EXTENSION OF TWO THEOREMS ON COVARIANTS

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[Communicated March 12th, 1903.—Received March 18th, 1903.]

1. Two of the best known elementary results in invariant algebra are the following:—

(i.) The Jacobian of a Jacobian of two binary forms with a third form is reducible.

(ii.) The product of two Jacobians can be expressed as an aggregate of products each containing three factors.

Involved in each statement is the condition that all the forms considered are of order 2 at least; accordingly, in extending the theorems I shall deal with perpetuants, and here content myself with the remark that the corresponding limitation for the extended theorems is easily discovered.

The first theorem has been extended by Jordan,* but, perpetuants not being considered by him, the extended theorem is not in its simplest form.

2. Consider now the first of the two results. Suppressing factors of the type a_x, it is equivalent to saying that a symbolical product of the form $(ab)(ac)$ is reducible, or, in other words, that, if an irreducible covariant contain three symbols, it must have at least three factors of the type (ab).

The extension is now clear, for it is known that a product containing ι symbols is reducible unless it contains at least $2^{\iota-1}-1$ factors of the determinantal type.

But the Jacobian theorem may be stated in a more general form, viz., if a product contain any number of symbols divided into three sets, and the sum of the exponents of factors containing two symbols belonging to different sets be less than three, then the form is reducible.

* See *Liouville*, 1879. Jordan's results in this connection are correct for forms of order not greater than 12, but after this point his upper limit to the order of an irreducible form is too large. I do not know of a case in which the highest order of an irreducible covariant of a system of binary forms is not given by choosing the greatest of the integers n, $2n-2$, $3n-6$, $4n-14$, $5n-30$, $6n-62$, ... where n is the greatest order of forms in the system. I have examined the cases where $n \ngtr 30$, and Mr. A. P. Thompson has, I believe, gone considerably further.

More generally, if there be ι sets of symbols, and the number of factors containing two symbols belonging to different sets be less than $2^{\iota-1}-1$, then the product is a reducible one.

3. To make this point clear it is convenient to introduce the idea of generalized transvectants. The r-th transvectant of $f = a_x^m$, $\phi = b_x^n$ is $(ab)^r a_x^{m-r} b_x^{n-r}$, that is to say, a numerical multiple of

$$\left(\frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial x_2 \partial y_1}\right)^r f_x \phi_y,$$

y being replaced by x after the operations are performed.

The latter form of definition is applicable when f and ϕ are symbolical products, and we have the fundamental theorem that any term in a transvectant can be expressed as an aggregate of transvectants, of index equal to or less than that of the original transvectant, of forms derived from the original forms by convolution. To extend this idea we note that the covariant $(bc)^\lambda (ca)^\mu (ab)^\nu a_x^{l-\mu-\nu} b_x^{m-\nu-\lambda} c_x^{n-\lambda-\mu}$ of a_x^l, b_x^m, c_x^n differs only numerically from

$$\left(\frac{\partial^2}{\partial y_1 \partial z_2} - \frac{\partial^2}{\partial z_1 \partial y_2}\right)^\lambda \left(\frac{\partial^2}{\partial z_1 \partial x_2} - \frac{\partial^2}{\partial x_1 \partial z_2}\right)^\mu \left(\frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial x_2 \partial y}\right)^\nu a_x^l b_y^m c_z^n,$$

y and z being replaced by x in the result.

This latter I call the generalized transvectant of a_x^l, b_x^m, c_x^n, having indices λ, μ, ν; the definition is applicable even when the forms are symbolical products.

In this way we can define a generalized transvectant of any number of forms, and, following the lines of the proof of the ordinary transvectant theorem quoted above, it is easy to show that any term in a transvectant can be expressed as an aggregate of transvectants, having indices equal to or less than the corresponding indices of the original transvectant, of forms derived from the original form by convolution.

4. Now suppose that P is a symbolical product containing ι sets of symbols, that P_1 is the product of all the factors containing only letters of the first set, P_2 the product of those containing only letters of the second set, &c., and let factors of the type a_x be suppressed.

Then, if the number of factors containing letters belonging to different sets be w, it is clear that P is a term in a generalized transvectant of $P_1, P_2, \ldots, P_\iota$, the sum of the indices being w.

Hence P can be expressed as an aggregate of transvectants of $\overline{P}_1, \overline{P}_2, \ldots, P_r, \ldots, \overline{P}_\iota$, where $\overline{P}_r$ is derived by convolution from P_r, and in each such transvectant the sum of the indices is equal to or less than w.

But, if $w < 2^{\iota-1}-1$, each of these transvectants is reducible and not conventionally so, but actually expressible in terms of forms of lower degree.

It follows at once that P is reducible. For example, consider the product $(ab)(bc)^9(cd)^3(de)^2$; it contains 15 factors, and, as 15 is the least possible number for an irreducible product containing five letters, this is at first sight a stable product, and, although in the complete system we should express it in terms of $(ab)^8(bc)^4(cd)^2(de)$ and reducible forms, this is only a conventional reduction. On the other hand, if we divide the letters into the four sets a; b, c; d; e, the sum of the exponents of factors containing letters of different sets is only 6, which is less than the minimum for irreducibility; hence the form in question can be completely expressed in terms of forms of lower degree.

5. We now proceed to the extension of the theorem relating to the product of two Jacobians.

Consider the product of $(ab)^2(bc)^2$ and (de), both of which are irreducible: it is

$$(ab)^2(bc)^2(de) = (ab)^2(bc)^2(ce)-(ab)^2(bc)^2(cd).$$

Now, in the product $(ab)^2(bc)^2(ce)$, d no longer explicitly occurs; therefore the corresponding form is a factor, and further, since there are only five factors and four letters, it follows that $(ab)^2(bc)^2(ce)$ can be expressed in terms of forms of lower degree.

Hence the product of the irreducible forms $(ab)^2(bc)^2$ and (de) can be expressed as an aggregate of products each containing three factors, and one of these factors is the form d or the form e.

Generally consider the product of P and $(\alpha\beta)$, when P contains ι letters and w factors, so that, if P be irreducible, we must have

$$w \geqslant 2^{\iota-1}-1.$$

Using just the same argument as above, it follows that, if $w < 2^{\iota}-1$, the product is expressible as an aggregate of products each containing either the form α or the form β and two other factors.

This is the extension contemplated; it is easy to construct examples, but rather tedious to calculate the right-hand side of the resulting syzygy.

[*Added October 11th*, 1903.—The formula for the maximum order of an irreducible covariant given on p. 153 is rigorously established in a paper of Mr. A. Young's to be published shortly in these *Proceedings*.]