reflection due to varying depth.

If it be assumed that, as a train of waves advances from deep into shallow water, the depth varies so slowly that the effect of reflection at the bottom may be neglected, it is evident that the constancy of the time-period and of the rate of transmission of energy are sufficient data from which to determine the change in the height and the length of the waves. The results thus obtained are markedly different from those in the case of "long" waves.

2. If β is the amplitude (*i.e.*, half the height from trough to crest) of a train of waves of length $2\pi/m$, and time-period $2\pi/n$, in water of depth h, the velocity function is

$$\phi = \frac{\beta n}{m} \operatorname{cosech} mh \cosh m(y-h) \cos(mx-nt),$$

where

$$n^2 = gm \tanh mh. \tag{i}$$

The mean rate of transmission of energy, *i.e.*, the mean value of

$$-\rho \int_0^h \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial y}\, dy$$

is

$$\frac{g^2\beta^2\rho}{8n} \frac{2mh+\sinh 2mh}{\cosh^2 mh}.$$

In the absence of reflection, the length and amplitude of the waves in water of depth h is given by the two equations

$$\frac{g^2\beta^2\rho}{8n}\,\frac{2mh+\sinh 2mh}{\cosh^2 mh} = \text{constant},$$

$$n^2-gm\tanh mh = 0,$$

where $2\pi/n$ is the constant time-period.

If a is the amplitude in deep water ($h=\infty$), and L the wave-length in deep water,

$$L = \frac{2\pi g}{n^2},$$

and
$$\beta^2\,\frac{2mh+\sinh 2mh}{2\cosh^2 mh} = a^2. \tag{ii}$$

Now the function
$$\frac{2\theta+\sinh 2\theta}{2\cosh^2\theta}$$

is zero when $\theta=0$, and unity when $\theta=\infty$. It has a single maximum when

$$\theta\tanh\theta = 1.$$

The corresponding value of θ, to two significant figures, is

$$\theta = 1.2,$$

and then
$$\frac{2\theta+\sinh 2\theta}{2\cosh^2\theta} = 1.2.$$

When
$$mh = 1.2, \qquad \tanh mh = .83,$$

and, from (i),
$$\frac{2\pi}{m} = .83L.$$

Hence as the wave-length diminishes with diminishing depth, the amplitude begins by diminishing, and reaches a minimum given by

$$\beta = \frac{a}{\sqrt{(1.2)}} = .91a,$$

when the depth has so far diminished that the wave-length is $.83L$.

After this, with diminishing depth the wave-length steadily diminishes, while the amplitude increases.

The solution of the equation

$$\frac{2\theta+\sinh 2\theta}{2\cosh^2\theta} = 1,$$

other than $\theta = \infty$, is $$\theta = .64,$$

giving $$\tanh \theta = .56.$$

Hence, when the wave-length has diminished to $.56L$, the amplitude has increased up to the original amplitude. At this stage, the ratio of the depth to the actual wave-length, viz., $mh/2\pi$, is about .1, and the waves are still far from being "long" waves in the ordinary sense of the term.

When $\beta = 2a$, *i.e.*, when the original amplitude is doubled, $\theta = .125$, and the ratio of the depth to the actual wave-length is .02 so that the motion is practically one of "long" waves. Beyond this point Green's formula is certainly applicable.

3. The interest these results have depends on how far the assumption of no reflection is justified. To test this directly in the hydrodynamical problem would probably be difficult. In the somewhat analogous problem of the transverse vibrations of a tense cord, Lord Rayleigh ("On the Reflection of Vibrations at the Confines of Two Media between which the Transition is Gradual," *Proc. London Math. Soc.*, Ser. 1, Vol. XI, pp. 51–56) has shewn there is sensibly no reflection if the change of density of the cord is sufficiently slow. It can hardly be doubted that a similar result is true in the hydrodynamical problem if the depth changes with sufficient slowness.